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# Monotone Method for Finite Systems of Nonlinear Riemann-Liouville Fractional Integro-Differential Equations

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**Abstract:** In this paper we develop the monotone method for nonlinear finite $N$-systems of Riemann-Liouville integro-differential equations of order $0 < q < 1$. The iterative technique approximates maximal and minimal coupled quasisolutions to the nonlinear system using sequences of linear systems that are constructed via coupled lower and upper solutions of varying types. Preliminary existence and comparison theorems are presented and proven where appropriate. Finally, we present a numerical example.

**Keywords:** *monotone method; Riemann-Liouville fractional integro-differential equations; finite systems.*

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## 1 Introduction

Fractional differential equations have various applications in widespread fields of science, such as engineering [5], chemistry [14,15], physics [1,8], and others [9,10]. Despite there being a number of existence theorems for nonlinear fractional differential equations, much as in the integer order case, this does not necessarily imply that calculating a solution explicitly will be routine, or even possible. Therefore, it may be necessary to employ an iterative technique to numerically approximate a needed solution. In this paper we

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construct such a method. For some existence results on fractional differential equations we refer the reader to the papers [6,7] and the books [9,16] along with references therein.

The iterative technique we construct in this paper is a generalization of the monotone method. Put simply, this method constructs two sequences from upper and lower solutions that converge monotonically and uniformly to maximal and minimal solutions. The advantage of the monotone method is that solutions of nonlinear differential equations are approximated by solutions of linear differential equations. Further, the interval of existence for the solution is guaranteed due to the nature of the upper and lower solutions and the method is valid whether the original DE has a unique solution or not. There are complications that arise when developing the monotone method for Riemann-Liouville equations. A major wrinkle comes from the fact that the constructed sequences do not converge uniformly themselves, but instead the weighted sequences $\{t^{1-q}v_n\}, \{t^{1-q}w_n\}$ converge uniformly to weighted maximal and minimal solutions, where $q$ is the order of the system.

The monotone method has been constructed for various forms of differential equations, in this paper we extend the method to approximate Riemann-Liouville fractional integro-differential systems. Integro-differential equations generalize the problem by incorporating an integral transformation within the forcing function of the problem, e.g. $f(t, x, \int_0^t K(x, s)x(s)ds)$, and therefore generalize the possibilities of models, see [13]. A generalized monotone method for the scalar form of this problem was constructed in [2], and in this paper we extend the problem to an $N$-system of these equations. Moving to finite systems allows for generalizations that include many combinations of monotone properties along with upper and lower solution constructions. For example, we can reorder the variables within $f$ for each iterate so that it increases in some variables and decreases in others, e.g. $f_i(t, x) = f_i(t, [x]_{s_i}, [x]_{r_i})$ where $f_i$ increases in $[x]_{s_i}$ and decreases in $[x]_{r_i}$. When combined with an integral transformation $T$ we establish the generalized system of the form

$$D^q x_i = f_i(t, x_i, [x]_{r_i}, [x]_{s_i}, [Tx]_{\rho_i}, [Tx]_{\sigma_i}),$$

where $f_i$ is split into components where it is increasing and decreasing respectively.

There is more nuance to these generalizations than described here, and we will go into more detail in Sections 2 and 3. In the final section we will develop numerical examples which exemplify our main results. The monotone method for more standard Riemann-Liouville fractional differential systems and multi-order systems was established in [3,4], and more information on the monotone method for ordinary differential equations and systems can be found in [11].

## 2 Preliminary Results

In this section, we will first consider basic results regarding scalar Riemann-Liouville (R-L) differential equations of order $q$, $0 < q < 1$. We will recall basic definitions and results in this case for simplicity, and we note that many of these results carry over naturally to finite systems. In the next section, we will apply these preliminary results to develop the monotone method for nonlinear fractional integro-differential systems. Note, for simplicity we only consider results on the interval $J = (0, T]$, where $T > 0$. Further, we will let $J_0 = [0, T]$, that is $J_0 = \bar{J}$.

**Definition 2.1** Let $p = 1 - q$, a function $\phi(t) \in C(J, R)$ is a $C_p$ continuous function if $t^p\phi(t) \in C(J_0, R)$. The set of $C_p$ continuous functions is denoted $C_p(J, R)$. Further, given a function $\phi(t) \in C_p(J,R)$, we call the function $t^p\phi(t)$ the continuous extension of $\phi(t)$.

Now we define the R-L integral and derivative of order $q$ on the interval $J$.

**Definition 2.2** Let $\phi \in C_p(J,R)$, then $D_t^q\phi(t)$ is the $q$-th R-L derivative of $\phi$ with respect to $t \in J$ defined as

$$D_t^q\phi(t) = \frac{1}{\Gamma(1-q)}\frac{d}{dt}\int_0^t (t-s)^{-q}\phi(s)ds,$$

and $I_t^q\phi(t)$ is the $q$-th R-L integral of $\phi$ with respect to $t \in J$ defined as

$$I_t^q\phi(t) = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\phi(s)ds.$$

Note that in cases where the initial value may be different or ambiguous, we will write out the definition explicitly. The next definition is related to the solution of linear R-L fractional differential equations and is also of great importance in the study of the R-L derivative.

**Definition 2.3** The Mittag-Leffler function with parameters $\alpha, \beta \in R$, denoted $E_{\alpha,\beta}$, is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},$$

which is entire for $\alpha, \beta > 0$.

For fractional differential equations we utilize the weighted $C_p$ version of the Mittag-Leffler function $t^{q-1}E_{q,q}(t^q)$, since it is its own $q$-th derivative. Further, it attains a convergence result we mention in the following remark.

**Remark 2.1** The $C_p$ weighted Mittag-Leffler function

$$t^{q-1}E_{q,q}(\lambda t^q) = \sum_{k=0}^{\infty}\frac{\lambda^k t^{kq+q-1}}{\Gamma(kq+q)},$$

where $\lambda$ is a constant, converges uniformly on compacta of $J$. Further

$$D^q\left[t^{q-1}E_{q,q}(\lambda t^q)\right] = \lambda t^{q-1}E_{q,q}(\lambda t^q),$$

and

$$I^q\left[t^{q-1}E_{q,q}(\lambda t^q)\right] = \tfrac{1}{\lambda}t^{q-1}E_{q,q}(\lambda t^q) - \tfrac{1}{\lambda\Gamma(q)}t^{q-1}.$$

The next result gives us that the $q$-th R-L integral of a $C_p$ continuous function is also a $C_p$ continuous function. This result will give us that the solutions of R-L differential equations are also $C_p$ continuous.

