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Weak Solutions to Implicit Differential Equations Involving the Hilfer Fractional Derivative

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Abstract: In this paper, the authors present some existence results for weak solutions to some functional implicit fractional differential equations of Hilfer type, by applying Mönch's fixed point theorem associated with the technique of measure of weak noncompactness.

Keywords: *functional differential equation; left-sided mixed Pettis Riemann-Liouville integral of fractional order; Hilfer fractional derivative; implicit; weak solution; fixed point.*

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1 Introduction

Fractional differential equations have recently been applied in various areas of engineering, mathematics, physics, bio-engineering, and other applied sciences [15,24]. For some fundamental results in the theory of fractional calculus and fractional differential equations, we refer the reader to the monographs of Abbas *et al.* [1–3], Samko *et al.* [23], Kilbas *et al.* [18], and Zhou [27].

The notion of a measure of weak noncompactness was introduced by De Blasi [13]. The strong measure of noncompactness was developed first by Banaȿ and Goebel [7] and subsequently developed and used in many papers; see, for example, Akhmerov *et al.* [5], Alvàrez [6], Benchohra *et al.* [11], Guo *et al.* [14], and the references therein. In [11,21], the authors considered some existence results by applying the techniques of the measure

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of noncompactness. Recently, several researchers obtained other results by applying the technique of measure of weak noncompactness; see [3, 9, 10], and the references therein.

Implicit functional differential equations have been considered by many authors [4, 8, 26]. Our intention is to extend the results to implicit differential equations of fractional order. Recently, considerable attention has been given to the existence of solutions of initial and boundary value problems for fractional differential equations with a Hilfer fractional derivative; see, for example, [15–17, 25]. In this paper, we discuss the existence of weak solutions to the problem of implicit Hilfer fractional differential equation of the form

$$\begin{cases} (D_0^{\alpha,\beta}u)(t) = f(t, u(t), (D_0^{\alpha,\beta}u)(t)), \ t \in I := [0, T], \\ (I_0^{1-\gamma}u)(t)|_{t=0} = \phi, \end{cases} \tag{1}$$

where $\alpha \in (0,1)$, $\beta \in [0,1]$, $\gamma = \alpha + \beta - \alpha\beta$, $T > 0$, $\phi \in E$, $f : I \times E \times E \to E$ is a given continuous function, E is a real (or complex) Banach space with norm $\|\cdot\|_E$ and dual space E^*, such that E is the dual space of a weakly compactly generated Banach space X, $I_0^{1-\gamma}$ is the left-sided mixed Riemann-Liouville integral of order $1-\gamma$, and $D_0^{\alpha,\beta}$ is the generalized Riemann-Liouville derivative operator of order α and type β introduced by Hilfer in [15]. Our goal in this work is to give some existence results for implicit Hilfer fractional differential equations in Banach spaces.

2 Preliminaries

Let C be the Banach space of all continuous functions v from I into E with the supremum (uniform) norm

$$\|v\|_\infty := \sup_{t \in I} \|v(t)\|_E.$$

As usual, $AC(I)$ denotes the space of absolutely continuous functions from I into E. We denote by $AC^1(I)$, the space defined by

$$AC^1(I) := \{w : I \to E : \frac{d}{dt}w(t) \in AC(I)\}.$$

By $C_\gamma(I)$ and $C^1_\gamma(I)$, we mean the weighted spaces of continuous functions defined by

$$C_\gamma(I) = \{w : (0, T] \to E : t^{1-\gamma}w(t) \in C\}$$

with the norm

$$\|w\|_{C_\gamma} := \sup_{t \in I} \|t^{1-\gamma}w(t)\|_E,$$

and

$$C^1_\gamma(I) = \{w \in C : \frac{dw}{dt} \in C_\gamma\}$$

with the norm

$$\|w\|_{C^1_\gamma} := \|w\|_\infty + \|w'\|_{C_\gamma}.$$

In what follows, we denote $\|w\|_{C_\gamma}$ by $\|w\|_C$. Let $(E, w) = (E, \sigma(E, E^*))$ be the Banach space E with its weak topology.

