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Stability, Boundedness and Square Integrability of Solutions to Certain Third-Order Vector Differential Equations

D. Beldjerd and M. Remili *

Department of Mathematics, University of Oran 1 Ahmed Ben Bella, 31000 Oran, Algeria.

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Abstract: In this paper, we establish some new sufficient conditions which guarantee the stability and the boundedness of solutions of certain third order vector differential equations. Sufficient conditions are also established for square integrability of solutions and their derivatives. By this work, we extend and improve some stability and boundedness results in the literature.

Keywords: *Lyapunov functional; third-order vector differential equation; boundedness; stability; square integrability.*

Mathematics Subject Classification (2010): 34C10, 34C11, 34D05, 34D20, 34D40.

1 Introduction

In recent years much attention has been drawn to the stability and boundedness of solutions of ordinary scalar and vector nonlinear differential equations of third order. See Afuwape [1, 2],Omeike [9, 10] Ezeilo [4, 5], Remili [11–14] and the references cited therein for a comprehensive treatment of the subject. Lyapunov's second (direct) method has been used as a basic tool to verify the results established in these works.

In 2009, Tunç [17] proved two results, for the cases P = 0 and P ≠ 0, respectively, on the stability and boundedness of solutions to the vector differential equations of third order

$$X'''(t) + \Psi(X'(t))X''(t) + BX'(t) + cX(t) = P(t). \tag{1}$$

* Corresponding author: mailto:remilimous@gmail.com

Recently, in 2014, for the same cases, Omeike [9] discussed the global asymptotic stability and boundedness of solutions to nonlinear vector differential equations of third order

$$X'''(t) + \Psi(X'(t))X''(t) + \Phi(X(t))X'(t) + cX(t) = P(t). \tag{2}$$

The purpose of this paper is to study the uniform asymptotic stability, boundedness and square integrability of solutions of the third order nonlinear vector differential equations of the form

$$(\Omega(X(t)))X'(t))'' + \Psi(X'(t))X''(t) + G(X(t))X'(t) + cX(t) = P(t), \tag{3}$$

where $X \in \mathbb{R}^n$, $t \in \mathbb{R}$ and c is a positive constant, Ψ and G are $n \times n-$symetric and differentiable matrix functions; Ω is an $n \times n-$symetric differentiable and inversible matrix function. $P : \mathbb{R} \to \mathbb{R}^n$ is a continuous function with respect to t. Let

$$\Omega' = \Omega'(X(t)) = \frac{d}{dt}(\mu_{i,j}(X(t)), \text{and } G' = G'(X(t)) = \frac{d}{dt}(g_{i,j}(X(t)) \quad (i, j = 1, 2, ..., n),$$

where $\mu_{i,j}(X(t))$ and $g_{i,j}(X(t))$ are the components of $\Omega(X)$ and $G(X)$ respectively. On the other hand $X(t)$, $Y(t)$, $Z(t)$, $\Omega(X(t)))$, $G(X(t)))$ and $\Psi(X'(t)))$ are, respectively, abbreviated as X, Y, Z, Ω, G and Ψ throughout the paper. Additionally, the symbol $\langle X, Y\rangle$ corresponding to any pair X and Y in $\mathbb{R}^n$ stands for the usual scalar product $\sum\limits_{i=1}^{n} x_i y_i$, that is, $\langle X, Y\rangle = \sum\limits_{i=1}^{n} x_i y_i$, Thus $\langle X, X\rangle = \|X\|^2$.

Let us, for convenience, replace (3) by the equivalent differential system

$$\begin{cases} X' = \Omega^{-1}(X)Y, \\ Y' = Z, \\ Z' = -\Psi\Omega^{-1}(X)Z - \Psi\theta Y - G\Omega^{-1}(X)Y - cX + P(t), \end{cases} \tag{4}$$

which was obtained by setting

$$\begin{aligned} X' &= \Omega^{-1}(X)Y, \\ X'' &= \theta(t)Y + \Omega^{-1}(X)Z, \end{aligned}$$

where

$$\theta(t) = \left(\Omega^{-1}(X)\right)' = -\Omega^{-1}(X)\Omega'(X)\Omega^{-1}(X). \tag{5}$$

This paper is organized as follows: in Section 2, we will recall briefly some basic definitions and preliminary facts which will be used throughout the following sections. In Section 3 we give stability results. In Section 4 boundedness of solutions is discussed. Finally, in Section 5 sufficient conditions for the square integrability of solutions are given.

2 Preliminaries

In order to reach our main results, we dispose some well-known algebraic results which will be required in the proofs.

Lemma 2.1 *[4] Let D be a real symmetric positive definite $n \times n$ matrix. Then for any X in $\mathbb{R}^n$, we have*

$$\delta_d \parallel X \parallel^2 \leq \langle DX, X\rangle \leq \Delta_d \parallel X \parallel^2,$$

where δ_d, Δ_d are the least and the greatest eigenvalues of D, respectively.