**Lemma 2.1** *Let $f \in C_p(J,R)$, then $I^qf(t) \in C_p(J,R)$, i.e. the $q$-th integral of a $C_p$ continuous function is $C_p$ continuous.*

Note the proof of this theorem for $q \in R^+$ can be found in [4].

**Remark 2.2** In [9] and [12] it was proven that if $0 < q < 1$, $G \subset R$ is an open set, and $f : J \times G \to R$ is such that for any $x \in G$, $f \in C_p(J, G)$, then $x$ satisfies the fractional differential equation

$$D^q x = f(t, x), \quad \text{with initial condition} \quad t^p x\big|_{t=0} = x_0, \tag{1}$$

if and only if it satisfies the Volterra fractional integral equation

$$x(t) = x_0 t^{q-1} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x) ds. \tag{2}$$

This relationship is especially true if $f : [0, T] \times G \to R$ is continuous.

Now we consider results for the nonhomogeneous linear R-L differential equation,

$$D_t^q x(t) = \lambda x(t) + z(t), \quad t^p x(t)\big|_{t=0} = x^0, \tag{3}$$

where $x^0$ is a constant and $x, z \in C_p(J, R)$, which has unique solution

$$x(t) = x^0 \Gamma(q) t^{q-1} E_{q,q}(\lambda t^q) + \int_0^t (t-s)^{q-1} E_{q,q}(\lambda (t-s)^q) z(s)\, ds.$$

Next, we recall a result we will utilize extensively in our proceeding comparison and existence results, and likewise in the construction of the monotone method. We note that this result is similar to the well known comparison result found in literature, as in [12], but we do not require the function to be Hölder continuous of order $\lambda > q$.

**Lemma 2.2** *Let $m \in C_p(J, R)$ be such that for some $t_1 \in J$ we have $m(t_1) = 0$ and $m(t) \leq 0$ for $t \in (0, t_1]$. Then*

$$D_t^q m(t)\big|_{t=t_1} \geq 0.$$

The proof of this lemma can be found in [4], along with further discussion as to why and how we weaken the Hölder continuous requirement. We use this lemma in the proof of the later main comparison result, which will be critical in the construction of the monotone method.

Now we will consider results for finite $N$-systems of R-L integro-differential equations. For simplicity, we will henceforth assume that $i \in \{1, 2, 3, \ldots, N\}$, and that for any $N$ element vectors $x, y$, $x \leq y$ implies $x_i \leq y_i$ for all $i$. We extend the concept of $C_p$ continuous functions to $R^N$ in the natural way

$$C_p(J, R^N) = \{\phi \in C(J, R^N) \,|\, t^p \phi_i \in C(J_0, R),\, 1 \leq i \leq N\}.$$

For simplicity we introduce the following notation for the scalar multiplication form of the continuous extension $x_p(t) = t^p x(t)$, so that $t^p x\big|_{t=0}$ becomes $x_p(0)$. The system we consider is

$$D^q x_i = f_i(t, x, Tx), \quad x_{p_i}(0) = x_i^0, \tag{4}$$

where each $x_i^0$ is a constant, and $Tx$ is a simplified notation for

$$Tx = \{T_1 x_1, T_2 x_2, T_3 x_3, \ldots T_N x_N\}, \quad T_i x_i = \int_0^t K_i(s, t) x_i(s) ds,$$

and where $K_i$ is continuous and positive on $J_0$ for each $i$.

**Remark 2.3** Notice that each $K_i$ is bounded on $J_0$ so letting $\widehat{K}$ be a bound for each $K_i$, and using Remark 2.1, for each $i$ we have

$$\begin{aligned} T_i t^{q-1} E_{q,q}(\lambda t^q) &\le \frac{\widehat{K}}{\Gamma(q)} \int_0^t \frac{\Gamma(q)(t-s)^p}{(t-s)^p} s^{q-1} E_{q,q}(\lambda s^q) ds \\ &\le \widehat{K} T^p \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} s^{q-1} E_{q,q}(\lambda s^q) ds < \widehat{K} T^p \Gamma(q) \tfrac{1}{\lambda} t^{q-1} E_{q,q}(\lambda t^q). \end{aligned}$$

Now, we introduce the concept of quasimonotonicity, which will be a generalization of monotonicity for our main results.

**Definition 2.4** A function $\phi : R^N \to R^N$ is said to be quasimonotone increasing if for each $i$, $x \le y$ and $x_i = y_i$ implies $\phi_i(x) \le \phi_i(y)$. Naturally, $\phi$ is quasimonotone decreasing if we reverse the inequalities.

From now on, if we wish to designate standard monotonicity we will state that a function increases or decreases traditionally. In our main result, we construct our iterative technique from lower and upper solutions. Further, many of our preliminary results stem from these solutions which we define below.

**Definition 2.5** Let $v, w \in C_p(J, R^N)$, then $v, w$ are lower and upper solutions of (4) respectively if

$$D^q v_i \le f_i(t, v, Tv), \quad D^q w_i \ge f_i(t, w, Tw), \quad v_{p_i}(0) \le x_i^0 \le w_{p_i}(0).$$

Now we present the main comparison theorem that will form the base of our remaining results. This result gives us conditions for when lower and upper solutions behave in a natural way, i.e. when $v \le w$ on $J$. Specifically, if $f$ is quasimonotone in $x$ and monotone in $Tx$ and satisfies a one-sided Lipschitz condition, then $v \le w$. The result is given below.

