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Stability Analysis of Impulsive Hopfield-Type Neuron System on Time Scale

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Abstract: Impulsive Hopfield-type neural systems on time scale are investigated. Sufficient conditions for the existence and uniqueness of the equilibrium state are obtained. Based on the generalized Lyapunov function method the sufficient conditions of global exponential stability are established for the neuron system under investigation. Efficiency of the obtained sufficient conditions is illustrated by a numerical example.

Keywords: *stability; time scale; Hopfield neural networks; impulsive system; exponential stability; Lyapunov function.*

Mathematics Subject Classification (2010): 92B20, 93D05, 93D30, 34K45, 34N05.

1 Introduction

Hopfield neural networks and their generalizations are important models of biological processes that are widely used now for solution of the applied problems in different areas of the modern technologies such as the optoelectronics, image reconstruction, speech synthesis, computer vision [1]– [6], and in the solution of different optimization problems, see also [7, 8].

Neural networks with impulses, both continuous and discrete ones, are widely used in the modeling of artificial intelligence, in robotics and electronics, and are intensively studied lately [9]– [13], with the most results obtained for neural networks with continuous time. Therefore, it makes sense to consider impulsive neural systems on time scale, which allows a simultaneous description of the system dynamics both in the discrete and the

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continuous case. In addition, this approach allows us to obtain new results for discrete neural systems, similar to those already known for the continuous case.

An extensive literature is devoted to the differential systems with impulsive action on general time scale [14]– [16] while the neural networks with pulses on the time scale are not well studied [17].

The purpose of this paper is to obtain the sufficient conditions of the global exponential stability of the equilibrium state for the impulsive neural Hopfield network on time scale. The study was carried out within the framework of the generalized second Lyapunov method on the basis of the scalar non-autonomous function on time scale.

2 Main Definitions and Necessary Theorems

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the set of real numbers $\mathbb{R}$. Fundamental notions and theorems of mathematical analysis on time scale, as well as the definitions of the derivative and the integral, the rules of differentiation and integration, the definitions and properties of rd-contiguous function, regressive function, the jump operator $\sigma(t)$, the graininess of the time scale $\mu(t)$ and the exponential function are explicity given in [18]– [20].

We need the following properties of the Δ-derivative.

Theorem 2.1 *Assume that f, g are Δ-differentiable at $t \in \mathbb{T}^k$. Then the following assertions are true:*

(1) *the product fg is Δ-differentiable at $t \in \mathbb{T}^k$ and*

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t) = f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t));$$

(2) $f(\sigma(t)) = f(t) + \mu(t)f^{\Delta}(t)$;

(3) *if $f^{\Delta}(t) \geq 0$ then f is non-decreasing on $\mathbb{T}^k$.*

We denote by $e_p(t, t_0)$ an exponential function on time scale. Further the following properties of an exponential function will be used.

Theorem 2.2 *If $p \in \mathcal{R}^+$, $\lambda > 0$, then for all $t, t_0 \in \mathbb{T}$ and $t \geq t_0$*

(1) $e_p(t_0, t_0) = 1$, $e_p(t, t_0) > 0$;

(2) $e_p(t, t_0) = 1/e_p(t_0, t)$;

(3) $e_p^{\Delta}(t, t_0) = p(t)e_p(t, t_0)$;

(4) $e_p(\sigma(t), t_0) = (1 + \mu(t)p(t))e_p(t, t_0)$;

(5) $\frac{1}{e_p(t,t_0)} = e_{\ominus p}(t, t_0)$, *where* $\ominus p \in \mathcal{R}^+$, $(\ominus p)(t) = -\frac{p(t)}{1+\mu(t)p(t)}$;

(6) $e_{\ominus\lambda}(t, t_0) \leq 1$, $\lim_{t\to+\infty} e_{\ominus\lambda}(t, t_0) = 0$ (*see* [21]);

(8) *if* $\mathbb{T} = \mathbb{R}$, *that* $e_{\ominus\lambda}(t, t_0) = e^{-\lambda(t-t_0)}$;

(9) *if* $\mathbb{T} = \mathbb{Z}$, *that* $e_{\ominus\lambda}(t, t_0) = (1 + \lambda)^{-(t-t_0)}$.

Here $\mathcal{R}^+$ is the set of all rd-continuous and positively regressive functions $f : \mathbb{T} \to \mathbb{R}$.

We denote by $\|x\| = (\sum_{i=1}^{n} x_i^2)^{1/2}$ an Euclidean vector norm of the vector $x \in \mathbb{R}^n$, $\|A\| = (\lambda_M(A^{\mathrm{T}}A))^{1/2}$ denotes a matrix norm of the matrix $A = \{a_{ij}\} \in \mathbb{R}^{n\times n}$, $\lambda_M(A)$ is a maximal eigenvalue of the matrix A, $[a,b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \le t \le b\}$ for $a, b \in \mathbb{T}$, the intervals $[a,b)_{\mathbb{T}}$, $(a,b]_{\mathbb{T}}$, $[a,+\infty)_{\mathbb{T}}$ are defined similarly.