Lemma 2.2 *[4] Let Q, D be any two real $n \times n$ commuting matrices. Then*

(i) *The eigenvalues $\lambda_i (QD) \, (i = 1, 2..., n)$ of the product matrix QD are all real and satisfy*

$$\min_{1 \leq j,k \leq n} \lambda_j (Q) \lambda_k (D) \leq \lambda_i (QD) \leq \max_{1 \leq j,k \leq n} \lambda_j (Q) \lambda_k (D) .$$

(ii) *The eigenvalues $\lambda_i (Q + D) \, (i = 1, 2..., n)$ of the sum of matrix Q and D are all real and satisfy*

$$\min_{1 \leq j \leq n} \lambda_j (Q) + \min_{1 \leq k \leq n} \lambda_k (D) \leq \lambda_i (Q + D) \leq \max_{1 \leq j \leq n} \lambda_j (Q) + \max_{1 \leq k \leq n} \lambda_k (D) .$$

Lemma 2.3 *[4] Let H be a continuous matrix function with $H(0) = 0$. Then*

$$\frac{d}{dt} \int_0^1 \sigma \langle H(\sigma X) X, X \rangle d\sigma = \langle H(X), \frac{dX}{dt} \rangle.$$

Lemma 2.4 *Let $H(X)$ be a continuous vector function with $H(0) = 0$. Then*

$$\delta_h \parallel X \parallel^2 \leq \int_0^1 \langle H(\sigma X), X \rangle d\sigma \leq \Delta_h \parallel X \parallel^2,$$

where δ_h, Δ_h are the least and the greatest eigenvalues of $J_h(X)$ (Jacobian matrix of H), respectively.

Definition 2.1 We define the spectral radius $\rho (A)$ of a matrix A by

$$\rho (A) = \max \{ |\lambda| : \ \ \lambda \text{ is the eigenvalue of } A \} .$$

Lemma 2.5 *For any $A \in \mathbb{R}^{n \times n}$, we have the norm $\|A\| = \sqrt{\rho (A^T A)}$. If A is symmetric, then $\|A\| = \rho (A)$.*

We shall note all the equivalent norms by the same notation $\|X\|$ for $X \in \mathbb{R}^n$ and $\|A\|$ for a matrix $A \in \mathbb{R}^{n \times n}$.

In the sequel we will assume :

H_1) There are positive constants $\omega_0, \ \omega_1, \ a_0, \ a_1, \ b_0, \ b_1$ such that the following conditions are satisfied

$$b_0 \leq \lambda_i(G) \leq b_1, \ \ a_0 \leq \lambda_i(\Psi) \leq a_1, \ \ \omega_0 \leq \lambda_i(\Omega) \leq \omega_1.$$

H_2) The $n \times n$ differentiable matrices $\Omega, \ \Omega^{-1}, \ \Psi$ and G are symmetric, associative and commute pairwise.

3 Stability

Our study of (3) here is concerned primarily with the problems of the stability for the case $P(t) = 0$. For the ease of exposition throughout this paper we will adopt the following notation :

$$\delta(t) = \parallel \Omega'(X(t)) + G'(X(t)) \parallel . \tag{6}$$

Theorem 3.1 *In addition to the fundamental assumptions imposed on* Ω, Ψ *and* G, *we suppose there exist positive constants* β *and* δ_0 *such that*
i) $\dfrac{c}{a_0 b_0} < \beta < \dfrac{1}{\omega_1}$,
ii) $\int_0^{+\infty} \delta(s)ds \le \delta_0 < \infty$.

Then every solution of (4) satisfies

$$\lim_{t\to\infty} X(t) = \lim_{t\to\infty} Y(t) = \lim_{t\to\infty} Z(t) = 0.$$

Proof. To prove this theorem, we define a Lyapunov functional $W = W(t, X, Y, Z)$ as

$$W = V \exp(-\mu(t)), \tag{7}$$

where

$$\mu(t) = \frac{1}{d}\int_0^t \delta(s)ds,$$

$$\begin{aligned} V \;=\; & \frac{1}{2}\langle cX, cX\rangle + \frac{1}{2}\beta b_0 \left\langle Y, G\Omega^{-1}Y\right\rangle + \beta\frac{b_0}{2}\langle\, Z, Z\rangle + \langle c\Omega^{-1}Y, Z\rangle \\ & +\beta\langle cX, b_0\; Y\rangle + \int_0^1 \sigma\langle c\Psi(\sigma\Omega^{-1}Y)\Omega^{-1}Y, \Omega^{-1}Y\rangle d\sigma, \end{aligned} \tag{8}$$

d is some positive constant which will be specified later. It is clear by (8) that $W(t,0,0,0) = 0$. Note that $\omega_0 \le \lambda_i(\Omega) \le \omega_1$ implies that $\frac{1}{\omega_1} \le \lambda_i(\Omega^{-1}) \le \frac{1}{\omega_0}$. Hence by (H_1), Lemma 2.1 and Lemma 2.2, we have

$$c\int_0^1 \sigma\langle\Psi(\sigma\Omega^{-1}Y)\Omega^{-1}Y, \Omega^{-1}Y\rangle d\sigma \;\;\ge\;\; \frac{ca_0}{2\omega_1^2} \parallel Y \parallel^2$$

and

$$\frac{1}{2}\beta b_0 \left\langle Y, G\Omega^{-1}Y\right\rangle \ge \frac{\beta b_0^2}{2\omega_1} \parallel Y \parallel^2 .$$