Definition 2.1 A Banach space X is called weakly compactly generated (WCG, for short) if it contains a weakly compact set whose linear span is dense in X.

Definition 2.2 A function $h : E \to E$ is said to be weakly sequentially continuous if h takes each weakly convergent sequence in E to a weakly convergent sequence in E (i.e., for any $\{u_n\}$ in E with $u_n \to u$ in (E, w), we have $h(u_n) \to h(u)$ in (E, w)).

Definition 2.3 ([22]) The function $u : I \to E$ is said to be Pettis integrable on I if and only if there is an element $u_J \in E$ corresponding to each $J \subset I$ such that $\phi(u_J) = \int_J \phi(u(s))ds$ for all $\phi \in E^*$, where the integral on the right-hand side is assumed to exist in the sense of Lebesgue (by definition, $u_J = \int_J u(s)ds$).

Let $P(I, E)$ be the space of all $E-$valued Pettis integrable functions on I, and $L^1(I, E)$ be the Banach space of Lebesgue integrable functions $u : I \to E$. Define the class $P_1(I, E)$ by

$$P_1(I, E) = \{u \in P(I, E) : \varphi(u) \in L^1(I, E) \ for\ every\ \varphi \in E^*\}.$$

The space $P_1(I, E)$ is normed by

$$\|u\|_{P_1} = \sup_{\varphi \in E^*,\ \|\varphi\| \le 1} \int_0^T |\varphi(u(x))| d\lambda x,$$

where λ stands for a Lebesgue measure on I.

The following result is due to Pettis (see [22, Theorem 3.4 and Corollary 3.41]).

Proposition 2.1 ([22]) *If $u \in P_1(I, E)$ and h is a measurable and essentially bounded E-valued function, then $uh \in P_1(I, E)$.*

For all that follows, the symbol "$\int$" denotes the Pettis integral. Now, we give some results and properties of fractional calculus.

Definition 2.4 ([2,18,23]) The left-sided mixed Riemann-Liouville integral of order $r > 0$ of a function $w \in L^1(I)$ is defined by

$$(I_\theta^r w)(t) = \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} w(s)ds \ for\ a.e.\ t \in I,$$

where $\Gamma(\cdot)$ is the (Euler's) Gamma function defined by

$$\Gamma(\xi) = \int_0^\infty t^{\xi-1} e^{-t} dt,\ \xi > 0.$$

Notice that for all r, r_1, $r_2 > 0$ and each $w \in C$, we have $I_0^r w \in C$, and

$$(I_0^{r_1} I_0^{r_2} w)(t) = (I_0^{r_1+r_2} w)(t) \ for\ a.e.\ t \in I.$$

Definition 2.5 ([2,18,23]) The Riemann-Liouville fractional derivative of order $r \in (0, 1]$ of a function $w \in L^1(I)$ is defined by

$$\begin{aligned} (D_0^r w)(t) &= \left(\frac{d}{dt} I_0^{1-r} w\right)(t) \\ &= \frac{1}{\Gamma(1-r)} \frac{d}{dt} \int_0^t (t-s)^{-r} w(s)ds \ for\ a.e.\ t \in I. \end{aligned}$$

Let $r \in (0,1]$, $\gamma \in [0,1)$, and $w \in C_{1-\gamma}(I)$. Then the following expression leads to the left inverse operator given by

$$(D_0^r I_0^r w)(t) = w(t) \ for\ all\ t \in (0,T].$$

Moreover, if $I_0^{1-r} w \in C_{1-\gamma}^1(I)$, then the composition

$$(I_0^r D_0^r w)(t) = w(t) - \frac{(I_0^{1-r} w)(0^+)}{\Gamma(r)} t^{r-1} \text{ for all } t \in (0,T]$$

is proved in [23].