Further we shall need the following result.

Lemma 2.1 *Let* $\widetilde{C} = \widetilde{C}[a,b]_{\mathbb{T}}$ *be a set of all continuous on* $[a,b]_{\mathbb{T}}$ *functions* $f\colon [a,b]_{\mathbb{T}} \to \mathbb{R}^n$, $p > 0$ *and* $\|\cdot\|^{\sim}$ *be a norm defined on* $\widetilde{C}$ *by the formula*

$$\|f\|^{\sim} = \sup_{t\in[a,b]_{\mathbb{T}}} \{p^{-1}\, e_{\ominus p}(t,a)\|f(t)\|\}.$$

Then $\big(\widetilde{C}, \|\cdot\|^{\sim}\big)$ *is a Banach space.*

Proof. Let $\|\cdot\|_1$ be a norm given on the set of $\widetilde{C}$ by the formula $\|f\|_1 = \sup_{t\in[a,b]_{\mathbb{T}}} \|f(t)\|$. We show that the norms $\|\cdot\|^{\sim}$ and $\|\cdot\|_1$ are equivalent. For all $f \in \widetilde{C}$ and $t \in [a,b]_{\mathbb{T}}$ we have $p^{-1}e_{\ominus p}(t,a)\|f(t)\| \le p^{-1}\|f(t)\| \le p^{-1}\|f\|_1$ or $\|f\|^{\sim} \le p^{-1}\|f\|_1$. Since the function $e_p(t,a)$ is continuous on $[a,b]_{\mathbb{T}}$, there exists a constant $\mathcal{E} > 0$ such that $e_p(t,a) \le \mathcal{E}$ for all $t \in [a,b]_{\mathbb{T}}$, whence for any $f \in \widetilde{C}$ we have $\|f(t)\| = p\, e_p(t,a)p^{-1}e_{\ominus p}(t,a)\|f\| \le p\,\mathcal{E}p^{-1}e_{\ominus p}(t,a)\|f(t)\| \le p\,\mathcal{E}\|f\|^{\sim}$ or $\|f\|_1 \le p\,\mathcal{E}\|f\|^{\sim}$. Thus, the norms $\|\cdot\|^{\sim}$ and $\|\cdot\|_1$ are equivalent.

As is known from the mathematical analysis, since $[a,b]_{\mathbb{T}}$ is a compact set, the space $\big(\widetilde{C}, \|\cdot\|_1\big)$ is a Banach space. Consequently, $\big(\widetilde{C}, \|\cdot\|^{\sim}\big)$ is also a Banach space. Lemma 2.1 is proved.

3 Impulsive Neural Network on Time Scale

Let $\mathbb{T}$ be an arbitrary time scale, $\sup \mathbb{T} = +\infty$, the sequence $\{t_k\}_{k=1}^{+\infty} \subset \mathbb{T}$ so that $t_1 < t_2 < ...$, $t_k \to +\infty$ and $k \to +\infty$ and points t_k are dense.

We consider the impulsive neural system

$$x^{\Delta}(t) = -Bx(t) + Ts(x(t)) + u, \quad t \in \mathbb{T}_{\tau}, \quad t \ne t_k, \tag{1}$$

$$x(t_k^+) = x(t_k) + I_k(x(t_k)), \quad k \in \mathbb{N}, \tag{2}$$

with the initial condition

$$x(t_0) = x_0, \quad t_0 \in \mathbb{T}_{\tau}, \quad x_0 \in \mathbb{R}^n. \tag{3}$$

Here $\mathbb{T}_{\tau} = [\tau,+\infty)_{\mathbb{T}}$, $\tau \in \mathbb{T}$, $\tau < t_1$, $x = (x_1, x_2, ..., x_n)^{\mathrm{T}} \in \mathbb{R}^n$, x_i is the activation of the i-th neuron, $T = \{t_{ij}\} \in \mathbb{R}^{n\times n}$, the components t_{ij} describe the interaction between the ith and jth neurons, $s : \mathbb{R}^n \to \mathbb{R}^n$, $s(x) = (s_1(x_1), s_2(x_2), \dots, s_n(x_n))^{\mathrm{T}}$, the function s_i describes the response of the ith neuron, $B \in \mathbb{R}^{n\times n}$, $B = \operatorname{diag}\{b_1, b_2, \dots, b_n\}$, $b_i > 0$, $i = 1, 2, \dots, n$, $u \in \mathbb{R}^n$ is a constant external input vector, the function $I_k\colon \mathbb{R}^n \to \mathbb{R}^n$ describes impulsive perturbations of the neural system.

By the solution of the impulsive system (1), (2) we mean the function $x(t)$, which for $t \ne t_k$ satisfies the equation (1) and for $t = t_k$ satisfies the equation (2), where $x(t_k^+) = \lim_{t\to t_k+0,\ t\in\mathbb{T}} x(t)$, $x(t_k) = x(t_k^-) = \lim_{t\to t_k-0,\ t\in\mathbb{T}} x(t)$ are one-sided right-hand and left-hand limits of the function $x(t)$, respectively.