Hence

$$V \ge \frac{c^2}{2} \parallel X \parallel^2 +\beta\langle cX, b_0 Y\rangle + \beta\frac{b_0}{2} \parallel Z \parallel^2 +\langle c\Omega^{-1}Y, Z\rangle + \Big(\frac{\beta b_0^2}{2\omega_1} + \frac{ca_0}{2\omega_1^2}\Big) \parallel Y \parallel^2 .$$

Thus, we clearly have

$$\frac{c^2}{2} \parallel X \parallel^2 +\beta\langle cX, b_0 Y\rangle = \frac{1}{2}\|cX + \beta b_0 Y\|^2 - \frac{\beta^2 b_0^2}{2} \parallel Y \parallel^2$$

and

$$\begin{aligned} \frac{\beta b_0}{2} \parallel Z \parallel^2 +\langle c\Omega^{-1}Y, Z\rangle \;\;&=\;\; \frac{\beta b_0}{2}\|Z + \frac{c}{\beta b_0}\Omega^{-1}Y\|^2 - \frac{c^2}{2\beta b_0}\langle\Omega^{-1}Y, \Omega^{-1}Y\rangle \\ &\ge\;\; \frac{\beta b_0}{2}\|Z + \frac{c}{\beta b_0}\Omega^{-1}Y\|^2 - \frac{c^2}{2\beta\omega_1^2 b_0} \parallel Y \parallel^2 . \end{aligned}$$

Combining the preceding estimates, we find

$$V \geq \frac{1}{2} \parallel cX + \beta b_0 Y \parallel^2 + \frac{\beta b_0}{2} \parallel Z + \frac{c}{\beta b_0}\Omega^{-1}Y \parallel^2 + \Delta \parallel Y \parallel^2,$$

where

$$\Delta = \frac{\beta b_0^2}{2\omega_1} + \frac{c a_0}{2\omega_1^2} - \frac{\beta^2 b_0^2}{2} - \frac{c^2}{2\beta\omega_1^2 b_0}.$$

Condition (i) implies

$$\Delta = c\frac{\beta\ a_0 b_0 - c}{2\beta b_0 \omega_1^2} + \beta b_0^2 (\frac{1}{2\omega_1} - \frac{\beta}{2}) \geq \frac{c}{2\beta b_0 \omega_1^2}(\beta\ a_0 b_0 - c) > 0.$$

It is evident, from the terms included in the last inequality, that there exists a sufficiently small positive constant k_0 such that

$$V \geq k_0\big(\parallel X \parallel^2 + \parallel Y \parallel^2 + \parallel Z \parallel^2 \big). \tag{9}$$

Finally, by condition (ii) and (7) we get

$$W \geq K_0\big(\parallel X \parallel^2 + \parallel Y \parallel^2 + \parallel Z \parallel^2, \tag{10}$$

where $K_0 = k_0 \exp(-\dfrac{\delta_0}{d})$.

Now, we show that $W'_{(4)}$ is negative definite function.
First, by Lemma 2.3, from the integral term in (8) we have the following derivative

$$\frac{d}{dt}\int_0^1 \sigma\langle c\Psi(\sigma\Omega^{-1}Y)\Omega^{-1}Y, \Omega^{-1}Y\rangle d\sigma = c\langle \Psi\Omega^{-1}Y, \theta Y + \Omega^{-1}Z\rangle.$$

Hence, the time derivative of functional V along the system (4) leads to

$$V'_{(4)} = V_1 + V_2 + V_3,$$

where

$$\begin{aligned} V_1 &= \beta c b_0 \langle \Omega^{-1}Y, Y\rangle - c\left\langle Y, G\Omega^{-2}Y\right\rangle, \\ V_2 &= c\left\langle \Omega^{-1}Z, Z\right\rangle - \beta b_0 \langle Z, \Psi\Omega^{-1}Z\rangle, \\ V_3 &= c\left\langle \theta Y, Z\right\rangle - \beta b_0 \langle Z, \Psi\theta Y\rangle + \frac{1}{2}\beta b_0 \left\langle Y, G\theta Y\right\rangle + \frac{1}{2}\beta b_0 \left\langle Y, G'\Omega^{-1}Y\right\rangle. \end{aligned}$$

By virtue of (H_1), Lemma 2.1 and Lemma 2.2 it follows

$$\begin{aligned} V_1 &= \left\langle Y, (\beta c b_0 I - cG\Omega^{-1})\Omega^{-1}Y\right\rangle \leq -\frac{c b_0}{\omega_0}(\frac{1}{\omega_1} - \beta) \parallel Y \parallel^2, \\ V_2 &= \left\langle Z, (cI - \beta b_0 \Psi)\Omega^{-1}Z\right\rangle \leq -\frac{1}{\omega_0}(\beta a_0 b_0 - c) \parallel Z \parallel^2. \end{aligned}$$

