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Lie Symmetry Reductions of a Coupled Kdv System of Fractional Order

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Abstract: In this paper, we investigate the coupled KdV system of fractional order, which describes a resonant interaction of two wave modes in shallow stratified liquid. The Lie group analysis method is applied for this coupled system. Then the corresponding invariant solutions are obtained using infinitesimal generators. Finally, we determined the reduced fractional ODE system corresponding to the fractional PDE system.

Keywords: *coupled KdV system; Lie symmetry method; Riemann-Liouville derivative; group- invariant solutions; reduced fractional system.*

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1 Introduction

Fractional partial differential equations (FPDEs) are becoming increasingly popular due to their practical applications in various fields of science and engineering, such as polymer physics, viscoelasticity materials, control theory, signal processing, systems identification and electrochemistry [1–5].

So it is necessary to obtain exact solutions or numerical solutions for FPDEs. During last few decades several analytical numerical and semi-analytical methods have been used for solving FPDEs [6, 7, 9, 10, 20].

Lie group analysis originally advocated by Sophus Lie has proven to be an efficient approach for PDEs [8], with the increasing applications of FPDEs, principle procedure of Lie group analysis was extended to FPDEs for finding the exact solution of the equation [11–13].

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Jafari et al. [14, 15] applied Lie group method to solve the time-fractional Kaup-Kupershmidt equation and time-fractional Boussinesq equation. In [18], Adem and Khalique have applied Lie symmetry analysis for Korteweg-de Vries(KdV) system given by

$$\begin{cases} u_t + u_{xxx} - \frac{7}{4}uu_x - vv_x + \frac{5}{4}(uv)_x = 0, \\ v_t + v_{xxx} - \frac{5}{4}uu_x - \frac{7}{4}vv_x + 2(uv)_x = 0. \end{cases} \tag{1}$$

The result for time fractional KdV-type equation has been obtained by Hu et al. [16]. Chen and Jiang [17] have applied the methods to simplify successfully two classes of FPDEs.

In this paper, we study Lie group method for solving the KdV system of fractional order

$$\begin{cases} D_t^\alpha u + u_{xxx} - \frac{7}{4}uu_x - vv_x + \frac{5}{4}(uv)_x = 0, \\ D_t^\alpha v + v_{xxx} - \frac{5}{4}uu_x - \frac{7}{4}vv_x + 2(uv)_x = 0, \end{cases} \tag{2}$$

where α $(0 < \alpha \leqslant 1)$ is a parameter describing the order of the fractional derivative, when $\alpha = 1$, the KdV system (2) becomes the KdV system (1).

The paper is organized as follows. In Section 2, we present the analysis of the Lie symmetry group of FPDEs system. We obtain the Lie point symmetries of fractional KdV system in Section 3. Then, in Section 4, we obtain invariant solutions and reduced equations of this system. Finally, conclusions are given in Section 5.

2 Preliminaries

We give some basic definitions and properties of the fractional Lie group method for finding infinitesimal function of the PDE system of fractional order.

Definition 2.1 The Riemann-Liouville fractional derivative of order α [2, 19], is defined by

$$D_t^\alpha u(x,t) = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \begin{cases} \frac{\partial^n u(x,t)}{\partial t^n}; & n \in \mathbb{N}, \\ \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{u(x,\tau)}{(t-\tau)^{\alpha+1-n}} \,\mathrm{d}\tau; & n-1 < \alpha < n. \end{cases}$$

For fractional PDE system with two independent variables we have

$$\begin{aligned} \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} &= F(x,t,u,v,u_{(1)},v_{(1)},\cdots), \qquad 0 < \alpha < 1, \\ \frac{\partial^\alpha v(x,t)}{\partial t^\alpha} &= G(x,t,u,v,u_{(1)},v_{(1)},\cdots). \end{aligned}$$

According to Lie's algorithm, the infinitesimal generator of the symmetry group admitted by (2) is given by

$$X = \xi^x(x,t,u,v)\frac{\partial}{\partial x} + \xi^t(x,t,u,v)\frac{\partial}{\partial t} + \eta^u(x,t,u,v)\frac{\partial}{\partial u} + \eta^v(x,t,u,v)\frac{\partial}{\partial v}, \tag{3}$$

in which ξ^x,ξ^t,η^u,η^v are infinitesimal functions of the group variables.