Concerning the system (1) and the time scale $\mathbb{T}$, we introduce the following assumptions.

H_1. There are positive constants $l_i > 0$, $i = 1, 2, \dots, n$ such that $|s_i(u) - s_i(v)| \le l_i|u - v|$ for all $u, v \in \mathbb{R}$.

H_2. There is a constant $\mu^* > 0$ such that $\mu(t) \le \mu^*$ for all $t \in \mathbb{T}_\tau$.

Existence conditions for a unique equilibrium state of the system (1) without impulses be given by the following theorem, the proof of which is similar to the proof of Theorem 1 from [22].

Theorem 3.1 *Let the assumption* H_1 *be valid and there exist a constant* $d_i > 0$, $i = 1, 2, \dots, n$, *such that the inequalities*

$$b_i - \frac{1}{2}\sum_{j=1}^{n}\big(l_j|t_{ij}| + \frac{d_j}{d_i}\, l_i|t_{ji}|\big) > 0, \quad i = 1, 2, \dots, n, \tag{4}$$

are true. Then there is a unique equilibrium state of the system (1).

The equilibrium state of the system (1), (2) will be referred to as the constant function $x(t) \equiv x^*$, which is the solution of the system (1), (2). Using Theorem 3.1 it is easy to get the following result.

Theorem 3.2 *Let the assumption* H_1 *and inequalities* (4) *be valid and let* $x(t) \equiv x^*$ *be an equilibrium state of the system* (1). *If* $I_k(x^*) = 0$ *for all* $k \in \mathbb{N}$, *then* $x(t) \equiv x^*$ *is a unique equilibrium state of the system* (1), (2).

We prove the following theorem on the existence and uniqueness of the solution of impulsive neural system.

Theorem 3.3 *Let the assumption* H_1 *be valid, then there exists a unique solution of problem* (1) – (3) *on* $[t_0, +\infty)_{\mathbb{T}}$ *for all initial data* $(t_0, x_0) \in \mathbb{T}_\tau \times \mathbb{R}^n$.

Proof. For an arbitrary $t_0 \in \mathbb{T}_\tau$ two cases are possible: $t_0 \in [\tau, t_1)_{\mathbb{T}}$ or $t_0 \in [t_{k-1}, t_k)_{\mathbb{T}}$ for some $k = 2, 3, \dots$. We first choose $t_0 \in [\tau, t_1)_{\mathbb{T}}$ and we denote $L = \max\{l_1, l_2, \dots, l_n\}$, $\gamma = \|B\| + L\|T\|$, $p = \gamma + 1$.

Let $\widetilde{C}_1 = \widetilde{C}_1[t_0, t_1]$ be the space of continuous functions $f\colon [t_0, t_1]_{\mathbb{T}} \to \mathbb{R}^n$ with the norm

$$\|f\|_1^{\sim} = \sup_{t \in [t_0,\, t_1]_{\mathbb{T}}} \{p^{-1}\, e_{\ominus p}(t, t_0)\|f(t)\|\}.$$

Consider the operator $F_1\colon \widetilde{C}_1 \to \widetilde{C}_1$ acting according to the formula

$$F_1(x)(t) = x_{01} + \int_{t_0}^{t} [-Bx(\lambda) + Ts(x(\lambda)) + u]\Delta\lambda,$$

where $x_{01} = x_0$. The function $-Bx(t) + Ts(x(t)) + u$ is continuous on the segment $[t_0, t_1]_{\mathbb{T}}$, hence it is rd-continuous on $[t_0, t_1]_{\mathbb{T}}$. In accordance with Theorem 1.74 from [18] the function $F_1(x)(t)$ is differentiable on $[t_0, t_1]_{\mathbb{T}}$ (and, as a consequence, it is continuous on $[t_0, t_1]_{\mathbb{T}}$) and

$$[F_1(x)(t)]^{\Delta} = -Bx(t) + Ts(x(t)) + u, \quad t \in [t_0, t_1]_{\mathbb{T}}.$$

We verify the fulfillment of the conditions of the contraction map principle. For any $x, y \in \widetilde{C}_1$ for all $t \in [t_0, t_1]_{\mathbb{T}}$ we obtain the inequalities

$$
\begin{aligned}
\|F_1(x)(t) - F_1(y)(t)\| &= \| \int_{t_0}^{t} [-B(x(\lambda) - y(\lambda)) + Ts(x(\lambda) - y(\lambda))]\Delta\lambda \| \leq \\
&\leq \|B\| \int_{t_0}^{t} \|x(\lambda) - y(\lambda)\|\Delta\lambda + L\|T\| \int_{t_0}^{t} \|x(\lambda) - y(\lambda)\|\Delta\lambda = \\
= \gamma \int_{t_0}^{t} \|x(\lambda) - y(\lambda)\|\Delta\lambda &= \gamma \int_{t_0}^{t} p\, e_p(\lambda, t_0) p^{-1} e_{\ominus p}(\lambda, t_0) \|x(\lambda) - y(\lambda)\|\Delta\lambda \leq \\
&\leq \gamma \sup_{\lambda \in [t_0,\, t_1]_{\mathbb{T}}} \{p^{-1} e_{\ominus p}(\lambda, t_0) \|x(\lambda) - y(\lambda)\|\} \int_{t_0}^{t} p\, e_p(\lambda, t_0)\Delta\lambda = \\
&= \gamma \|x - y\|_1^{\sim} (e_p(t, t_0) - 1) \leq \gamma\, e_p(t, t_0) \|x - y\|_1^{\sim},
\end{aligned}
$$