Finally, by (5), Lemma 2.5 and the inequality $2 \parallel UV \parallel \leq \parallel U \parallel^2 + \parallel V \parallel^2$ we get

$$\parallel \theta(t) \parallel = \parallel \Omega^{-1}(X)\Omega'(X)\Omega^{-1}(X) \parallel \leq \frac{1}{\omega_0^2} \parallel \Omega'(X) \parallel, \tag{11}$$

$$\begin{aligned} V_3 &= c\left\langle \theta Y, Z\right\rangle - \beta b_0 \langle Z, \Psi\theta Y\rangle + \frac{1}{2}\beta b_0 \left\langle Y, G\theta Y\right\rangle + \frac{1}{2}\beta b_0 \left\langle Y, G'\Omega^{-1}Y\right\rangle \\ &\leq \Big[\frac{1}{\omega_0^2}\left(\frac{c}{2k_0} + \frac{\beta b_0 a_1}{2k_0} + \frac{1}{2k_0}\beta b_0 b_1\right) \parallel \Omega' \parallel + \frac{1}{2k_0}\beta b_0 \parallel G' \parallel \Big] V \\ &\leq K_1 \delta(t) V, \end{aligned}$$

where $K_1 = \max\left\{\frac{1}{2k_0\omega_0^2}\big(c+\beta b_0 a_1+\beta b_0 b_1\big);\ \frac{\beta b_0}{2k_0}\right\}$. Hence, we conclude that

$$V'_{(4)} \leq -M \parallel Z \parallel^2 -N \parallel Y \parallel^2 + K_1\delta(t)V. \tag{12}$$

Clearly, from condition (i) of Theorem 3.1 we have

$$N = \frac{cb_0}{\omega_0}\left(\frac{1}{\omega_1}-\beta\right) > 0 \quad \text{and} \quad M = \frac{1}{\omega_0}(\beta a_0 b_0 - c) > 0.$$

Now, from (7) and (12) we obtain

$$\begin{aligned} W'_{(4)} &= \left[V' - \frac{1}{d}\delta(t)V\right]\exp(-\mu(t)) \\ &\leq \left[-M \parallel Z \parallel^2 -N \parallel Y \parallel^2 +(K_1-\frac{1}{d})\delta(t)V\right]\exp(-\mu(t)). \end{aligned}$$

Choosing $K_1 - \dfrac{1}{d} = 0$, the last inequality becomes

$$W'_{(4)} \leq -C(\parallel Z \parallel^2 + \parallel Y \parallel^2), \tag{13}$$

where $C = \exp(-\frac{\delta_0}{d})\min\{M,\ N\}$. In view of (10) and (13), it follows that the solution $\big(X(t),Y(t),Z(t)\big)$ of (4) is uniformly stable.

Now $E = \{(X,Y,Z) :\ W'_{(4)}(X,Y,Z) = 0\} = \{(X,0,0) : X \in \mathbb{R}^n\}$ and the largest invariant set contained in E is $F = \{(0,0,0)\}$. By LaSalle's invariance principe

$$\lim_{t\to\infty} X(t) = \lim_{t\to\infty} Y(t) = \lim_{t\to\infty} Z(t) = 0.$$

This fact completes the proof of Theorem 3.1.

4 Boundedness

Our main theorem in this section is stated with respect to $P(t) \neq 0$ as follows :

Theorem 4.1 *Assume that all the conditions of Theorem 3.1 are satisfied and there exist positive constants d_1 and D_1 such that :*

I_1) $\parallel P(t) \parallel \leq \lambda(t) < d_1$,

I_2) $\int_0^t \lambda(s)ds < D_1$,

I_3) $\lim\limits_{t\to\infty} \parallel \Omega'(X(t)) \parallel$ *exists.*

Then there exists a positive constant D_5 such that any solution $X(t)$ of (3) and their derivatives $X'(t)$, and $X''(t)$ satisfy

$$\parallel X(t) \parallel \leq D_5,\ \parallel X'(t) \parallel \leq D_5,\ \parallel X''(t) \parallel \leq D_5. \tag{14}$$

Proof. For the case $P(t) \neq 0$, on differentiating (8) along the system (4) we obtain

$$\begin{aligned} V'_{(4)} &\leq -J + K_1\delta(t)V + c\langle\Omega^{-1}Y, P(t)\rangle + \langle\beta b_0 Z, P(t)\rangle \\ &\leq -J + K_1\delta(t)V + \lambda(t)\Big(c \parallel \Omega^{-1} \parallel \parallel Y \parallel + \beta b_0 \parallel Z \parallel\Big). \end{aligned}$$

Using Lemma 2.5 we get

$$V'_{(4)} \leq -J + K_1\delta(t)V + K_2\lambda(t)(\| Y \| + \| Z \|),$$

where $K_2 = \max\left\{\dfrac{c}{\omega_0}, \beta b_0\right\}$ and $J = M \| Z \|^2 + N \| Y \|^2$.