**Theorem 2.1** *Let $v, w \in C_p(J, R^N)$ be lower and upper solutions of (4). If $f$ is quasimonotone increasing in $x$ and traditionally increasing in $Tx$, and satisfies the Lipschitz condition:*

$$f_i(t, x, Tx) - f_i(t, y, Ty) \le \sum_{k=1}^{N} L_i(x_k - y_k) + M_i T_k(x_k - y_k),$$

*then $v \le w$ on $J$.*

***Proof.*** We start by assuming that one of the inequalities is strict, $D^q v_i < f_i(t, v, Tv)$ for each $i$, and $v_p(0) < w_p(0)$, and we will show that $v < w$ on $J$. Suppose to the contrary that our claim is not true, then the set

$$Z = \bigcup_{i=1}^{N} \{t \in J \,:\, v_i(t) = w_i(t)\}$$

is nonempty. So let $\tau = \inf Z$, and suppose without loss of generality, via reordering if necessary, that $v_1(\tau) = w_1(\tau)$. Now by the continuity of $v_p$ and $w_p$ on $J_0$ and since

$v_p(0) < w_p(0)$, we have that $v_p < w_p$ on $[0, \tau)$, and thus giving us that $v \le w$ on $(0, \tau]$. This also gives us that $T_i v_i(\tau) \le T_i w_i(\tau)$ for each $i$.

Letting $m = v - w$ we have by Lemma 2.2 that $D^q m|_{t=\tau} \ge 0$. Now, using this and the quasimonotone and traditional monotone properties of $f$ we obtain:

$$\begin{aligned} f_1(\tau, v(\tau), Tv(\tau)) > D^q v_j\big|_{t=\tau} &\ge D^q w_j\big|_{t=\tau} \ge f_1(\tau, w(\tau), Tw(\tau)) \\ &= f_1(\tau, v_1(\tau), w_2(\tau), w_3(\tau), \ldots w_N(\tau), Tw(\tau)) \ge f_1(\tau, v(\tau), Tv(\tau)), \end{aligned}$$

which is a contradiction. Therefore, $v < w$ on $J$.

Now, to prove the theorem as given we will use the strict inequality case. To do so let $\varepsilon > 0$, and construct functions

$$v_{\varepsilon i} = v_i - \varepsilon\varphi, \quad w_{\varepsilon i} = w_i + \varepsilon\varphi,$$

where $\varphi(t) = t^{q-1} E_{q,q}((N+1)\mathcal{L} t^q)$ and $\mathcal{L}$ is defined as

$$\mathcal{L} = \max_{1 \le i \le N} \{\widehat{K} T^p \Gamma(q), L_i, M_i\},$$

where $\widehat{K}$ is defined as in Remark 2.3. Note that by definition $v_\varepsilon < v$ and $w_\varepsilon > w$ on $J$ since $\varphi > 0$ on $J$. To start with, note that for each $i$

$$v^0_{\varepsilon i} = t^p v_{\varepsilon i}\big|_{t=0} = v^0_i - \varepsilon E_{q,q}(0) = v^0_i - \frac{\varepsilon}{\Gamma(q)} < v^0_i,$$

so $v^0_\varepsilon < v^0$. Then, for each $i$, we have

$$\begin{aligned} D^q v_{\varepsilon i} &\le f_i(t, v, Tv) - \varepsilon(N+1)\mathcal{L}\varphi \\ &= f_i(t, v_\varepsilon, Tv_\varepsilon) + f_i(t, v, Tv) - f_i(t, v_\varepsilon, Tv_\varepsilon) - \varepsilon(N+1)\mathcal{L}\varphi \\ &\le f_i(t, v_\varepsilon, Tv_\varepsilon) + \sum_{k=1}^{N} \Big[L_k(v_k - v_{\varepsilon k}) + M_k T(v_k - v_{\varepsilon k})\Big] - \varepsilon(N+1)\mathcal{L}\varphi \\ &\le f_i(t, v_\varepsilon, Tv_\varepsilon) + N\mathcal{L}\varepsilon\varphi + N\mathcal{L}T\varepsilon\varphi - \varepsilon(N+1)\mathcal{L}\varphi \\ &< f_i(t, v_\varepsilon, Tv_\varepsilon) + \frac{N\mathcal{L}}{(N+1)}\varepsilon\varphi - \varepsilon\mathcal{L}\varphi < f_i(t, v_\varepsilon, Tv_\varepsilon). \end{aligned}$$

We note that the penultimate inequality came from the application of Remark 2.3. Further, we can similarly show that $D^q w_{\varepsilon i} > f_i(t, w_\varepsilon, Tw_\varepsilon)$. Therefore, by the previous work involving strict inequalities we have that $v_\varepsilon < w_\varepsilon$ on $J$. Then letting $\varepsilon \to 0$ we obtain $v \le w$ on $J$, which completes the proof. This result can be extended to linear systems utilizing the following corrolary, the result follows from the Lipschitz nature of linear systems.

**Corollary 2.1** *If $g$ is a continuous function and $v, w \in C_p$ satisfy the following properties*

$$D^q v_i \le \lambda v_i + g_i(t), \quad D^q w_i \ge \lambda w_i + g_i(t), \quad v_{p_i}(0) \le w_{p_i}(0),$$

*then $v \le w$ on $J$.*

## 3 Monotone Method

For this section we expand our general system to cover more cases. To do so we split $\{x_i\}$ and $\{T_i x_i\}$ within each $f_i$ to isolate variables where each $f_i$ is increasing or decreasing in each $i$. So, for each $i$, let $r_i, s_i, \rho_i, \sigma_i$ be such that $r_i + s_i = N - 1$ and $\rho_i + \sigma_i = N$. Then, for each $i$, reorder $x$ and $Tx$, using the following notation

$$\begin{aligned} x =&\{x_1, x_2, x_3, \ldots, x_N\} = \{x_i, [x]_{r_i}, [x]_{s_i}\}, \\ Tx =&\{T_1x_1, T_2x_2, T_3x_3, \ldots, T_Nx_N\} = \{[Tx]_{\rho_i}, [Tx]_{\sigma_i}\}. \end{aligned}$$

This reordering allows us to isolate the variables where each $f_i$ increases or decreases, and each $r_i, s_i, \rho_i, \sigma_i$ represents the number of $x$ terms with each monotone property, and yields the following definition regarding $f$.

