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Local Existence and Uniqueness of Solution for Hilfer-Hadamard Fractional Differential Problem

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Abstract: This paper deals with the local existence and uniqueness results for the solution of fractional differential equations involving Hilfer-Hadamard fractional derivative. Using Picard's approximations and generalizing the restrictive conditions imposed on nonlinear function, the iterative scheme for uniformly approximating solution is constructed. An example is given to illustrate the main results.

Keywords: *Picard iterative technique; fractional differential equation; convergence.*

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1 Introduction

Fractional differential equations (FDEs) occur in control of dynamical systems, physical and biological sciences, see for details [14, 19, 23] and references therein. Nowadays, many people have given attention to the existence theory of nonlinear FDEs of various types [2–13, 15–18, 21, 22]. Recently, existence and uniqueness of weak solutions for some class of Hilfer-Hadamard and Hilfer fractional differential equations are obtained in [1]. Further, some attractivity and Ulam stability results are obtained [1] by applying the fixed point theory, also one can see [12, 20].

Kassim and Tatar [16] obtained the well-posedness of Cauchy-type problem

$$\begin{cases} {}_H\mathcal{D}^{\alpha,\beta}_{a^+}x(t) = f(t,x), \quad t > a > 0, \\ {}_H\mathcal{J}^{1-\gamma}_{a^+}x(a) = c, \qquad \gamma = \alpha + \beta(1-\alpha), \end{cases} \tag{1}$$

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where $c \in \mathbb{R}$ and ${}_H\mathcal{D}_{a^+}^{\alpha,\beta}$ is the Hilfer-Hadamard fractional derivative [15] of order $\alpha(0 < \alpha < 1)$ and type $\beta(0 \leq \beta \leq 1)$, in the weighted space of continuous functions $C_{1-\gamma}^{\alpha,\beta}[a,b]$ defined by

$$C_{1-\gamma,\mu}^{\alpha,\beta}[a,b] = \big\{x \in C_{1-\gamma,\log}[a,b] | {}_H\mathcal{D}_{a^+}^{\alpha,\beta} x \in C_{\mu,\log}[a,b]\big\}, \quad 0 \leq \mu < 1, \gamma = \alpha + \beta(1-\alpha), \tag{2}$$

where

$$C_{\gamma,log}[a,b] = \left\{g : (a,b] \to \mathbb{R} | \big(\log \frac{t}{a}\big)^{\gamma} g(t) \in C[a,b]\right\}, \quad 0 \leq \gamma < 1. \tag{3}$$

They obtained the equivalence of initial value problem (IVP) (1) and integral equation

$$x(t) = \frac{c}{\Gamma(\gamma)}\big(\log \frac{t}{a}\big)^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_a^t \big(\log \frac{t}{s}\big)^{\alpha-1} f(s, x(s))\frac{ds}{s}, \quad t > a,\, c \in \mathbb{R}. \tag{4}$$

Existence result for IVP (1) is proved in [16] using Banach fixed point theorem.

Motivated by these works, to avoid ambiguity of fixed point theory, we adopted the method of successive approximations. In this paper, we study the IVP for fractional differential equation involving Hilfer-Hadamard fractional derivative

$$\begin{cases} {}_H\mathcal{D}_1^{\alpha,\beta} x(t) = f(t,x), & 0 < \alpha < 1, 0 \leq \beta \leq 1, \\ \lim\limits_{t\to 1} \big(\log t\big)^{1-\gamma} x(t) = x_0, & \gamma = \alpha + \beta(1-\alpha). \end{cases} \tag{5}$$

In this paper we prove the existence and uniqueness results for IVP (5), using some well-known convergence criterion and Picard sequence functions [18, 24]. The computable iterative scheme as well as the uniform convergence criterion for solution are also developed.

The rest of the paper is organised as follows. The next section covers the useful prerequisites which include definitions and lemmas. The main results are proved in Section 3 with the supporting illustrative example.

2 Preliminaries

We need the following basic definitions and properties from fractional calculus [19].

Definition 2.1 [19] Let $(1,b), 1 < b \leq \infty$, be a finite or infinite interval of the half-axis $\mathbb{R}^+$ and let $\alpha > 0$. The left-sided Hadamard fractional integral ${}_H\mathcal{I}_1^{\alpha} f$ of order $\alpha > 0$ is defined by

$$({}_H\mathcal{I}_1^{\alpha} f)(t) = \frac{1}{\Gamma(\alpha)}\int_1^t (\log t)^{\alpha-1}\frac{f(s)ds}{s}, \quad 1 < t < b, \tag{6}$$

provided that the integral exists. When $\alpha = 0$, we set ${}_H\mathcal{I}_1^0 f = f$.