Definition 2.6 ([2,18,23]) The Caputo fractional derivative of order $r \in (0,1]$ of a function $w \in L^1(I)$ is defined by

$$\begin{aligned}({}^cD_0^r w)(t) &= \left(I_0^{1-r}\frac{d}{dt}w\right)(t)\\ &= \frac{1}{\Gamma(1-r)}\int_0^t (t-s)^{-r}\frac{d}{ds}w(s)ds \ for\ a.e.\ t \in I.\end{aligned}$$

In [15], Hilfer studied applications of a generalized fractional operator having the Riemann-Liouville and the Caputo derivatives as special cases (see also [16,17,25]).

Definition 2.7 (Hilfer derivative) Let $\alpha \in (0,1)$, $\beta \in [0,1]$, $w \in L^1(I)$, and $I_0^{(1-\alpha)(1-\beta)} \in AC^1(I)$. The Hilfer fractional derivative of order α and type β of w is defined as

$$(D_0^{\alpha,\beta} w)(t) = \left(I_0^{\beta(1-\alpha)}\frac{d}{dt}I_0^{(1-\alpha)(1-\beta)}w\right)(t) \ \ for\ a.e.\ t \in I. \tag{2}$$

Properties of the Hilfer derivative. Let $\alpha \in (0,1)$, $\beta \in [0,1]$, $\gamma = \alpha+\beta-\alpha\beta$, and $w \in L^1(I)$.

1. The operator $(D_0^{\alpha,\beta} w)(t)$ can be written as

$$(D_0^{\alpha,\beta} w)(t) = \left(I_0^{\beta(1-\alpha)}\frac{d}{dt}I_0^{1-\gamma}w\right)(t) = \left(I_0^{\beta(1-\alpha)}D_0^{\gamma}w\right)(t) \ \ for\ a.e.\ t \in I.$$

   Moreover, the parameter γ satisfies

$$\gamma \in (0,1],\ \gamma \geq \alpha,\ \gamma > \beta, \quad \text{and} \quad 1-\gamma < 1-\beta(1-\alpha).$$

2. The generalization (2) for $\beta = 0$ coincides with the Riemann-Liouville derivative, and for $\beta = 1$, with the Caputo derivative:

$$D_0^{\alpha,0} = D_0^{\alpha} \quad \text{and} \quad D_0^{\alpha,1} = {}^cD_0^{\alpha}.$$

3. If $D_0^{\beta(1-\alpha)} w$ exists and is in $L^1(I)$, then

$$(D_0^{\alpha,\beta} I_0^{\alpha} w)(t) = (I_0^{\beta(1-\alpha)} D_0^{\beta(1-\alpha)} w)(t) \ for\ a.e.\ t \in I.$$

   Furthermore, if $w \in C_\gamma(I)$ and $I_0^{1-\beta(1-\alpha)} w \in C_\gamma^1(I)$, then

$$(D_0^{\alpha,\beta} I_0^{\alpha} w)(t) = w(t) \ for\ a.e.\ t \in I.$$

4. If $D_0^{\gamma}w$ exists and is in $L^1(I)$, then

$$(I_0^{\alpha}D_0^{\alpha,\beta}w)(t)=(I_0^{\gamma}D_0^{\gamma}w)(t)=w(t)-\frac{I_0^{1-\gamma}(0^+)}{\Gamma(\gamma)}t^{\gamma-1}\ for\ a.e.\ t\in I.$$

Corollary 2.1 *Let $h\in C_{\gamma}(I)$. A function $u\in L^1(I,E)$ is a solution of the problem*

$$\begin{cases}(D_0^{\alpha,\beta}u)(t)=h(t), & t\in I:=[0,T],\\ (I_0^{1-\gamma}u)(t)|_{t=0}=\phi,\end{cases}$$

if and only if u satisfies the Volterra integral equation

$$w(t)=\frac{\phi}{\Gamma(\gamma)}t^{\gamma-1}+(I_0^{\alpha}h)(t).$$

From the above corollary and Lemma 5.1 in [4], we have the following lemma.