**Definition 3.1** We say $f$ possesses the mixed quasimonotonicity property if for each $i$

$$f_i(t, x, Tx) = f_i(t, x_i, [x]_{r_i}, [x]_{s_i}, [Tx]_{\rho_i}, [Tx]_{\sigma_i}),$$

and where $f_i$ is quasimonotone increasing in $[x]_{r_i}$, quasimonotone decreasing in $[x]_{s_i}$, traditionally increasing in $[Tx]_{\rho_i}$, and traditionally decreasing in $[Tx]_{\sigma_i}$.

**Remark 3.1** Definition 3.1 generalizes standard monotone cases for system (4), since if $s_i = \sigma_i = 0$, Definition 3.1 reduces down to $f(t, x, Tx)$ which is quasimonotone increasing in $x$ and traditionally increasing in $Tx$. Similarly, Definition 3.1 reduces to quasimonotone decreasing in $x$ and traditionally decreasing in $Tx$ when $r_i = \rho_i = 0$.

Now, the final fractional integro-differential system we construct the monotone method for is:

$$D^q x_i = f_i(t, x_i, [x]_{r_i}, [x]_{s_i}, [Tx]_{\rho_i}, [Tx]_{\sigma_i}), \quad x_{p_i}(0) = x_i^0, \tag{5}$$

where $f$ has the mixed quasimonotonicity property. This new formulation allows us to define new types of coupled upper and lower quasisolutions. We still have natural upper and lower solutions as defined in Definition 2.5, but in the following definition we introduce coupled, i.e. mixed, forms of the lower and upper solutions.

**Definition 3.2** $v, w \in C_p$ are Type I coupled lower and upper quasisolutions of (5) if

$$\begin{aligned} D^q v_i &\le f_i(t, v_i, [v]_{r_i}, [w]_{s_i}, [Tv]_{\rho_i}, [Tw]_{\sigma_i}), \quad v_{p_i}(0) = v_i^0 \le x_i^0 \\ D^q w_i &\ge f_i(t, w_i, [w]_{r_i}, [v]_{s_i}, [Tw]_{\rho_i}, [Tv]_{\sigma_i}), \quad w_{p_i}(0) = w_i^0 \ge x_i^0. \end{aligned}$$

$v, w \in C_p$ are Type II coupled lower and upper quasisolutions of (5) if

$$\begin{aligned} D^q v_i &\le f_i(t, w_i, [w]_{r_i}, [v]_{s_i}, [Tw]_{\rho_i}, [Tv]_{\sigma_i}), \quad v_{p_i}(0) = v_i^0 \le x_i^0 \\ D^q w_i &\ge f_i(t, v_i, [v]_{r_i}, [w]_{s_i}, [Tv]_{\rho_i}, [Tw]_{\sigma_i}), \quad w_{p_i}(0) = w_i^0 \ge x_i^0. \end{aligned}$$

If the inequalities in above definitions are replaced with equal signs, then they become coupled Type I or II quasisolutions of (5) and minimal and maximal coupled Type I or II quasisolutions are defined in the natural way given these definitions.

In an effort to simplify our manuscript and make it more readable we introduce the following notation, for each $i$, let $[v,w]_i$ be such that

$$f_i(t,x_i,[v,w]_i) = f_i(t,x_i,[v]_{r_i},[w]_{s_i},[Tv]_{\rho_i},[Tw]_{\sigma_i}).$$

Thus the first component of $[v,w]_i$ corresponds with the "increasing" portion of $f$ and the second component corresponds with the "decreasing" portion of $f$. So for example, the above coupled lower and upper quasisolutions can be rewritten as

$$\begin{aligned} \text{Type I:} \quad & D^q v_i \le f_i(t,v_i,[v,w]_i), \quad D^q w_i \ge f_i(t,w_i,[w,v]_i), \\ \text{Type II:} \quad & D^q v_i \le f_i(t,w_i,[w,v]_i), \quad D^q w_i \ge f_i(t,v_i,[v,w]_i). \end{aligned}$$

Now, if we know of the existence of lower and upper solutions $v$ and $w$ such that $v \le w$, we can prove the existence of a solution in the set

$$\Omega = \{(t,y) \,:\, v(t) \le y \le w(t),\, t \in J\}.$$

We consider this result in the following theorem.

**Theorem 3.1** *Let $v,w \in C_p(J,R^N)$ be Type I lower and upper solutions of (5) such that $v \le w$ on $J$ and let $f \in C(\Omega,R^N)$, where $\Omega$ is defined as above. Then there exists a solution $x \in C_p(J,R^N)$ of (4) such that $v \le x \le w$ on $J$.*

This theorem is proved in the same way as seen in [4], with only minor additions to incorporate the transformation $T$. In the next theorem we establish our main result. Essentially, if there are Type I lower and upper quasisolutions that satisfy their natural inequalities, that is $v \le w$ on $J$, and if $f$ satisfies the described conditions, then we can construct sequences of linear R-L systems, where the $C_p$ continuous extensions converge uniformly and monotonically to maximal and minimal Type I quasisolutions.

**Theorem 3.2** *Let $f$ possess the mixed quasimonotone property. Let $v_0,w_0 \in C_p(J,R^N)$ be Type I coupled lower and upper quasisolutions of (5) such that $v_0 \le w_0$ on $J$. For each $i$ suppose $f_i$ satisfies the following one-sided Lipschitz condition in the $x_i$ component:*

$$f_i(t,x_i,[x,x]_i) - f_i(t,y_i,[x,x]_i) \ge -M_i(x_i - y_i),$$

*whenever $v_0 \le x \le w_0$, and $v_{0_i} \le y_i \le x_i \le w_{0_i}$ on $J$ and $M_i \ge 0$. Then there exist monotone sequences $\{v_n\}$ and $\{w_n\}$ such that*

$$t^p v_n \to t^p v, \quad t^p w_n \to t^p w,$$

*uniformly and monotonically on $J_0$, where $v$ and $w$ are Type I coupled minimal and maximal quasisolutions of (5) on $J$ for solutions $v_0 \le x \le w_0$.*