whence we have

$$
\begin{aligned}
\frac{1}{p}\, e_{\ominus p}(t, t_0) \|F_1(x)(t) - F_1(y)(t)\| &\leq \frac{\gamma}{p} \|x - y\|_1^{\sim}, \\
\sup_{t \in [t_0,\, t_1]_{\mathbb{T}}} \{p^{-1} e_{\ominus p}(t, t_0) \|F_1(x)(t) - F_1(y)(t)\|\} &\leq \frac{\gamma}{p} \|x - y\|_1^{\sim}, \\
\|F_1(x) - F_1(y)\|_1^{\sim} &\leq \frac{\gamma}{\gamma + 1} \|x - y\|_1^{\sim}.
\end{aligned}
$$

Thus, the map F_1 is a contraction and consequently, there exists a unique fixed point $\widetilde{x}_1 \in \widetilde{C}_1$ of the operator F_1 for which we have

$$
\begin{aligned}
&[\widetilde{x}_1(t)]^{\Delta} = -B\widetilde{x}_1(t) + Ts(\widetilde{x}_1(t)) + u, \quad t \in [t_0, t_1]_{\mathbb{T}}, \\
&\widetilde{x}_1(t_0) = x_{01}.
\end{aligned}
$$

Now let $\widetilde{C}_2 = \widetilde{C}_2[t_1, t_2]$ be a space of continuous functions $f\colon [t_1,\, t_2]_{\mathbb{T}} \to \mathbb{R}^n$ with the norm

$$
\|f\|_2^{\sim} = \sup_{t \in [t_1,\, t_2]_{\mathbb{T}}} \{p^{-1} e_{\ominus p}(t, t_1) \|f(t)\|\}.
$$

Consider the operator $F_2\colon \widetilde{C}_2 \to \widetilde{C}_2$ acting according to the formula

$$
F_2(x)(t) = x_{02} + \int_{t_1}^{t} [-Bx(\lambda) + Ts(x(\lambda)) + u]\Delta\lambda,
$$

where $x_{02} = \widetilde{x}_1(t_1) + I_1(\widetilde{x}_1(t_1))$. As above, there exists a unique fixed point $\widetilde{x}_2 \in \widetilde{C}_2$ of the operator F_2 for which we obtain that

$$
\begin{aligned}
&[\widetilde{x}_2(t)]^{\Delta} = -B\widetilde{x}_2(t) + Ts(\widetilde{x}_2(t)) + u, \quad t \in [t_1, t_2]_{\mathbb{T}}, \\
&\widetilde{x}_2(t_1) = x_{02}.
\end{aligned}
$$

Similarly, at the kth step, let $\widetilde{C}_k = \widetilde{C}_k[t_{k-1}, t_k]$ be a space of continuous functions $f\colon [t_{k-1},\, t_k]_{\mathbb{T}} \to \mathbb{R}^n$ with the norm

$$
\|f\|_k^{\sim} = \sup_{t \in [t_{k-1},\, t_k]_{\mathbb{T}}} \{p^{-1} e_{\ominus p}(t, t_{k-1}) \|f(t)\|\},
$$

and $F_k\colon \widetilde{C}_k \to \widetilde{C}_k$ is the operator acting according to the formula

$$F_k(x)(t) = x_{0k} + \int_{t_{k-1}}^{t} [-Bx(\lambda) + Ts(x(\lambda)) + u]\Delta\lambda,$$

where $x_{0k} = \widetilde{x}_{k-1}(t_{k-1}) + I_{k-1}(\widetilde{x}_{k-1}(t_{k-1}))$. As above, there exists a unique fixed point $\widetilde{x}_k \in \widetilde{C}_k$ of the operator F_k and

$$\begin{aligned}
&[\widetilde{x}_k(t)]^{\Delta} = -B\widetilde{x}_k(t) + Ts(\widetilde{x}_k(t)) + u, \quad t \in [t_{k-1}, t_k]_{\mathbb{T}},\\
&\widetilde{x}_k(t_1) = x_{0k}.
\end{aligned}$$

We now consider the function

$$x(t) = \begin{cases} \widetilde{x}_1(t), & t \in [\tau, t_1]_{\mathbb{T}},\\ \widetilde{x}_k(t), & t \in (t_{k-1}, t_k]_{\mathbb{T}}, \quad k = 2, 3, \dots. \end{cases} \tag{5}$$

It is clear that the function (5) is a solution of the Cauchy problem (1)-(3) on $[t_0, +\infty)_{\mathbb{T}}$ and moreover, it is unique.