Lemma 2.1 *Let $f:I\times E\times E\to E$ be such that $f(\cdot,u(\cdot),v(\cdot))\in C_{\gamma}(I)$ for any u, $v\in C_{\gamma}(I)$. Then problem (1) is equivalent to the problem of obtaining the solution of the equation*

$$g(t)=f\left(t,\frac{\phi}{\Gamma(\gamma)}t^{\gamma-1}+(I_0^{\alpha}g)(t),g(t)\right);$$

moreover, if $g(\cdot)\in C_{\gamma}$ is the solution of this equation, then

$$u(t)=\frac{\phi}{\Gamma(\gamma)}t^{\gamma-1}+(I_0^{\alpha}g)(t).$$

Remark 2.1 Let $h\in P_1([I,E)$. For every $\varphi\in E^*$, we have

$$\varphi(I_0^{\alpha}h)(t)=(I_0^{\alpha}\varphi h)(t)\ for\ a.e.\ t\in I.$$

Definition 2.8 ([13]) Let E be a Banach space, Ω_E be the class of all bounded subsets of E, and B_1 be the unit ball in E. The De Blasi measure of weak noncompactness is the map $\beta:\Omega_E\to[0,\infty)$ defined by

$$\beta(X)=\inf\{\epsilon>0:\text{there exists a weakly compact }\Omega\subset E\text{ such that }X\subset\epsilon B_1+\Omega\}.$$

The De Blasi measure of weak noncompactness satisfies the following properties:

(a) $A\subset B$ implies $\beta(A)\le\beta(B)$;

(b) $\beta(A)=0$ if and only if A is weakly relatively compact;

(c) $\beta(A\cup B)=\max\{\beta(A),\beta(B)\}$;

(d) $\beta(\overline{A}^{\omega})=\beta(A)$ where $\overline{A}^{\omega}$ denotes the weak closure of A;

(e) $\beta(A+B)\le\beta(A)+\beta(B)$;

(f) $\beta(\lambda A)=|\lambda|\beta(A)$;

(g) $\beta(conv(A))=\beta(A)$;

(h) $\beta(\cup_{|\lambda|\leq h}\lambda A) = h\beta(A)$.

The next result follows directly from the Hahn-Banach theorem.

Proposition 2.2 *Let E be a normed space and let $x_0 \in E$ with $x_0 \neq 0$. Then, there exists $\varphi \in E^*$ with $\|\varphi\| = 1$ and $\varphi(x_0) = \|x_0\|$.*

For a given set V of functions $v : I \to E$, let

$$V(t) = \{v(t) : v \in V\},\ t \in I,$$

and

$$V(I) = \{v(t) : v \in V,\ t \in I\}.$$

Lemma 2.2 ([14]) *Let H be a bounded and equicontinuous subset of C. Then the function $t \to \beta(H(t))$ is continuous on I,*

$$\beta_C(H) = \max_{t\in I} \beta(H(t))$$

and

$$\beta\left(\int_I u(s)ds\right) \leq \int_I \beta(H(s))ds,$$

where $H(s) = \{u(s) : u \in H,\ s \in I\}$, and β_C is the De Blasi measure of weak noncompactness defined on the bounded sets of C.

For our purposes, we will need the following fixed point theorem.

Theorem 2.1 ([20]) *Let Q be a nonempty, closed, convex, and equicontinuous subset of a metrizable locally convex vector space $C(I, E)$ such that $0 \in Q$. Suppose $T : Q \to Q$ is weakly-sequentially continuous. If the implication*

$$\overline{V} = \overline{conv}(\{0\} \cup T(V)) \text{ implies } V \text{ is relatively weakly compact,} \tag{3}$$

holds for every subset $V \subset Q$, then the operator T has a fixed point.

3 Existence of Weak Solutions

Let us start by defining what we mean by a weak solution of the problem (1).