Now, the inequalities $\| Y \| \leq \| Y \|^2 + 1$ and $\| Z \| \leq \| Z \|^2 + 1$ lead to

$$V'_{(4)} \leq -J + K_1\delta(t)V + K_2\lambda(t)(\| Y \|^2 + \| Z \|^2 + 2). \tag{15}$$

From (7) we have

$$W'_{(4)} = \left[V' - \frac{1}{d}\delta(t)V\right]\exp(-\mu(t)). \tag{16}$$

Since $K_1 - \dfrac{1}{d} = 0$, it follows that

$$W'_{(4)} \leq \left[-J + K_2\lambda(t)(\| Y \|^2 + \| Z \|^2 + 2)\right]\exp(-\mu(t)).$$

In view of (13) and (10), the above estimates imply that

$$W'_{(4)} \leq -C(\| Y \|^2 + \| Z \|^2) + \frac{K_2}{K_0}\lambda(t)\ W + K_3\lambda(t), \tag{17}$$

with $K_3 = 2K_2$. Integrating both sides (17) from 0 to t, one can easily obtain

$$W(t) - W(0) \leq K_3\int_0^t \lambda(s)ds + \frac{K_2}{K_0}\int_0^t W(s)\lambda(s)ds.$$

Let

$$D_2 = W(0) + K_3D_1. \tag{18}$$

Thus

$$W(t) \leq D_2 + \frac{K_2}{K_0}\int_0^t W(s)\lambda(s)ds.$$

By the Gronwall inequality it follows

$$W(t) \leq D_2\exp\Big(\frac{K_2}{K_0}\int_0^t \lambda(s)ds\Big) \leq D_3, \tag{19}$$

where $D_3 = D_2\exp\left(\frac{K_2}{K_0}D_1\right)$. This result implies that there exists a constant D_4 such that

$$\| X(t) \|\leq D_4,\ \| Y(t) \|\leq D_4, \| Z(t) \|\leq D_4.$$

From (4) we have

$$\begin{aligned} \| X'(t) \| &= \| \Omega^{-1}Y(t) \| \\ &\leq \left\|\Omega^{-1}\right\| \| Y(t) \| \\ &\leq \frac{D_4}{\omega_0}. \end{aligned}$$

Since $\lim_{t\to\infty} \parallel \Omega'(X(t)) \parallel$ exists, we have

$$\parallel \Omega'\big(X(t)\big) \parallel < q_1, \tag{20}$$

for some positive constant q_1. So, from (11) we get

$$\parallel \theta(t) \parallel \leq \frac{q_1}{\omega_0^2}. \tag{21}$$

Hence

$$\begin{aligned} \parallel X^{''}(t) \parallel &= \parallel \theta(t)Y(t) + \Omega^{-1}Z(t) \parallel \\ &\leq \parallel \theta(t)Y(t) \parallel + \parallel \Omega^{-1}Z(t) \parallel \\ &\leq \Big(\frac{q_1}{\omega_0^2} + \frac{1}{\omega_0}\Big)D_4. \end{aligned}$$

Therefore, there exists a positive constant D_5 such that

$$\parallel X(t) \parallel \leq D_5, \ \parallel X'(t) \parallel \leq D_5, \ \parallel X''(t) \parallel \leq D_5, \tag{22}$$

for all $t \geq 0$, where $D_5 = \max\big\{\big(\frac{q_1}{\omega_0^2} + \frac{1}{\omega_0}\big)D_4, \ D_4\big\}$. This completes the proof of Theorem 4.1.

5 Square Integrability

Our next result concerns the square integrability of solutions of equation (3).

Theorem 5.1 *In addition to the assumptions of Theorem 4.1, we assume that*

$I_4)$ $c - (\frac{a_1 + b_1}{2}) > 0.$

Then all the solutions of (3) and their derivatives are elements of $L^2[0, +\infty)$.

Proof. Define H(t) as

$$H(t) = W(t) + \varepsilon \int_0^t (\parallel Z(s) \parallel^2 + \parallel Y(s) \parallel^2) ds, \tag{23}$$

where $\varepsilon > 0$ is a constant to be specified later. By differentiating H(t) and using (17) we obtain

$$H'(t) \leq (\varepsilon - C)(\parallel Z(t) \parallel^2 + \parallel Y(t) \parallel^2) + \Big(K_2 W + K_3\Big)\lambda(t).$$

If we choose $\varepsilon - C < 0$, then from (19) we get

$$H'(t) \leq K_4 \lambda(t), \tag{24}$$

where $K_4 = K_2 D_3 + K_3$. Integrating (24) from 0 to t, $t \geq 0$, and using condition (I_2) of Theorem 4.1 we obtain

$$H(t) - H(0) = \int_0^t H'(s) ds \leq K_4 D_1.$$

Using (18) and equality $H(0) = W(0)$ we get

$$H(t) \leq K_4 D_1 + D_2 - K_3 D_1.$$

We can conclude by (23) that

$$\int_0^t (\parallel Z(s) \parallel^2 + \parallel Y(s) \parallel^2) ds < \frac{K_4 D_1 + D_2 - K_3 D_1}{\varepsilon},$$

which implies the existence of positive constants σ_1 and σ_2 such that

$$\int_0^t \parallel Z(s) \parallel^2 ds \leq \sigma_2 \text{ and } \int_0^t \parallel Y(s) \parallel^2 ds \leq \sigma_1.$$