The case $t_0 \in [t_{k-1}, t_k)_{\mathbb{T}}$ for some $k = 2, 3, \dots$ is investigated similarly. Theorem 3.3 is proved.

4 Stability of the Neural Network

Let $x(t) \equiv x^*$ be an isolated equilibrium state of the system (1), (2).

Definition 4.1 The equilibrium state $x(t) \equiv x^*$ of the system (1), (2) is called globally uniformly exponentially stable, if there exist constants $p > 0$, $\alpha > 0$ and $\mathcal{N} = \mathcal{N}(x_0) > 0$ such that $\|x(t; t_0, x_0) - x^*\| < \mathcal{N}(e_{\ominus p}(t, t_0))^{\alpha}$ for all $x_0 \in \mathbb{R}^n$, $t_0 \in \mathbb{T}_\tau$ and $t \in [t_0, +\infty)_{\mathbb{T}}$.

We make the change of variables $y(t) = x(t) - x^*$ and rewrite the initial problem (1)–(3) in the form

$$y^{\Delta}(t) = -By(t) + Tg(y(t)), \quad t \in \mathbb{T}_\tau, \quad t \neq t_k, \tag{6}$$

$$y(t_k^+) = y(t_k) + J_k(y(t_k)), \quad k \in \mathbb{N}, \tag{7}$$

$$y(t_0; t_0, y_0) = y_0, \quad t_0 \in \mathbb{T}_\tau, \quad y_0 \in \mathbb{R}^n,$$

where

$$y \in \mathbb{R}^n, \quad g(y) = (g_1(y_1), g_2(y_2), \dots, g_n(y_n))^{\mathrm{T}}, \;\; J_k(y) = (J_{k1}(y), J_{k2}(y), \dots, J_{kn}(y))^{\mathrm{T}},$$

$$g(y) = s(y + x^*) - s(x^*), \quad J_k(y) = I_k(y + x^*) - I_k(x^*).$$

It is clear that the behavior of the solution $x(t)$ of the system (1), (2) in the neighborhood of the equilibrium state x^* is equivalent to the behavior of solution $y(t)$ of the system (6), (7) in the neighborhood of zero.

If for the system (1) the assumption H_1 is valid, then for the system (6), (7) the following statements are true.

G_1. There are positive constants $l_i > 0$ such that $|g_i(r) - g_i(v)| \le l_i|\,r - v|$, for all $r, v \in \mathbb{R}$, $i = 1, 2, \dots, n$.

G_2. $g(0) = 0,\ J_k(0) = 0,\ k \in \mathbb{N}$.

Theorem 4.1 *We assume that assumptions* G_1, G_2 *and* H_2 *are valid. Let for all* $y \in \mathbb{R}^n$ *the inequalities*

$$J_{ki}^2(y) + 2y_i J_{ki}(y) \le 0, \quad k \in \mathbb{N}, \quad i = 1, 2, \dots, n,$$

are true and there exist constants $d_i > 0,\ i = 1, 2, \dots, n$ *such that the inequalities*

$$b_i - \frac{1}{2}\sum_{j=1}^{n}(l_j|t_{ij}| + \frac{d_j}{d_i} l_i |t_{ji}|) - \frac{1}{2}\mu^*(n+1)(b_i^2 + l_i^2 \sum_{j=1}^{n} t_{ji}^2) > 0, \quad i = 1, 2, \dots, n, \tag{8}$$

are valid. Then the equilibrium state $y(t) \equiv 0$ *of the system* (6), (7) *is globally uniformly exponentially stable.*

Proof. We denote

$$\xi_i = \sum_{j=1}^{n}(d_i l_j |t_{ij}| + d_j l_i |t_{ji}|),$$
$$\nu_i = (n+1)(d_i b_i^2 + \eta_i),$$
$$\eta_i = d_i l_i^2 \sum_{j=1}^{n} t_{ji}^2, \quad i = 1, 2, \dots, n,$$

and write the inequalities (8) in the form

$$2d_i b_i - \xi_i - \mu^* \nu_i > 0, \quad i = 1, 2, \dots, n.$$

Now choose a constant

$$0 < p < \min_{i=1,2,\dots,n}(2d_i b_i - \xi_i - \mu^*\nu_i) d_i^{-1}$$

and apply the Lyapunov function to the proof of the theorem

$$v(t, y) = \sum_{i=1}^{n} d_i y_i^2(t) e_p(t, t_0), \quad d_i > 0, \quad i = 1, 2, \dots, n.$$