***Proof.*** For the construction of the sequences let $\eta,\xi \in C_p(J,R^N)$ with $v_0 \le \eta,\xi \le w_0$ on $J$, then we start by considering the system

$$D^q x_i = f_i(t,\eta_i,[\eta,\xi]_i) - M_i(x_i - \eta_i), \quad x_{p_i}(0) = x_i^0. \tag{6}$$

We note that this system is an uncoupled linear system, therefore for each $\eta,\xi$ the system has a unique solution $x$. Thus we can define a transformation $A$ that yields the unique solution of (6) for each $\eta,\xi$, that is $A[\eta,\xi] = x$. We will construct our monotone sequences using this transformation, so we wish to show that $A$ has a mixed monotone property. $A$ is increasing in its first component and decreasing in its second component. To prove this, let $\eta, \widehat{\eta}, \xi \in C_p$ such that $v_0 \leq \eta, \widehat{\eta}, \xi \leq w_0$ and $\eta \geq \widehat{\eta}$ on $J$. Now suppose $x_a, x_b \in C_p$ such that $A[\eta, \xi] = x_a$ and $A[\widehat{\eta}, \xi] = x_b$.

Now, since $\eta \geq \widehat{\eta}$, we have that $T\eta \geq T\widehat{\eta}$, and then by the mixed quasimonotone property of $f$ we have that

$$f_i(t, \widehat{\eta}_i, [\eta, \xi]_i) \geq f_i(t, \widehat{\eta}_i, [\widehat{\eta}, \xi]_i).$$

So, using this, the definition of $x_a$ and the Lipschitz condition of $f$ we obtain

$$\begin{aligned} D^q x_{ai} &= f_i(t, \widehat{\eta}_i, [\eta, \xi]_i) + f_i(t, \eta_i, [\eta, \xi]_i) - f_i(t, \widehat{\eta}_i, [\eta, \xi]_i) - M_i(x_{ai} - \eta_i) \\ &\geq f_i(t, \widehat{\eta}_i, [\eta, \xi]_i) - M_i(x_{ai} - \widehat{\eta}_i) \geq f_i(t, \widehat{\eta}_i, [\widehat{\eta}, \xi]_i) - M_i(x_{ai} - \widehat{\eta}_i), \end{aligned}$$

and by definition of $x_b$

$$D^q x_{bi} = f_i(t, \widehat{\eta}_i, [\widehat{\eta}, \xi]_i) - M_i(x_{bi} - \widehat{\eta}_i).$$

Thus, by Theorem 2.1 we have that $x_b \leq x_a$, i.e. $A[\widehat{\eta}, \xi] \leq A[\eta, \xi]$ on $J$. Since $\eta, \widehat{\eta}$ were arbitrary, we have that $A$ is increasing in its first component. Similarly, we can show that $A$ is decreasing in its second component. Therefore A has a mixed monotone property, and with it we obtain the property that $A[\eta, \xi] \leq A[\xi, \eta]$ when $v_0 \leq \eta \leq \xi \leq w_0$ on $J$.

The sequences $\{v_n\}$ and $\{w_n\}$ we construct are unique solutions of the fractional systems

$$\begin{aligned} D^q v_{n+1_i} &= f_i(t, v_{ni}, [v_n, w_n]_i) - M_i(v_{n+1_i} - v_{ni}), \quad v^0_{n+1_i} = x^0_i, \\ D^q w_{n+1_i} &= f_i(t, w_{ni}, [w_n, v_n]_i) - M_i(w_{n+1_i} - w_{ni}), \quad w^0_{n+1_i} = x^0_i, \end{aligned}$$

where $v_0$ and $w_0$ are as defined in the hypothesis. That is, the sequences are defined as

$$v_{n+1} = A[v_n, w_n], \quad w_{n+1} = A[w_n, v_n].$$

With the transformation $A$ it is far more efficient to prove that the sequences are monotonic inductively, since if

$$v_0 \leq v_1 \leq v_2 \leq \cdots \leq v_{k-1} \leq v_k \leq w_k \leq w_{k-1} \leq \cdots \leq w_2 \leq w_1 \leq w_0$$

up to some $k$, then

$$A[v_{k-1}, w_{k-1}] \leq A[v_k, w_k] \leq A[w_k, v_k] \leq A[w_{k-1}, v_{k-1}],$$

implying $v_k \leq v_{k+1} \leq w_{k+1} \leq w_k$ on $J$, and giving us the monotonicity of the constructed sequences.

Now we will prove that the weighted sequences $\{t^p v_n\}$, $\{t^p w_n\}$ converge uniformly, to do so we will invoke the Arzela-Ascoli theorem. To begin, note that for all $i$ we have that

$$|t^p(v_i)| \leq |t^p(v_i - v_0)| + |t^p v_0| \leq |t^p(w_0 - v_0)| + |t^p v_0|,$$

giving us that $\{t^p v_n\}$ is uniformly bounded. Now we wish to show that the weighted sequence is uniformly continuous. For simplicity, for each $i$ and $n$, let

$$F_i(v_{n+1}) = f_i(t, v_{ni}, [v_n, w_n]_i) - M_i(v_{n+1_i} - v_{ni}),$$

then since each $v_n$ is $C_p$ continuous, $f$ is continuous over $J_0$, and since $\{t^p v_n\}$ is uniformly bounded, we can choose $\mu > 0$ such that $|t^p F_i(v_n)| \leq \mu$ for all $i$ and all $n$. Also, our preceding argument requires analysis of the function

$$\varphi(t) = t^p (t-s)^{-p},$$

for $0 \leq s \leq t \leq T$, specifically we note that $\varphi$ is decreasing in $t$, to show why consider

$$\tfrac{d}{dt}\varphi = pt^{p-1}(t-s)^{-p-1}(-s) \leq 0.$$

Now, let $\varepsilon > 0$, and let $t, \tau \in (0, T]$ such that, without loss of generality, $0 < t \leq \tau$ and $\tau - t < \varepsilon^{1/q}$. Further, suppose $\varepsilon$ is sufficiently small enough such that $1 \leq \frac{\tau}{t} < 2$. Then via Remark 2.2, utilizing that $\varphi(\tau) \leq \varphi(t)$, we have for each $i$ and $n$,