Definition 3.1 By a weak solution of the problem (1) we mean a measurable function $u \in C_\gamma$ that satisfies the condition $(I_0^{1-\gamma}u)(0^+) = \phi$ and the equation $(D_0^{\alpha,\beta}u)(t) = f(t, u(t), (D_0^{\alpha,\beta}u)(t))$ on I.

The following hypotheses will be used in the sequel.

(H_1) For a.e. $t \in I$, the functions $v \to f(t, v, w)$ and $w \to f(t, v, w)$ are weakly sequentially continuous.

(H_2) For each $v,\ w \in E$, the function $t \to f(t, v, w)$ is Pettis integrable a.e. on I.

(H_3) There exists $p \in C(I, [0,\infty))$ such that for all $\varphi \in E^*$, we have

$$|\varphi(f(t, u, v))| \leq \frac{p(t)\|\varphi\|}{1 + \|\varphi\| + \|u\|_E + \|v\|_E} \text{ for a.e. } t \in I \text{ and each } u,\ v \in E.$$

(H_4) For each bounded and measurable set $B \subset E$ and for each $t \in I$, we have

$$\beta(f(t, B, D_0^{\alpha,\beta} B) \leq t^{1-r} p(t)\beta(B),$$

where $D_0^{\alpha,\beta} B = \{D_0^{\alpha,\beta} w : w \in B\}$.

Set

$$p^* = \sup_{t \in I} p(t).$$

Theorem 3.1 *Assume that conditions (H_1)–(H_4) hold. If*

$$L := \frac{p^* T^{1-\gamma+\alpha}}{\Gamma(1+\alpha)} < 1, \tag{4}$$

then the problem (1) has at least one weak solution defined on I.

Proof. Consider the operator $N : C_\gamma \to C_\gamma$ defined by

$$(Nu)(t) = \frac{\phi}{\Gamma(\gamma)} t^{\gamma-1} + (I_0^\alpha g)(t), \tag{5}$$

where $g \in C_\gamma$ is given by

$$g(t) = f\left(t, \frac{\phi}{\Gamma(\gamma)} t^{\gamma-1} + (I_0^\alpha g)(t), g(t)\right).$$

First notice that by hypothesis, for each $g \in C_\gamma$, the function

$$t \mapsto (t-s)^{\alpha-1} g(s)$$

is Pettis integrable over I, and the function

$$t \mapsto f\left(t, \frac{\phi}{\Gamma(\gamma)} t^{\gamma-1} + (I_0^\alpha g)(t), g(t)\right) \ for \ a.e. \ t \in I,$$

is Pettis integrable. Thus, the operator N is well defined. Let $R > 0$ be such that

$$R > \frac{p^* T^{1-\gamma+\alpha}}{\Gamma(1+\alpha)},$$

and consider the set

$$\begin{aligned} Q = \Big\{ u \in C_\gamma : \|u\|_C \leq R \text{ and } & \|t_2^{1-\gamma} u(t_2) - t_1^{1-\gamma} u(t_1)\|_E \\ \leq \frac{p^* T^{1-\gamma+\alpha}}{\Gamma(1+\alpha)} (t_2 - t_1)^\alpha + \frac{p^*}{\Gamma(\alpha)} & \int_0^{t_1} |t_2^{1-\gamma}(t_2-s)^{\alpha-1} - t_1^{1-\gamma}(t_1-s)^{\alpha-1}| ds \Big\}. \end{aligned}$$

Clearly, the subset Q is closed, convex, and equicontinuous. We shall show that the operator N satisfies all the assumptions of Theorem 2.1. The proof will be given in several steps.