Let $t \neq t_k$. For convenience, in what follows we shall write $y_i,\ \sigma,\ \mu_i$ and $g(y_i)$ instead of $y_i(t),\ \sigma(t),\ \mu_i(t)$ and $g(y_i(t))$ respectively. Since

$$(y_i^2)^\Delta = y_i^\Delta y_i + y_i^\Delta y_i(\sigma) = 2y_i^\Delta y_i + \mu(y_i^\Delta)^2,$$

for the derivative of the function y_i^2 along the solutions of the system (6) at the point t we have the estimate

$$
\begin{aligned}
(y_i^2)^\Delta|_{(6)} &= 2y_i\big(-b_i y_i + \sum_{j=1}^{n} t_{ij} g_j(y_j)\big) + \mu\big(-b_i y_i + \sum_{j=1}^{n} t_{ij} g_j(y_j)\big)^2 \le \\
&\le -2b_i y_i^2 + 2\sum_{j=1}^{n} |t_{ij}||y_i||g_j(y_j)| + \mu(n+1)\big(b_i^2 y_i^2 + \sum_{j=1}^{n} t_{ij}^2 g_j^2(y_j)\big) \le \\
&\le -2b_i y_i^2 + 2\sum_{j=1}^{n} l_j |t_{ij}||y_i||y_j| + \mu(n+1)\big(b_i^2 y_i^2 + \sum_{j=1}^{n} l_j^2 t_{ij}^2 y_j^2\big) = \\
&\le (-2b_i y_i^2 + \mu(n+1) b_i^2) y_i^2 + 2\sum_{j=1}^{n} l_j |t_{ij}||y_i||y_j| + \mu(n+1)\sum_{j=1}^{n} l_j^2 t_{ij}^2 y_j^2.
\end{aligned}
$$

Further, using the formula

$$[y_i^2 e_p(t,t_0)]^\Delta = \big[p y_i^2 + (y^\Delta)^2(1+\mu p)\big] e_p(t,t_0),$$

for the derivative of the function $y_i^2 e_p(t,t_0)$ along the solutions of system (6) we obtain the estimates

$$\begin{aligned}
[y_i^2 e_p(t,t_0)]^\Delta|_{(6)} \le \Big[& p y_i^2 + (1+\mu p)\big\{(-2b_i y_i^2 + \mu(n+1)b_i^2)y_i^2 + \\
&+ 2\sum_{j=1}^n l_j|t_{ij}||y_i||y_j| + \mu(n+1)\sum_{j=1}^n l_j^2 t_{ij}^2 y_j^2\big\}\Big] e_p(t,t_0) = \\
&= \Big[\big\{p + (1+\mu p)(-2b_i + \mu(n+1)b_i^2)\big\}y_i^2 + \\
&+ 2(1+\mu p)\sum_{j=1}^n l_j|t_{ij}||y_i||y_j| + \mu(n+1)(1+\mu p)\sum_{j=1}^n l_j^2 t_{ij}^2 y_j^2\Big] e_p(t,t_0).
\end{aligned}$$

Now we can estimate the derivative of the function $v(t,y(t))$ along solutions (6)

$$\begin{aligned}
v^\Delta(t,y(t))|_{(6)} &= \sum_{i=1}^n d_i [y_i^2 e_p(t,t_0)]^\Delta|_{(6)} \le \\
\le \sum_{i=1}^n d_i e_p(t,t_0)\Big[&\big\{p + (1+\mu p)(-2b_i + \mu(n+1)b_i^2)\big\}y_i^2 + \\
+2(1+\mu p)\sum_{j=1}^n l_j|t_{ij}||y_i||y_j| &+ \mu(n+1)(1+\mu p)\sum_{j=1}^n l_j^2 t_{ij}^2 y_j^2\Big] = \\
= e_p(t,t_0)\Big[\sum_{i=1}^n d_i &\big\{p + (1+\mu p)(-2b_i + \mu(n+1)b_i^2)\big\}y_i^2 + \\
+(1+\mu p)\sum_{i,j=1}^n 2d_i l_j|t_{ij}||y_i||y_j| &+ \mu(n+1)(1+\mu p)\sum_{i,j=1}^n d_i l_j^2 t_{ij}^2 y_j^2\Big].
\end{aligned} \tag{9}$$

Let us consider separately the last two double sums

$$\begin{aligned}
\sum_{i,j=1}^n 2d_i l_j|t_{ij}||y_i||y_j| &\le \sum_{i,j=1}^n 2d_i l_j|t_{ij}|\frac{y_i^2 + y_j^2}{2} = \sum_{i,j=1}^n \big(d_i l_j|t_{ij}|y_i^2 + d_i l_j|t_{ij}|y_j^2\big) = \\
&= \sum_{i,j=1}^n d_i l_j|t_{ij}|y_i^2 + \sum_{i,j=1}^n d_i l_j|t_{ij}|y_j^2 = \sum_{i,j=1}^n d_i l_j|t_{ij}|y_i^2 + \sum_{i,j=1}^n d_j l_i|t_{ji}|y_i^2 = \\
&= \sum_{i=1}^n\Big[\sum_{j=1}^n (d_i l_j|t_{ij}| + d_j l_i|t_{ji}|)\Big]y_i^2 = \sum_{i=1}^n \xi_i y_i^2, \\
\sum_{i,j=1}^n d_i l_j^2 t_{ij}^2 y_j^2 &= \sum_{i,j=1}^n d_j l_i^2 t_{ji}^2 y_i^2 = \sum_{i=1}^n l_i^2\big(\sum_{j=1}^n d_j t_{ji}^2\big)y_i^2 = \sum_{i=1}^n \eta_i y_i^2
\end{aligned}$$