$$\begin{aligned}
&|\tau^p v_{ni}(\tau) - t^p v_{ni}(t)| \\
&= \Big|\frac{1}{\Gamma(q)}\int_0^\tau \varphi(\tau)F_i(v_n)ds - \frac{1}{\Gamma(q)}\int_0^t \varphi(t)F_i(v_n)ds\Big| \\
&\leq \frac{1}{\Gamma(q)}\int_t^\tau \varphi(\tau)|F_i(v_n)|ds + \frac{1}{\Gamma(q)}\int_0^t |\varphi(\tau)-\varphi(t)||F_i(v_n)|ds \\
&\leq \frac{\mu}{\Gamma(q)}\tau^p t^{q-1}\int_t^\tau (\tau-s)^{q-1}ds + \frac{\mu}{\Gamma(q)}\int_0^t (\varphi(t)-\varphi(\tau))s^{q-1}ds. \\
&= \frac{\mu}{\Gamma(q+1)}\Big(\frac{\tau}{t}\Big)^p(\tau-t)^q + \frac{\mu\Gamma(q)}{\Gamma(2q)}t^q - \frac{\mu\tau^p}{\Gamma(q)}\int_0^t (\tau-s)^{q-1}s^{q-1}ds. \qquad (7)
\end{aligned}$$

From here we will evaluate the third term from (7) individually, and for simplicity without the constant $\frac{\mu}{\Gamma(q)}$. To do so we will use the integral form of the beta funtion $B(q,q)$,

$$B(q,q) = \frac{\Gamma(q)\Gamma(q)}{\Gamma(2q)} = \int_0^1 (1-\alpha)^{q-1}\alpha^{q-1}d\alpha.$$

Then we will apply the transformation $s = t\alpha$ to obtain

$$\begin{aligned}
-\tau^p\int_0^t (\tau-s)^{q-1}s^{q-1}ds &= -\tau^q B(q,q) + \tau^q B(q,q) - \tau^q\int_0^{t/\tau}(1-\alpha)^{q-1}\alpha^{q-1}d\alpha \\
&= -\tau^q B(q,q) + \tau^q\int_{t/\tau}^1 (1-\alpha)^{q-1}\alpha^{q-1}d\alpha \\
&\leq -\tau^q B(q,q) + \tau^q (t/\tau)^{q-1}\int_{t/\tau}^1 (1-\alpha)^{q-1}d\alpha \\
&= -\tau^q B(q,q) + \frac{1}{q}\Big(\frac{\tau}{t}\Big)^p(\tau-t)^q.
\end{aligned}$$

Putting this result back into (7) we obtain

$$|\tau^p v_{ni}(\tau) - t^p v_{ni}(t)| < \frac{2^{p+1}\mu}{\Gamma(q+1)}(\tau-t)^q + \frac{\mu\Gamma(q)}{\Gamma(2q)}(t^q-\tau^q) < \frac{2^{p+1}\mu}{\Gamma(q+1)}\varepsilon.$$

Thus $t^p v_{ni}(t)$ is continuous at $t > 0$ since the case when $t \geq \tau$ will follow in a similar manner. For the case when $t = 0$, consider

$$|\tau^p v_{ni}(\tau) - x_i^0| \leq \frac{\mu\tau^p}{\Gamma(q)}\int_0^\tau (\tau-s)^{q-1}s^{q-1}ds = \frac{\mu\Gamma(q)}{\Gamma(2q)}\tau^q < \frac{\mu\Gamma(q)}{\Gamma(2q)}\varepsilon,$$

so $t^p v_{ni}(t)$ is continuous at $t = 0$. Further, since $\varepsilon$ did not depend on the arbitrary choices of $n$ or $i$, we have that the weighted sequence $\{t^p v_n\}$ is equicontinuous on $J_0$.

Now, since $\{t^p v_n\}$ is monotonic, uniformly bounded, and equicontinuous on $J_0$, by the Arzela-Ascoli theorem we have that $\{t^p v_n\}$ converges uniformly on $J_0$. Note we can prove the same result for $\{t^p w_n\}$, thus both weighted sequences converge uniformly. Now, suppose that $v, w \in C_p(J, R^N)$ such that $t^p v_n \to t^p v$ and $t^p w_n \to t^p w$ uniformly on $J_0$. We wish to show that $Tv_n \to Tv$ and $Tw_n \to Tw$ uniformly on $J_0$. To do so, let $\varepsilon > 0$ and choose $\mathcal{M}$ such that for $n \geq \mathcal{M}$, $|t^p(v_n - v)| < \frac{\varepsilon q}{\widehat{K} T^q}$, where $\widehat{K}$ is defined as in Remark 2.3. Then for all $t \in J_0$ and for all $n \geq \mathcal{M}$

$$|Tv_n - Tv| \leq \widehat{K} \int_0^t |v_n - v| ds < \frac{\varepsilon q}{T^q} \int_0^t s^{q-1} ds = \frac{\varepsilon t^q}{T^q} \leq \varepsilon.$$

Therefore $Tv_n \to Tv$ uniformly on $J_0$, similarly $Tw_n \to Tw$ uniformly on $J_0$.

Now, due to the fact that $f_i$ is continuous and bounded on $J_0$ and the nature of $C_p$ continuous functions, for each $i$ there exists a function $\mathcal{F}$ such that

$$\mathcal{F}_i(t, t^p x_i, [t^p x]_{r_i}, [t^p x]_{s_i}, [Tx]_{\rho_i}, [Tx]_{\sigma_i}) = f_i(t, x_i, [x]_{r_i}, [x]_{s_i}, [Tx]_{\rho_i}, [Tx]_{\sigma_i}).$$

So, due to all of the convergence properties we have that

$$t^p v_{n+1_i} = \tfrac{1}{\Gamma(q)} + t^p \mathcal{F}_i(t, t^p v_{ni}, [t^p v_n]_{r_i}, [t^p w_n]_{s_i}, [Tv_n]_{\rho_i}, [Tw_n]_{\sigma_i}) - M_i t^p(v_{n+1_i} - v_{ni})$$

converges uniformly to

$$t^p v_i = \tfrac{1}{\Gamma(q)} + t^p \mathcal{F}_i(t, t^p v_i, [t^p v]_{r_i}, [t^p w]_{s_i}, [Tv]_{\rho_i}, [Tw]_{\sigma_i})$$

on $J_0$, giving us that

$$v_i = \tfrac{1}{\Gamma(q)} t^{q-1} + f_i(t, v_i, [v]_{r_i}, [w]_{s_i}, [Tv]_{\rho_i}, [Tw]_{\sigma_i})$$

on $J$ and implying that $v$ is a coupled quasisolution of (5), and we have the similar result for $w$ as well.