Since the KdV system of fractional order has at most α-order derivatives, the α-prolongation of the generator should be considered in the form

$$\begin{aligned} X^{(\alpha)} &= \xi^x(x,t,u,v)\frac{\partial}{\partial x} + \xi^t(x,t,u,v)\frac{\partial}{\partial t} + \eta^u(x,t,u,v)\frac{\partial}{\partial u} + \eta^v(x,t,u,v)\frac{\partial}{\partial v} \\ &+ \eta_i^{(1)u}(x,t,u,v,u_{(i)},v_{(i)})\frac{\partial}{\partial u_i} + \eta_i^{(1)v}(x,t,u,v,u_{(i)},v_{(i)})\frac{\partial}{\partial v_i} + \cdots \\ &+ \eta_{i_1\cdots i_k}^{(k)u}(x,t,u,v,u_{(1)},v_{(1)},\cdots,u_{(k)},v_{(k)})\frac{\partial}{\partial u_{i_1,\cdots,i_k}} \\ &+ \eta_{i_1\cdots i_k}^{(k)v}(x,t,u,v,u_{(1)},v_{(1)},\cdots,u_{(k)},v_{(k)})\frac{\partial}{\partial v_{i_1,\cdots,i_k}} \\ &+ \eta_t^{(\alpha)u}(x,t,u,v,\cdots,u_{(\alpha)},\cdots)\frac{\partial}{\partial u_t^\alpha} + \eta_t^{(\alpha)v}(x,t,u,v,\cdots,v_{(\alpha)},\cdots)\frac{\partial}{\partial v_t^\alpha}, \end{aligned} \tag{4}$$

where

$$\begin{aligned} \eta_t^{(\alpha)u} &= D_{1t}^\alpha(\eta^u) + \xi^x D_{1t}^\alpha(u_x) - D_{1t}^\alpha(\xi^x u_x) + D_{1t}^\alpha(D_{1t}(\xi^t)u) - D_{1t}^{\alpha+1}(\xi^t u) + \xi^t D_{1t}^{\alpha+1}u, \\ \eta_t^{(\alpha)v} &= D_{2t}^\alpha(\eta^v) + \xi^x D_{2t}^\alpha(v_x) - D_{2t}^\alpha(\xi^x v_x) + D_{2t}^\alpha(D_{2t}(\xi^t)v) - D_{2t}^{\alpha+1}(\xi^t v) + \xi^t D_{2t}^{\alpha+1}v. \end{aligned}$$

D_{1t} and D_{2t} are the total derivative operators defined as

$$\begin{aligned} D_{1t} &= \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{xt}\frac{\partial}{\partial u_x} + u_{tt}\frac{\partial}{\partial u_t} + u_{xxt}\frac{\partial}{\partial u_{xx}} + \cdots, \\ D_{2t} &= \frac{\partial}{\partial t} + v_t\frac{\partial}{\partial v} + v_{xt}\frac{\partial}{\partial v_x} + v_{tt}\frac{\partial}{\partial v_t} + v_{xxt}\frac{\partial}{\partial v_{xx}} + \cdots. \end{aligned}$$

Definition 2.2 A vector X given by (3) is said to be Lie point symmetry vector field for system (2), if

$$\begin{aligned} X^{(\alpha)}\Big[D_t^\alpha u + u_{xxx} - \frac{7}{4}uu_x - vv_x + \frac{5}{4}(uv)_x\Big] &= 0, \\ X^{(\alpha)}\Big[D_t^\alpha v + v_{xxx} - \frac{5}{4}uu_x - \frac{7}{4}vv_x + 2(uv)_x\Big] &= 0. \end{aligned}$$

3 Lie Symmetry for Coupled KdV System of Fractional Order

In this section, we investigate the infinitesimal generator of the KdV system of fractional order (2).