From (4) we have

$$\begin{aligned} \int_0^t \parallel X'(s) \parallel^2 ds &= \int \parallel \Omega^{-1} Y(s) \parallel^2 ds \\ &\leq \int \left\|\Omega^{-1}\right\|^2 \parallel Y(s) \parallel^2 ds \\ &\leq \frac{\sigma_1}{\omega_0^2} = \beta_1. \end{aligned} \tag{25}$$

Also

$$\begin{aligned} \int_0^t \parallel X''(s) \parallel^2 ds &= \int_0^t \Big(\parallel \theta(s) Y(s) + \Omega^{-1} Z(s) \parallel^2 \Big) ds \\ &\leq \int_0^t \big(\parallel \theta(s) \parallel^2 + \parallel \theta(s) \parallel \parallel \Omega^{-1} \parallel \big) \parallel Y(s) \parallel^2 ds \\ &\quad + \int_0^t \big(\parallel \Omega^{-1} \parallel^2 + \parallel \theta(s) \parallel \parallel \Omega^{-1} \parallel \big) \parallel Z(s) \parallel^2 ds. \end{aligned}$$

From (21) and (20) we have

$$\begin{aligned} \int_0^t \big(\parallel \theta(s) \parallel^2 + \parallel \theta(s) \parallel \parallel \Omega^{-1} \parallel \big) \parallel Y(s) \parallel^2 ds &\leq \frac{q_1}{\omega_0^2} (\frac{q_1}{\omega_0^2} + \frac{1}{\omega_0}) \int_0^t \parallel Y(s) \parallel^2 ds \\ &\leq \frac{q_1}{\omega_0^2} (\frac{q_1}{\omega_0^2} + \frac{1}{\omega_0}) \sigma_1, \end{aligned}$$

and

$$\begin{aligned} \int_0^t \big(\parallel \Omega^{-1} \parallel^2 + \parallel \theta(s) \parallel \parallel \Omega^{-1} \parallel \big) \parallel Z(s) \parallel^2 ds &\leq \frac{1}{\omega_0} (\frac{1}{\omega_0} + \frac{q_1}{\omega_0^2}) \int_0^t \parallel Y(s) \parallel^2 ds \\ &\leq \frac{1}{\omega_0} (\frac{1}{\omega_0} + \frac{q_1}{\omega_0^2}) \sigma_2. \end{aligned}$$

It follows

$$\int_0^t \parallel X''(s) \parallel^2 ds \leq \frac{q_1}{\omega_0^2} (\frac{q_1}{\omega_0^2} + \frac{1}{\omega_0}) \sigma_1 + \frac{1}{\omega_0} (\frac{1}{\omega_0} + \frac{q_1}{\omega_0^2}) \sigma_2 \ = \beta_2. \tag{26}$$

Next, multiplying (3) by $X(t)$, we obtain

$$\Big\langle \big(\Omega(X)X'\big)'', X\Big\rangle + \langle \Psi(X')X'', X\rangle + \langle G(X)X', X\rangle + c \parallel X \parallel^2 = \langle X, P(t)\rangle. \tag{27}$$

Integrating (27) from 0 to t we have

$$c\int_0^t \parallel X(s) \parallel^2 ds = L_1(t) + L_2(t) + L_3(t), \tag{28}$$

where

$$\begin{aligned} L_1(t) &= -\int_0^t \Big\langle \big(\Omega(X(s))X'(s)\big)'', X(s)\Big\rangle\, ds,\\ L_2(t) &= -\int_0^t \Big\langle \Big(\Psi(X'(s))X''(s) + G(X(s))X'(s)\Big), X(s)\Big\rangle\, ds,\\ L_3(t) &= \int_0^t \langle X(s), P(s)\rangle ds. \end{aligned}$$

Integrating by parts and using (25) and (26), we obtain

$$\begin{aligned} L_1(t) &= -\langle \Omega' X'(t), X(t)\rangle - \langle \Omega X''(t), X(t)\rangle \\ &\quad + \langle \Omega X'(t), X'(t)\rangle - \int_0^t \langle \Omega X'(s), X''(s)\rangle\, ds \\ &\leq \mid -\langle \Omega' X'(t), X(t)\rangle - \langle \Omega X''(t), X(t)\rangle + \langle \Omega X'(t), X'(t)\rangle \mid \\ &\quad + \int_0^t \frac{\omega_1}{2}\big(\parallel X'(s) \parallel^2 + \parallel X''(s) \parallel^2\big)\, ds \\ &\leq \mid -\langle \Omega' X'(t), X(t)\rangle - \langle \Omega X''(t), X(t)\rangle + \langle \Omega X'(t), X'(t)\rangle \mid + \frac{\omega_1}{2}(\beta_1 + \beta_2). \end{aligned}$$