Finally, we wish to prove that $v, w$ are minimal and maximal coupled quasisolutions of (5). To do so, let $x$ be any solution of (5) with $v_0 \leq x \leq w_0$, we know such a solution exists thanks to Theorem 3.1. Then note that

$$v_1 = A[v_0, w_0] \leq A[x, x] \leq A[w_0, v_0] = w_1,$$

giving us that $v_1 \leq x \leq w_1$, and continuing this process inductively we can show that $v_n \leq x \leq w_n$ for all $n$, which implies that $v \leq x \leq w$ on $J$. Therefore, $v, w$ are minimal and maximal mixed quasisolutions of (5), which completes the proof.

Note that in the case that (5) has a unique solution, e.g. $f$ is fully Lipschitz, then $v = x = w$ on $J$. Further, this acts as a generalization for the monotone method constructed with natural upper and lower solutions to the system (4) where $f(t, x, Tx)$ is quasimonotone increasing in $x$ and traditionally increasing in $Tx$. This follows directly from considering the previous theorem where $s_i = \sigma_i = 0$.

We can also construct the monotone method beginning with Type II coupled lower and upper quasisolutions. To do so requires a further assumption that $v_0 \leq w_1$ and $v_1 \leq w_0$, further we get intertwined montone sequences that still converge to minimal and maximal quasisolutions.

**Theorem 3.3** *Suppose $f$ satisfies the same properties as in Theorem 3.2. Let $v_0, w_0 \in C_p(J, R^N)$ be Type II lower and upper quasisolutions of (5) such that $v_0 \le w_0$. Let $\{v_n\}$ and $\{w_n\}$ be sequences defined by*

$$D^q v_{n+1_i} = f_i(t, w_{n_i}, [w_n, v_n]_i) - M_i(v_{n+1_i} - w_{n_i}), \quad v_{n+1_i}^0 = x_i^0,$$
$$D^q w_{n+1_i} = f_i(t, v_{n_i}, [v_n, w_n]_i) - M_i(w_{n+1_i} - v_{n_i}), \quad w_{n+1_i}^0 = x_i^0,$$

*for $n \ge 1$ and where $v_0, w_0$ are the given lower and upper solutions. If $v_0 \le w_1$ and $v_1 \le w_0$, then the sequences have the following intertwined monotonic property*

$$v_0 \le w_1 \le v_2 \le \dots v_{2n} \le w_{2n+1} \le v_{2n+1} \le w_{2n} \le \dots \le w_2 \le v_1 \le w_0,$$

*and the weighted sequences*

$$t^p v_{2n}, t^p w_{2n+1} \to t^p \alpha, \quad t^p v_{2n+1}, t^p w_{2n} \to t^p \beta$$

*uniformly on $J_0$, where $\alpha$ and $\beta$ are Type II coupled minimal and maximal quasisolutions of (5) on $J$ for solutions $v_0 \le x \le w_0$.*

The proof of this theorem follows in a similar manner as that of Theorem 3.2, even with the intertwined nature the proof only requires minor adjustments for incorporating the Type II sequences. This theorem is also a generalization for the monotone method constructed with natural upper and lower solutions to the system (4) where $f(t, x, Tx)$ is quasimonotone decreasing in $x$ and traditionally decreasing in $Tx$. As before, this follows directly from considering $s_i = \sigma_i = 0$.

In the next section we will construct a numerical example that will exemplify our results. In the example we will look at a system when $N = 2$ and $q = 1/2$.

## 4 Numerical Example

We finish this work by illustrating the result of Theorem 3.2 with an example. Consider the fractional system of the form (5) with $q = \frac{1}{2}$,

$$\begin{aligned} D^{1/2}x_1 &= \tfrac{1}{2} + \tfrac{5}{8}t + \tfrac{1}{32}\left(x_1^2 - \tfrac{1}{4}x_2\right) + \tfrac{1}{16}\int_0^t (1+s)x_1 ds, & x_{p_1}(0) &= 0, \\ D^{1/2}x_2 &= \tfrac{1}{6} + \tfrac{1}{5}t + \tfrac{1}{20}(x_1 - x_2) - \tfrac{1}{20}\int_0^t (1+s)x_2 ds, & x_{p_2}(0) &= 0, \end{aligned} \tag{8}$$

where $p = \frac{1}{2}$, and for simplicity we will consider the same transformation

$$Tx_i(t) = \int_0^t (1+s)x_i(s)ds$$

for $i = 1, 2$, and further for simplicity call

$$f_1(t, x_1, x_2, Tx_1, Tx_2) = \tfrac{1}{2} + \tfrac{5}{8}t + \tfrac{1}{32}x_1^2 + \tfrac{1}{16}Tx_1, \ \ f_2(t, x_1, x_2, Tx_1, Tx_2) = \tfrac{1}{6} + \tfrac{1}{5}t + \tfrac{1}{20}x_1,$$

$$g_1(t, x_1, x_2, Tx_1, Tx_2) - \tfrac{1}{128}x_2, \ \ g_2(t, x_1, x_2, Tx_1, Tx_2) = -\tfrac{1}{20}x_2 - \tfrac{1}{20}Tx_2.$$

If $J = (0, 1]$ and $J_0 = [0, 1]$, then $f_i(t, x, Tx) + g_i(t, x, Tx)$ together, where $i = 1, 2$, satisfy the mixed quasimonotonicity property. Now let $v_{01}(t) = \frac{\sqrt{t}}{2}$, $v_{02}(t) = 0$, $w_{01}(t) = 3$ and $w_{02}(t) = 3 - t$.