Theorem 3.1 *Lie symmetries of (2) are*

1. *If $\alpha \neq \frac{1}{2}, \frac{1}{3}$, then we have:*

$$\begin{aligned} \xi^x(x,t,u,v) &= c_1 + c_2\alpha x, & \xi^t(x,t,u,v) &= 3c_2 t, \\ \eta^u(x,t,u,v) &= -2c_2\alpha u, & \eta^v(x,t,u,v) &= -2c_2\alpha v, \end{aligned}$$

where c_1 and c_2 are two arbitrary constants. Hence, the infinitesimal generators are given by

$$X_{a1} = \frac{\partial}{\partial x}, \qquad X_{a2} = \alpha x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t} - 2\alpha u\frac{\partial}{\partial u} - 2\alpha v\frac{\partial}{\partial v}.$$

2. *If $\alpha = \frac{1}{2}$, then we have:*

$$\xi^x(x,t,u,v) = c_1 - c_2 x, \qquad \xi^t(x,t,u,v) = -6c_2 t,$$
$$\eta^u(x,t,u,v) = 2c_2\alpha u, \qquad \eta^v(x,t,u,v) = 2c_2\alpha v,$$

where c_1 and c_2 are two arbitrary constants. Hence

$$X_{b1} = \frac{\partial}{\partial x}, \qquad X_{b2} = -x\frac{\partial}{\partial x} - 6t\frac{\partial}{\partial t} + 2\alpha u\frac{\partial}{\partial u} + 2\alpha v\frac{\partial}{\partial v}.$$

3. *If $\alpha = \frac{1}{3}$, then we have:*

$$\xi^x(x,t,u,v) = c_1 + c_2 x, \qquad \xi^t(x,t,u,v) = 9c_2 t,$$
$$\eta^u(x,t,u,v) = -2c_2\alpha u, \qquad \eta^v(x,t,u,v) = -2c_2\alpha v,$$

where c_1 and c_2 are two arbitrary constants. Hence

$$X_{c1} = \frac{\partial}{\partial x}, \qquad X_{c2} = x\frac{\partial}{\partial x} + 9t\frac{\partial}{\partial t} - 2\alpha u\frac{\partial}{\partial u} - 2\alpha v\frac{\partial}{\partial v}.$$

Proof. Let us consider a one parameter Lie group of infinitesimal transformation in x, t, u, v given by

$$x \longrightarrow x + \epsilon\xi^x(x,t,u,v), \quad t \longrightarrow t + \epsilon\xi^t(x,t,u,v),$$
$$u \longrightarrow u + \epsilon\eta^u(x,t,u,v), \quad v \longrightarrow v + \epsilon\eta^v(x,t,u,v), \tag{5}$$

with a small parameter $\epsilon \ll 1$, and the symmetry group of KdV system will be generated by the vector field (3), now we find the coefficient functions $\xi^x, \xi^t, \eta^u, \eta^v$ in (5).

By applying the $X^{(\alpha)}$ to both sides of (2), we have

$$X^{(\alpha)}\Big[D_t^\alpha u + u_{xxx} - \frac{7}{4}uu_x - vv_x + \frac{5}{4}(uv)_x\Big] = 0,$$
$$X^{(\alpha)}\Big[D_t^\alpha v + v_{xxx} - \frac{5}{4}uu_x - \frac{7}{4}vv_x + 2(uv)_x\Big] = 0. \tag{6}$$

Expanding (6), and solving the obtained system using a mathematical software, we obtain the Lie point symmetries.