Step 1. *N maps Q into itself.*

Let $u \in Q$ and $t \in I$ and assume that $(Nu)(t) \neq 0$. Then there exists $\varphi \in E^*$ such that $\|t^{1-\gamma}(Nu)(t)\|_E = |\varphi(t^{1-\gamma}(Nu)(t))|$. Thus,

$$\|t^{1-\gamma}(Nu)(t)\|_E = \varphi\left(\frac{\phi}{\Gamma(\gamma)} + \frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s)ds\right),$$

where $g \in C_\gamma$ is given by

$$g(t) = f\left(t, \frac{\phi}{\Gamma(\gamma)}t^{\gamma-1} + (I_0^\alpha g)(t), g(t)\right).$$

Then,

$$\begin{aligned}
\|t^{1-\gamma}(Nu)(t)\|_E &\le \frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|\varphi(g(s))|ds \\
&\le \frac{p^*T^{1-\gamma}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds \\
&\le \frac{p^*T^{1-\gamma+\alpha}}{\Gamma(1+\alpha)} \\
&< R.
\end{aligned}$$

Next, take t_1, $t_2 \in I$ with $t_1 < t_2$, and let $u \in Q$ with

$$t_2^{1-\gamma}(Nu)(t_2) - t_1^{1-\gamma}(Nu)(t_1) \neq 0.$$

Then, there exists $\varphi \in E^*$ such that

$$\|t_2^{1-\gamma}(Nu)(t_2) - t_1^{1-\gamma}(Nu)(t_1)\|_E = |\varphi(t_2^{1-\gamma}(Nu)(t_2) - t_1^{1-\gamma}(Nu)(t_1))|$$

and $\|\varphi\| = 1$. Hence,

$$\begin{aligned}
&\|t_2^{1-\gamma}(Nu)(t_2) - t_1^{1-\gamma}(Nu)(t_1)\|_E = |\varphi(t_2^{1-\gamma}(Nu)(t_2) - t_1^{1-\gamma}(Nu)(t_1))| \\
&\le \varphi\left(t_2^{1-\gamma}\int_0^{t_2}(t_2-s)^{\alpha-1}\frac{g(s)}{\Gamma(\alpha)}ds - t_1^{1-\gamma}\int_0^{t_1}(t_1-s)^{\alpha-1}\frac{g(s)}{\Gamma(\alpha)}ds\right),
\end{aligned}$$

where $g \in C_\gamma$ satisfies

$$g(t) = f\left(t, \frac{\phi}{\Gamma(\gamma)}t^{\gamma-1} + (I_0^\alpha g)(t), g(t)\right).$$

Therefore,

$$\begin{aligned}
&\|t_2^{1-\gamma}(Nu)(t_2) - t_1^{1-\gamma}(Nu)(t_1)\|_E \le t_2^{1-\gamma}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\frac{|\varphi(g(s))|}{\Gamma(\alpha)}ds \\
&\quad + \int_0^{t_1}|t_2^{1-\gamma}(t_2-s)^{\alpha-1} - t_1^{1-\gamma}(t_1-s)^{\alpha-1}|\frac{|\varphi(g(s))|}{\Gamma(\alpha)}ds \\
&\le t_2^{1-\gamma}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\frac{p(s)}{\Gamma(\alpha)}ds \\
&\quad + \int_0^{t_1}|t_2^{1-\gamma}(t_2-s)^{\alpha-1} - t_1^{1-\gamma}(t_1-s)^{\alpha-1}|\frac{p(s)}{\Gamma(\alpha)}ds \\
&\le \frac{p^*T^{1-\gamma+\alpha}}{\Gamma(1+\alpha)}(t_2-t_1)^\alpha + \frac{p^*}{\Gamma(\alpha)}\int_0^{t_1}|t_2^{1-\gamma}(t_2-s)^{\alpha-1} - t_1^{1-\gamma}(t_1-s)^{\alpha-1}|ds.
\end{aligned}$$

Hence, $N(Q) \subset Q$.

Step 2. *N is weakly-sequentially continuous.*

Let $\{u_n\}$ be a sequence in Q such that $\{u_n(t)\} \to u(t)$ in (E, ω) for each $t \in I$. Fix $t \in I$; since f satisfies (H_1), we have $f(t, u_n(t), (D_0^{\alpha,\beta} u_n)(t))$ converges weakly uniformly to $f(t, u(t), (D_0^{\alpha,\beta} u)(t))$. Hence, by the Lebesgue dominated convergence theorem for Pettis integrals, $(Nu_n)(t)$ converges weakly uniformly to $(Nu)(t)$ in (E, ω) for each $t \in I$. Thus, $N(u_n) \to N(u)$, and so $N : Q \to Q$ is weakly-sequentially continuous.