We will illustrate graphically in Figures 1–4 that $v_{0i}(t)$ and $w_{0i}(t)$ are Type I coupled lower and upper quasisolutions.

First note,

$$v_{0p_i}(0) = w_{0p_i}(0) = 0 \text{ for } i = 1, 2.$$

Since $D^{1/2}v_{01} = \frac{\sqrt{\pi}}{4}$, we have

$$D^{1/2}v_{01} = \tfrac{\sqrt{\pi}}{4} \leq \tfrac{1}{2} + \tfrac{5}{8}t + \tfrac{1}{32}\Big(v_{01}^2 - \tfrac{1}{4}w_{02}\Big) + \tfrac{1}{16}Tv_{01} = f_1(t, v_0, Tv_0) + g_1(t, w_0, Tw_0).$$

Similarly,

$$D^{1/2}w_{01} = \tfrac{3}{\sqrt{\pi t}} \geq \tfrac{1}{2} + \tfrac{5}{8}t + \tfrac{1}{32}\Big(w_{01}^2 - \tfrac{1}{4}v_{02}\Big) + \tfrac{1}{16}Tw_{01} = f_1(t, w_0, Tw_0) + g_1(t, v_0, Tv_0),$$

$$D^{1/2}v_{02} = 0 \leq \tfrac{1}{6} + \tfrac{1}{5}t + \tfrac{1}{20}\big(v_{01} - w_{02}\big) - \tfrac{1}{20}Tw_{02} = f_2(t, v_0, Tv_0) + g_2(t, w_0, Tw_0),$$

$$D^{1/2}w_{02} = \tfrac{3-2t}{\sqrt{\pi t}} \geq \tfrac{1}{6} + \tfrac{1}{5}t + \tfrac{1}{20}\big(w_{01} - v_{02}\big) - \tfrac{1}{20}Tv_{02} = f_2(t, w_0, Tw_{01}, Tw_0) + g_2(t, v_0, Tv_0).$$

We now show the graphs of these lower and upper quasisolutions in Figures 1–4.

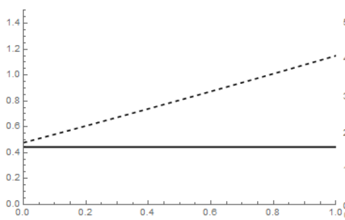

**Figure 1**: $D^q v_{01} \leq f_1 + g_1$.

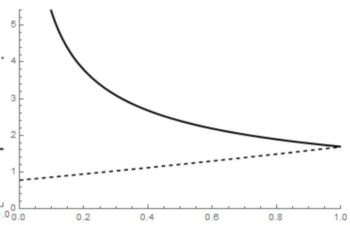

**Figure 2**: $D^q w_{01} \geq f_1 + g_1$.

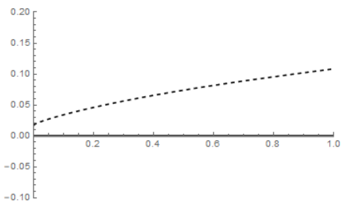

**Figure 3**: $D^q v_{02} \leq f_2 + g_2$.

After verifying that we have indeed Type I coupled lower and upper quasisolutions we computed four iterates of $\{t^{1/2}v_n\}$ and $\{t^{1/2}w_n\}$, for $i = 1, 2$, according to Theorem 3.2 for $t \in J_0 = [0, 1]$. The results are given in Figures 5 and 6 for $0 \leq n \leq 4$.

Finally we show a table of ten values of $\{t^{1/2}v_4\}$ and $\{t^{1/2}w_4\}$, for $i = 1, 2$, on the interval $[0, 1]$.

| $t$ | $t^{1/2}v_{41}$ | $t^{1/2}w_{41}$ | $t^{1/2}v_{42}$ | $t^{1/2}w_{42}$ | $t$ | $t^{1/2}v_{41}$ | $t^{1/2}w_{41}$ | $t^{1/2}v_{42}$ | $t^{1/2}w_{42}$ |
|---|---|---|---|---|---|---|---|---|---|
| 0.1 | 0.0612153 | 0.0612154 | 0.0208512 | 0.0208514 | 0.2 | 0.1322313 | 0.1322315 | 0.0452015 | 0.0452027 |
| 0.3 | 0.2132962 | 0.2132970 | 0.0729141 | 0.0729180 | 0.4 | 0.3046777 | 0.3046800 | 0.1039346 | 0.1039439 |
| 0.5 | 0.4066788 | 0.4066841 | 0.1382209 | 0.1382401 | 0.6 | 0.5196440 | 0.1757330 | 0.5196553 | 0.1757683 |
| 0.7 | 0.6439644 | 0.6439867 | 0.2164287 | 0.5702515 | 0.8 | 0.7800823 | 0.7801239 | 0.2602622 | 0.2603606 |
| 0.9 | 0.9284960 | 0.9285705 | 0.3071838 | 0.3073374 | 1.0 | 1.0897658 | 1.0898941 | 0.3571390 | 0.3573711 |

We used Mathematica to compute the iterates, the graphs and the tables.

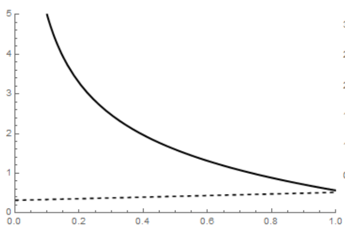

**Figure 4**: $D^q w_{02} \geq f_2 + g_2$.

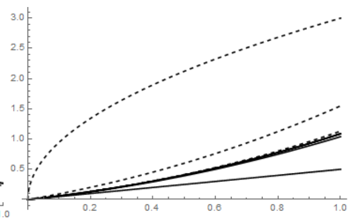

**Figure 5**: $t^p v_{n1} \leq t^p w_{n1}$.

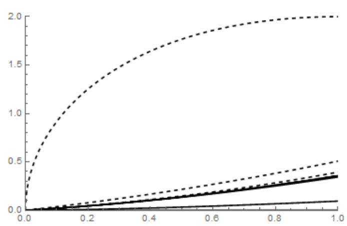

**Figure 6**: $t^p v_{n2} \leq t^p w_{n2}$.

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