1. If $\alpha \neq \frac{1}{2}, \frac{1}{3}$, then we have:

$$\xi^x(x,t,u,v) = c_1 + c_2\alpha x, \qquad \xi^t(x,t,u,v) = 3c_2 t,$$
$$\eta^u(x,t,u,v) = -2c_2\alpha u, \qquad \eta^v(x,t,u,v) = -2c_2\alpha v.$$

Hence, the infinitesimal generators are given by

$$X_{a1} = \frac{\partial}{\partial x}, \qquad X_{a2} = \alpha x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t} - 2\alpha u\frac{\partial}{\partial u} - 2\alpha v\frac{\partial}{\partial v}.$$

2. If $\alpha = \frac{1}{2}$, then

$$\xi^x(x,t,u,v) = c_1 - c_2 x, \qquad \xi^t(x,t,u,v) = -6c_2 t,$$
$$\eta^u(x,t,u,v) = 2c_2\alpha u, \qquad \eta^v(x,t,u,v) = 2c_2\alpha v.$$

Therefore

$$X_{b1} = \frac{\partial}{\partial x}, \qquad X_{b2} = -x\frac{\partial}{\partial x} - 6t\frac{\partial}{\partial t} + 2\alpha u\frac{\partial}{\partial u} + 2\alpha v\frac{\partial}{\partial v}.$$

3. If $\alpha = \frac{1}{3}$, then

$$\xi^x(x,t,u,v) = c_1 + c_2 x, \qquad \xi^t(x,t,u,v) = 9c_2 t,$$
$$\eta^u(x,t,u,v) = -2c_2\alpha u, \qquad \eta^v(x,t,u,v) = -2c_2\alpha v.$$

Therefore

$$X_{c1} = \frac{\partial}{\partial x}, \qquad X_{c2} = x\frac{\partial}{\partial x} + 9t\frac{\partial}{\partial t} - 2\alpha u\frac{\partial}{\partial u} - 2\alpha v\frac{\partial}{\partial v}.$$

4 Symmetry Reduction

In the previous section, we obtained the infinitesimal generators X_{ij} $(i = a, b, c, j = 1, 2)$. Here we want to obtain similarity variables and their reduction equations. Then by using these variables the system (2) transforms into a system of fractional ODE.

One has to solve the associated Lagrange equations

$$\frac{dx}{\xi^x(x,t,u,v)} = \frac{dt}{\xi^t(x,t,u,v)} = \frac{du}{\eta^u(x,t,u,v)} = \frac{dv}{\eta^v(x,t,u,v)}.$$

We consider the following cases.

- Case 1: $0 < \alpha < 1$, $\alpha \neq \frac{1}{2}, \frac{1}{3}$, $X_{a1} = \frac{\partial}{\partial x}$.
  In this case the symmetry X_{a1} gives rise to the group-invariant solution:
  $$r = t, \qquad u = F(r), \qquad v = G(r), \tag{7}$$
  substituting (7) into (2) results in the fact that $F(r)$ and $G(r)$ satisfy the following differential equations:
  $$\frac{d^\alpha F(t)}{dt^\alpha} = 0, \qquad \frac{d^\alpha G(t)}{dt^\alpha} = 0,$$
  by using Laplace transformation we get
  $$F(t) = \frac{k}{\Gamma(\alpha)} t^{\alpha-1}, \qquad G(t) = \frac{k}{\Gamma(\alpha)} t^{\alpha-1},$$
  where k is a constant, therefore
  $$u(x,t) = \frac{k}{\Gamma(\alpha)} t^{\alpha-1}, \qquad v(x,t) = \frac{k}{\Gamma(\alpha)} t^{\alpha-1}.$$

- Case 2: $0 < \alpha < 1$, $\alpha \neq \frac{1}{2}, \frac{1}{3}$, $X_{a2} = \alpha x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t} - 2\alpha u\frac{\partial}{\partial u} - 2\alpha v\frac{\partial}{\partial v}$.
  In this case, the group-invariant solution is:
  $$r = tx^{\frac{-3}{\alpha}}, \qquad u = F(r)x^{-2}, \qquad v = G(r)x^{-2}, \tag{8}$$
  substituting (8) into (2) leads to the following fractional ODE system:
  $$\begin{cases} D_r^\alpha F + k_1 F(r) + k_2 rF'(r) + k_3 r^2 F''(r) + k_4 r^3 F^{(3)}(r) + k_5 F^2(r) \\ +k_6 rF(r)F'(r) \ + k_7 G^2(r) + k_8 rG(r)G'(r) + k_9 F(r)G(r) \\ +k_{10} rF'(r)G(r) + k_{11} rF(r)G'(r) = 0, \\ D_r^\alpha G + k_1' G(r) + k_2' rG'(r) + k_3' r^2 G''(r) + k_4' r^3 G^{(3)}(r) + k_5' F^2(r) \\ +k_6' rF(r)F'(r) + k_7' G^2(r) + k_8' rG(r)G'(r) + k_9' F(r)G(r) \\ +k_{10}' rF'(r)G(r) + k_{11}' rF(r)G'(r) = 0, \end{cases}$$
  where $k_i = h_i(\alpha)$ and $k_i' = g_i(\alpha)$, $(i = 1, 2, \cdots, 11)$ are constants.