In view of (20) and (22) we get

$$|- \langle \Omega' X'(t), X(t)\rangle - \langle \Omega X''(t), X(t)\rangle + \langle \Omega X'(t), X'(t)\rangle| \leq D_5^2\big(q_1 + 2\omega_1\big),$$

for all $t \geq 0$. Consequently, there exists a constant l_1 such that $L_1(t) < l_1$, with $l_1 = D_5^2\big(q_1 + 2\omega_1\big) + \frac{\omega_1}{2}(\beta_1 + \beta_2)$. Similarly we have

$$\begin{aligned} L_2(t) &= -\int_0^t \big\langle \big(\Psi X''(s) - G X'(s)\big), X(s)\big\rangle\, ds \\ &\leq \int_0^t \Big(\parallel \Psi \parallel \parallel X''(s) \parallel + \parallel G \parallel \parallel X'(s) \parallel \Big) \parallel X(s) \parallel ds \\ &\leq \int_0^t \parallel \Psi \parallel \parallel X''(s) \parallel \parallel X(s) \parallel ds + \int_0^t \parallel G \parallel \parallel X'(s) \parallel \parallel X(s) \parallel ds \\ &\leq \frac{a_1}{2}\int_0^t \parallel X''(s) \parallel^2 ds + (\frac{a_1 + b_1}{2})\int_0^t \parallel X(s) \parallel^2 ds + \frac{b_1}{2}\int_0^t \parallel X'(s) \parallel^2 ds \\ &\leq \frac{a_1}{2}\beta_2 + \frac{b_1}{2}\beta_1 + (\frac{a_1 + b_1}{2})\int_0^t \parallel X(s) \parallel^2 ds. \end{aligned}$$

Next

$$\begin{aligned} L_3(t) &\leq \int_0^t \parallel X(s) \parallel \parallel P(s) \parallel ds \\ &\leq D_5 \int_0^t \lambda(s) ds \\ &\leq D_1 D_5. \end{aligned}$$

By (28) and condition (I_4) of the Theorem 5.1 we obtain

$$\left(c - (\frac{a_1 + b_1}{2})\right) \int_0^t \parallel X(s) \parallel^2 ds \leq K,$$

where $K = l_1 + \frac{a_1}{2}\beta_2 + \frac{b_1}{2}\beta_1 + D_1 D_5$. This fact completes the proof of theorem.

Example 5.1 As a special case consider the following equation

$$(\Omega(X(t))X'(t))'' + \Psi(X')X''(t) + G(X)X'(t) + cX(t) = P(t), \tag{29}$$

where

$$\begin{aligned} \Omega(X) &= \begin{pmatrix} \frac{\sin x}{1+x^2} + 2 & 0 \\ 0 & \frac{2}{10}\frac{\cos y}{1+y^2} + 2 \end{pmatrix}, \quad \Psi(Y) = \begin{pmatrix} 9 + \frac{1}{1+y^2} & 1 \\ 1 & 9 + \frac{1}{1+y^2} \end{pmatrix}, \\ G(X) &= \begin{pmatrix} \frac{1}{3+x^2} + 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad P(t) = \begin{pmatrix} \frac{\sin t}{1+t^2} \\ \frac{\cos t}{1+t^2} \end{pmatrix}, c = 7 \quad . \end{aligned}$$

Clearly, $\Psi(Y), G(X)$ and $\Omega(X)$ are symmetric matrices and commute pairwise. Then, by an easy calculation, we obtain eigenvalues of the matrices $\Psi(Y), G(X)$ and $\Omega(X)$ as follows:

$$\begin{aligned} \omega_0 &= 1 \leq \lambda_i(\Omega(X)) \leq 2.2 = \omega_1, \\ a_0 &= 8 \leq \lambda_i(\Psi(Y)) \leq 11 = a_1, \\ b_0 &= 2 \leq \lambda_i(G(X)) \leq \frac{7}{3} = b_1, \end{aligned}$$

for $i \in \{1, 2\}$. For $t \in [0, +\infty)$ a straightforward calculation gives

$$\begin{aligned} \int_0^t \parallel \Omega'(X(s)) \parallel du &= \int_0^t \left| (\frac{\cos x}{1+x^2} - \frac{2x \sin x}{(1+x^2)^2}) x'(s) \right| ds \\ &\quad + \int_0^t \left| (\frac{-\sin y}{1+y^2} - \frac{2y \cos y}{(1+y^2)^2}) y'(s) \right| ds \\ &\leq \int_{\theta_1(t)}^{\theta_2(t)} \left| (\frac{\cos u}{1+u^2} - \frac{2u \sin u}{(1+u^2)^2}) \right| du \\ &\quad + \int_{\varphi_1(t)}^{\varphi_2(t)} \left| (\frac{-\sin v}{1+v^2} - \frac{2v \cos v}{(1+v^2)^2}) \right| dv \\ &< \left(\int_{-\infty}^{+\infty} \left| \frac{1+u^2+2u}{(1+u^2)^2} \right| du + \int_{-\infty}^{+\infty} \left| \frac{1+u^2+2u}{(1+u^2)^2} \right| du \right) \\ &= (\pi + 2), \end{aligned}$$

where

$$\theta_1(t) = \min\{x(0), x(t)\}, \qquad \theta_2(t) = \max\{x(0), x(t)\},$$
$$\varphi_1(t) = \min\{y(0), y(t)\}, \qquad \varphi_2(t) = \max\{y(0), y(t)\}.$$