and continue the estimate (9)

$$v^{\Delta}(t,y(t))|_{(6)} \le e_p(t,t_0)\Big[\sum_{i=1}^{n} d_i\big\{p+(1+\mu p)(-2b_i+\mu(n+1)b_i^2)\big\}y_i^2+$$

$$+(1+\mu p)\sum_{i=1}^{n}\xi_i y_i^2+\mu(n+1)(1+\mu p)\sum_{i=1}^{n}\eta_i y_i^2\Big]=$$

$$=e_p(t,t_0)\sum_{i=1}^{n}\Big[d_i\big\{p+(1+\mu p)(-2b_i+\mu(n+1)b_i^2)\big\}+(1+\mu p)(\xi_i+\mu(n+1)\eta_i)\Big]y_i^2=$$

$$=e_p(t,t_0)\sum_{i=1}^{n}\Big[d_ip+(1+\mu p)\{d_i(-2b_i+\mu(n+1)b_i^2)+\xi_i+\mu(n+1)\eta_i\}\Big]y_i^2=$$

$$=e_p(t,t_0)\sum_{i=1}^{n}\Big[d_ip+(1+\mu p)\{-2d_ib_i+\mu(n+1)d_ib_i^2+\xi_i+\mu(n+1)\eta_i\}\Big]y_i^2=$$

$$=e_p(t,t_0)\sum_{i=1}^{n}\Big[d_ip+(1+\mu p)\{-2d_ib_i+\xi_i+\mu(n+1)(d_ib_i^2+\eta_i)\}\Big]y_i^2=$$

$$=e_p(t,t_0)\sum_{i=1}^{n}\Big[d_ip+(1+\mu p)\{-2d_ib_i+\xi_i+\mu\nu_i\}\Big]y_i^2. \tag{10}$$

For the quadratic trinomial $\psi_i(z)=\nu_i z^2-(2d_ib_i-\xi_i)z+d_i$, taking into account the fact that $\nu_i>0$ and the discriminant

$$\begin{aligned}D&=(2d_ib_i-\xi_i)^2-4\nu_id_i=\\&=4d_i^2b_i^2-4d_ib_i\xi_i+\xi_i^2-4d_i(n+1)(d_ib_i^2+\eta_i)=\\&=4d_i^2b_i^2-4d_ib_i\xi_i+\xi_i^2-4d_i^2(n+1)b_i^2-4d_i(n+1)\eta_i=\\&=-4nd_i^2b_i^2-\xi_i^2+\xi_i^2-4d_ib_i\xi_i+\xi_i^2-4d_i(n+1)\eta_i=\\&=-4nd_i^2b_i^2-\xi_i^2-2\xi_i(2d_ib_i-\xi_i)-4d_i(n+1)\eta_i<0,\end{aligned}$$

we have that $\psi_i(z)>0$ for all $z\in\mathbb{R}$. Thus, for all $i=1,2,\dots,n,\ t\ne t_k$

$$d_i-\mu(2d_ib_i-\xi_i-\mu\nu_i)=\nu_i\mu^2-(2d_ib_i-\xi_i)\mu+d_i>0$$

and, beside,

$$2d_ib_i-\xi_i-\mu\nu_i>2d_ib_i-\xi_i-\mu^*\nu_i>0.$$

Therefore, by the choice of the constant p the inequalities

$$0<p<\frac{2d_ib_i-\xi_i-\mu^*\nu_i}{d_i}\le\frac{2d_ib_i-\xi_i-\mu\nu_i}{d_i}\le\frac{2d_ib_i-\xi_i-\mu\nu_i}{d_i-\mu(2d_ib_i-\xi_i-\mu\nu_i)},$$

are true, whence we obtain

$$\begin{aligned}p(d_i-\mu(2d_ib_i-\xi_i-\mu\nu_i))&\le 2d_ib_i-\xi_i-\mu\nu_i,\\ d_ip+(1+\mu p)(-2d_ib_i+\xi_i+\mu\nu_i)&<0.\end{aligned}$$

Continuing the estimate (10), finally, for all $t \neq t_k$, we will have

$$v^{\Delta}(t, y(t))|_{(6)} \leq 0.$$

If $t \in [t_0, t_1]_{\mathbb{T}}$, then by Theorem 2.1 and definition of $y(t_k^+)$ we have

$$v(t, y(t)) \leq v(t_0, y_0). \tag{11}$$

Similarly, for all $t \in (t_k, t_{k+1}]_{\mathbb{T}}$, $k \in \mathbb{N}$, the inequality

$$v(t, y(t)) \leq v(t_k^+, y(t_k^+))$$

is true. Since

$$v(t_k^+, y(t_k^+)) - v(t_k, y(t_k)) = \sum_{i=1}^{n} d_i (y_i^2(t_k^+) - y_i^2(t_k)) e_p(t_k, t_0) =$$
$$= \sum_{i=1}^{n} d_i \Big[2y(t_k) J_{ki}(y(t_k)) + J_{ki}^2(y(t_k)) \Big] e_p(t_k, t_0) \leq 0,$$

we have

$$v(t, y(t)) \leq v(t_k, y(t_k)), \quad t \in (t_k, t_{k+1}]_{\mathbb{T}}, \;\; k \in \mathbb{N}.$$