- Case 3:$\alpha = \frac{1}{2}$, $X_{b2} = -x\frac{\partial}{\partial x} - 6t\frac{\partial}{\partial t} + 2u\frac{\partial}{\partial u} + 2v\frac{\partial}{\partial v}$.
  For this case, the group-invariant solution is:

$$r = tx^{-6}, \qquad u = F(r)x^{-2}, \qquad v = G(r)x^{-2}. \tag{9}$$

  Again by substituting (9) into (2), we have:

$$\begin{cases} D_r^{\frac{1}{2}}F - 24F(r) - 696rF'(r) - 405r^2F''(r) - 216r^3F^3(r) + \frac{7}{2}F^2(r) \\ +\frac{21}{2}rF(r)F'(r) + 2G^2(r) + 6rG(r)G'(r) - 5F(r)G(r) \\ -\frac{15}{2}rF'(r)G(r) - \frac{15}{2}rF(r)G'(r) = 0, \\ D_r^{\frac{1}{2}}G - 24G(r) - 696rG'(r) - 405r^2G''(r) - 216r^3G^3(r) + \frac{5}{2}F^2(r) \\ +\frac{15}{2}rF(r)F'(r) + \frac{7}{2}G^2(r) + \frac{21}{2}rG(r)G'(r) - 8F(r)G(r) \\ -12rF'(r)G(r) - 12rF(r)G'(r) = 0. \end{cases}$$

- Case 4:$\alpha = \frac{1}{3}$, $X_{c2} = x\frac{\partial}{\partial x} + 9t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u} - 2v\frac{\partial}{\partial v}$.
  In this case, the group-invariant solution is:

$$r = tx^{-9}, \qquad u = F(r)x^{-2}, \qquad v = G(r)x^{-2}, \tag{10}$$

  substituting (10) into (2) results in the fact that $F(r)$ and $G(r)$ satisfy the following fractional ODE system

$$\begin{cases} D_r^{\frac{1}{3}}F - 24F(r) - 1692.09rF'(r) - 2430r^2F''(r) - 729r^3F^3(r) + \frac{7}{2}F^2(r) \\ +\frac{63}{4}rF(r)F'(r) + 2G^2(r) + 9rG(r)G'(r) - 5F(r)G(r) - \frac{45}{4}rF'(r)G(r) \\ -\frac{45}{4}rF(r)G'(r) = 0, \\ D_r^{\frac{1}{3}}G - 24G(r) - 1692.09rG'(r) - 2430r^2G''(r) - 729r^3G^3(r) + \frac{5}{2}F^2(r) \\ +\frac{45}{4}rF(r)F'(r) + \frac{7}{2}G^2(r) + \frac{63}{4}rG(r)G'(r) - 8F(r)G(r) \\ -18rF'(r)G(r) - 18rF(r)G'(r) = 0. \end{cases}$$

Note. For $\alpha = 1$, the Lie point symmetries provide is similar results to those obtained by Adem and Khalique in [18].

5 Conclusion

In this paper, we carry out the Lie symmetry group analysis for a fractional PDE system. First, we apply Lie symmetries method for the KdV system of fractional order (2), and get its infinitesimal generators. Then, we use similarity variables to obtain reduction equations. Finally, we have shown that the KdV system of fractional order can be transformed into a fractional ODE system.

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