Step 3. *The implication (3) holds.*

Let V be a subset of Q such that $\overline{V} = \overline{conv}(N(V) \cup \{0\})$. Clearly,

$$V(t) \subset \overline{conv}(NV)(t)) \cup \{0\}) \text{ for each } \ t \in I.$$

Furthermore, since V is bounded and equicontinuous, by Lemma 3 in [12] the function $t \to v(t) = \beta(V(t))$ is continuous on I. From (H_3), (H_4), Lemma 2.2, and the properties of the measure β, for any $t \in I$, we have

$$\begin{aligned} t^{1-\gamma} v(t) &\leq \beta(t^{1-\gamma}(NV)(t) \cup \{0\}) \\ &\leq \beta(t^{1-\gamma}(NV)(t)) \\ &\leq \frac{T^{1-\gamma}}{\Gamma(\alpha)} \int_0^t |t-s|^{\alpha-1} p(s) \beta(V(s)) ds \\ &\leq \frac{T^{1-\gamma}}{\Gamma(\alpha)} \int_0^t |t-s|^{\alpha-1} s^{1-\gamma} p(s) v(s) ds \\ &\leq \frac{p^* T^{1-\gamma+\alpha}}{\Gamma(1+\alpha)} \|v\|_C. \end{aligned}$$

Thus,

$$\|v\|_C \leq L \|v\|_C.$$

From (4), we see that $\|v\|_C = 0$, that is, $v(t) = \beta(V(t)) = 0$ for each $t \in I$. By [19, Theorem 2], V is weakly relatively compact in C. Applying Theorem 2.1, we conclude that N has a fixed point that is a weak solution of the problem (1).

4 An Example

Let

$$E = l^1 = \left\{ u = (u_1, u_2, \ldots, u_n, \ldots) : \sum_{n=1}^{\infty} |u_n| < \infty \right\}$$

be our Banach space with the norm

$$\|u\|_E = \sum_{n=1}^{\infty} |u_n|.$$

As an application of our results, we consider the Hilfer fractional differential equation

$$\begin{cases} (D_0^{\frac{1}{2},\frac{1}{2}} u_n)(t) = f_n(t, u(t), (D_0^{\frac{1}{2},\frac{1}{2}} u_n)(t)), \quad t \in [0,1], \\ (I_0^{\frac{1}{4}} u)(t)|_{t=0} = (0, 0, \ldots, 0, \ldots), \end{cases} \tag{6}$$

where

$$f_n(t, u(t), v(t)) = \frac{ct^2}{1 + \|u(t)\|_E + \|v(t)\|_E} \frac{u_n(t)}{e^{t+4}}, \quad t \in [0, 1],$$

with

$$u = (u_1, u_2, \ldots, u_n, \ldots), \ and\ c := \frac{e^4}{8}\Gamma\left(\frac{1}{2}\right).$$

Set

$$f = (f_1, f_2, \ldots, f_n, \ldots).$$

Clearly, the function f is continuous.

For each $u,\ v \in E$ and $t \in [0, 1]$, we have

$$\|f(t, u, v)\|_E \leq ct^2 \frac{1}{e^{t+4}}.$$

Hence, condition (H_3) is satisfied with $p^* = ce^{-4}$. We shall show that condition (4) holds with $T = 1$. In fact,

$$\frac{p^* T^{1-\gamma+\alpha}}{\Gamma(1+\alpha)} = \frac{2ce^{-4}}{\Gamma(\frac{1}{2})} = \frac{1}{4} < 1.$$

Simple computations show that all the conditions of Theorem 3.1 are satisfied, and so problem (6) has at least one weak solution defined on $[0, 1]$.

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