In view of (11), the last estimate leads to the inequality

$$v(t, y(t)) \leq v(t_0, y_0) \;\text{ for all }\; t \in [t_0, +\infty]_{\mathbb{T}},$$

from which it is easy to obtain the following estimate

$$\|y(t)\| \leq m \|y_0\| (e_{\ominus p}(t, t_0))^{\frac{1}{2}},$$

where $m = (\max_{i=1,2,\dots,n}\{d_i\} / \min_{i=1,2,\dots,n}\{d_i\})^{1/2}$, for all $y_0 \in \mathbb{R}^n$, $t_0 \in \mathbb{T}_\tau$ and $t \in [t_0, +\infty)_{\mathbb{T}}$. Theorem 4.1 is proved.

We now consider the system (1), (2) in the particular case, when the impulsive action is given by a linear function.

Corollary 4.1 *Suppose that the assumptions* $\mathrm{H}_1, \mathrm{H}_2$ *are satisfied and there exist constants* $d_i > 0$, $i = 1, 2, \dots, n$ *such that the following inequalities hold*

$$b_i - \frac{1}{2} \sum_{j=1}^{n} (l_j |t_{ij}| + \frac{d_j}{d_i} l_i |t_{ji}|) - \frac{1}{2} \mu^* (n+1)(b_i^2 + l_i^2 \sum_{j=1}^{n} t_{ji}^2) > 0, \quad i = 1, 2, \dots, n.$$

Let $x(t) \equiv x^*$ *be the only equilibrium state of the systems* (1), (2) *and*

$$I_{ki}(x_i(t_k) - x^*) = -\gamma_{ik}(x_i(t_k) - x^*), \qquad k \in \mathbb{N}, \quad i = 1, 2, \dots, n,$$

where $0 < \gamma_{ik} \leq 2$. *Then the equilibrium state* $x(t) \equiv x^*$ *of the system* (1), (2) *is globally uniformly exponentially stable.*

5 Example

On the time scale

$$\mathbb{P}_{1,\beta} = \cup_{j=0}^{\infty} \big[j(1+\beta), 1 + j(1+\beta)\big]$$

we consider a three-component neural network with impulses

$$\begin{aligned} x_1(t)^{\Delta} &= -x_1(t) + 0,1s_1(x_1(t)) + 0,09s_2(x_2(t)) - 0,1s_3(x_3(t)) - 2,09,\\ x_2(t)^{\Delta} &= -x_2(t) + 0,05s_1(x_1(t)) - 0,1s_2(x_2(t)) + 0,1s_3(x_3(t)) + 1,25,\\ x_3(t)^{\Delta} &= -x_3(t) - 0,1s_1(x_1(t)) + 0,05s_2(x_2(t)) + 0,06s_3(x_3(t)) + 0,96,\\ &\qquad t \neq t_k, \end{aligned} \tag{12}$$

$$\begin{aligned} x_1(t_k^+) &= x_1(t_k) + \gamma(x_1(t_k) - 2),\\ x_2(t_k^+) &= x_2(t_k) + \gamma(x_2(t_k) + 1),\\ x_3(t_k^+) &= x_3(t_k) + \gamma(x_1(t_k) + 1,5), \quad k \in \mathbb{N}, \end{aligned} \tag{13}$$

where $x_1, x_2, x_3 \in \mathbb{R}$, $s_1(r) = s_2(r) = s_3(r) = \frac{1}{2}\big(|r+1| - |r-1|\big)$, $t_k = (k-1)(1+\beta)+0,5$. Since the inequalities (4) are satisfied with the constants $l_i = d_i = 1$, the state $x^* = (2; -1; -1,5)^{\mathrm{T}}$ is the only equilibrium state of the systems (12). In view of the fact that $\mu^* = \beta$, the inequalities (8) take the form

$$\begin{aligned} &0,32 - 2,045\beta > 0,\\ &0,805 - 2,0412\beta > 0,\\ &0,765 - 2,0472\beta > 0, \end{aligned}$$

from which we find $\beta < 0,1564$. According to Corollary 4.1 we conclude that for $\beta < 0,1564$ the equilibrium $x^* = (2; -1; -1,5)^{\mathrm{T}}$ of the system (12), (13) is globally uniformly exponentially stable.

6 Conclusion

In the framework of the approach proposed in the paper [23] sufficient conditions of global uniform exponential stability are obtained for the equilibrium state of a neural network with impulses on an arbitrary time scale. The case is considered when the impulse action is given by a linear function. We note that in [10] similar results are obtained for $\mathbb{T} = \mathbb{R}$ under the assumption that the functions s_i are bounded. Corollary 4.1 of the present paper for $\mathbb{T} = \mathbb{R}$ gives sufficient conditions under which such a restriction is absent. In addition, sufficient conditions for the existence of a unique equilibrium state of a neural impulsive system on time scale are obtained.

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