Similarly

$$\int_{-\infty}^{+\infty} \|G'(X(s))\| ds = \int_{-\infty}^{+\infty} \left| \frac{-2u}{(3+u^2)^2} \right| du = \frac{2}{3}.$$

Now, we have

$$\| P(t) \| = \sqrt{\frac{\sin^2 t}{1+t^2} + \frac{\cos^2 t}{1+t^2}} = \frac{1}{1+t^2} < \frac{2}{1+t^2} = \lambda(t) < 2 = d_1.$$

So,

$$\int_0^t \| \lambda(s) \| ds = \int_0^t \frac{2}{1+s^2} ds < \int_0^{+\infty} \frac{2}{1+s^2} ds = \pi = D_1.$$

By taking $\beta = 0.44$, it follows easily that

$$0.4375 = \frac{7}{16} = \frac{c}{a_0 b_0} < \beta < \frac{1}{\omega_1} = 0.45455.$$

We have also

$$c - \frac{a_1 + b_1}{2} = \frac{1}{3} > 0.$$

Thus, all the conditions of Theorem 5.1 are satisfied.

References

[1] Afuwape, A.U. Ultimate boundedness results for a certain system of third-order nonlinear differential equations. *J. Math. Anal. Appl.* **97** (1983) 140–150.

[2] Afuwape, A.U. and Carvajal, Y.E. Stability and ultimate boundedness of solutions of a certain third order nonlinear vector differential equation. *J. Nigerian Math. Soc.* **31** (2012) 69–80.

[3] Elmadssia,S. Saadaoui, K. and Benrejeb, M. Stability Conditions for a Class of Nonlinear Time Delay. *Nonlinear Dynamics and Systems Theory* **14** (3) (2014) 279–291.

[4] Ezeilo, J.O.C. n-dimensional extensions of boundedness and stability theorems for some third-order differential equations. *J. Math. Anal. Appl.* **18** (1967) 395–416.

[5] Ezeilo, J.O.C. and Tejumola, H.O. Further results for a system of third-order ordinary differential equations. *Atti. Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* **58** (1975) 143–151.

[6] Graef, J.R., Beldjerd, D. and M. Remili. On stability, ultimate boundedness, and existence of periodic solutions of certain third order differential equations with delay. *Pan-American Mathematical Journal* **25** (2015) 82–94.

[7] Graef, J.R., Oudjedi, L.D. and Remili, M. Stability and square integrability of solutions of nonlinear third order differential equations. *Dynamics of Continuous, Discrete and Impulsive Systems Series A: Mathematical Analysis* **22** (2015) 313–324.

[8] Meng, F.W. Ultimate boundedness results for a certain system of third-order nonlinear differential equations. *J. Math. Anal. Appl.* **177** (1993) 496–509.

[9] Omeike, M.O. Stability and boundedness of solutions of nonlinear vector differential equations of third order. *Arch. Math. (Brno)* **50** (2)(2014) 101–106.

[10] Omeike, M.O. Stability and Boundedness of Solutions of a Certain System of Third-order Nonlinear Delay Differential Equations. *Acta Univ. Palacki. Olomuc., Fac. rer. nat., Mathematica* **54** (1) (2015) 109–119.

[11] Remili, M. and Beldjerd, D. On the asymptotic behavior of the solutions of third order delay differential equations. *Rend. Circ. Mat. Palermo* **63** (3) (2014) 447–455.

[12] Remili, M. and Beldjerd. D. Stability and ultimate boundedness of solutions of some third order differential equations with delay. *Journal of the Association of Arab Universities for Basic and Applied Sciences* (2016), http://dx.doi.org/10.1016/j.jaubas.

[13] Remili, M. and Beldjerd. D. On ultimate boundedness and existence of periodic solutions of kind of third order delay differential equations. Acta Universitatis Matthiae Belii, series Mathematics Issue 2016, 1–15.

[14] Remili, M. and Oudjedi, D.L. Stability and boundedness of the solutions of non autonomous third order differential equations with delay. *Acta Univ. Palack. Olomuc. Fac. Rerum Natur. Math.* **53** (2) (2014) 139–147.

[15] Remili, M. and Rahmane, M. Boundedness and Square Integrability of Solutions of Nonlinear Fourth Order Differential Equations. *Nonlinear Dynamics and Systems Theory* **16** (2) (2016) 192–205.

[16] Tunç, C. The Boundedness of Solutions to Nonlinear Third Order Differential Equations. *Nonlinear Dynamics and Systems Theory* **10** (1) (2010) 97–102.

[17] Tunç, C. On the stability and boundedness of solutions of nonlinear vector differential equations of third order. *Nonlinear Analysis* **70** (6) (2009) 2232–2236.