Definition 2.2 [17, 19] The left-sided Hadamard fractional derivative of order $\alpha(0 \leq \alpha < 1)$ on $(1,b)$ is defined by

$$({}_H\mathcal{D}_1^{\alpha} f)(t) = \delta({}_H\mathcal{I}_1^{1-\alpha} f)(t), \qquad 1 < t < b, \tag{7}$$

where $\delta = t(d/dt)$. In particular, when $\alpha = 0$ we have ${}_H\mathcal{D}_1^0 f = f$.

Definition 2.3 [16] The left-sided Hilfer-Hadamard fractional derivative of order $\alpha(0 < \alpha < 1)$ and type $\beta(0 \leq \beta \leq 1)$ with respect to t is defined by

$$({}_H\mathcal{D}_1^{\alpha,\beta} f)(t) = ({}_H\mathcal{J}_1^{\beta(1-\alpha)} {}_H\mathcal{D}_1^{\alpha+\beta(1-\alpha)} f)(t), \tag{8}$$

of function f for which the expression on the right-hand side exists, where ${}_H\mathcal{D}_1^{\alpha+\beta(1-\alpha)}$ is the Hadamard fractional derivative.

Lemma 2.1 [19] *If* $\alpha > 0, \beta > 0$ *and* $1 < b < \infty$, *then*

$$\big({}_H\mathcal{J}_1^{\alpha}\big(\log s\big)^{\beta-1}\big)(t) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}\big(\log t\big)^{\beta+\alpha-1}, \tag{9}$$

$$\big({}_H\mathcal{D}_1^{\alpha}\big(\log s\big)^{\beta-1}\big)(t) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\big(\log t\big)^{\beta-\alpha-1}. \tag{10}$$

The following lemma plays a vital role in the proof of main results.

Lemma 2.2 [23] *Suppose that* $x > 0$. *Then* $\Gamma(x) = \lim_{m\to+\infty} \frac{m^x m!}{x(x+1)(x+2)\cdots(x+m)}$.

We denote $D = [1, 1+h], D_h = (1, 1+h], I = (1, 1+l]$ and $J = [1, 1+l]$, for $h > 0$. Here we choose

$$l = \min\left\{h, \big(\tfrac{b}{M}\tfrac{\Gamma(\alpha+k+1)}{\Gamma(k+1)}\big)^{\frac{1}{\mu+k}}\right\}, \ \mu = 1-\beta(1-\alpha).$$

Further $E = \{x : |x(\log t)^{1-\gamma} - x_0| \leq b\}$ for $b > 0$ and $t \in D_h$. A function $x(t)$ is said to be a solution of IVP (5) if there exists $l > 0$ such that $x \in C^0(I)$ satisfies the differential equation ${}_H\mathcal{D}_1^{\alpha,\beta} x(t) = f(t, x)$ almost everywhere on I along with the condition

$$\lim_{t\to 1} (\log t)^{1-\gamma} x(t) = x_0.$$

To prove our main results, we assume the following hypotheses:

(H1) $(t, x) \to f(t, (\log t)^{\gamma-1}x(t))$ is defined on $D_h \times E$ and satisfies:

(i) $x \to f(t, (\log t)^{\gamma-1}x(t))$ is continuous on E for all $t \in D_h$,
$t \to f(t, (\log t)^{\gamma-1}x(t))$ is measurable on D_h for all $x \in E$;

(ii) there exist $k > (\beta(1-\alpha)-1)$ and $M \geq 0$ such that the relation $|f(t, (\log t)^{\gamma-1}x(t))| \leq M(\log t)^k$ holds for all $t \in D_h$ and $x \in E$.

(H2) There exist $A > 0$ and $x_1, x_2 \in E$ such that

$$|f(t, (\log t)^{\gamma-1}x_1(t)) - f(t, (\log t)^{\gamma-1}x_2(t))| \leq A(\log t)^k |x_1 - x_2|, \quad \text{for all } t \in I.$$

3 Main Results

In this section, we state and prove the existence and uniqueness results for IVP (5) for Hilfer-Hadamard FDEs. We present the iterative scheme for approximating such a unique solution.

Lemma 3.1 *Suppose that (H1) holds. Then $x : J \to \mathbb{R}$ is the solution of IVP* (5) *if and only if $x : I \to \mathbb{R}$ is the solution of the Volterra integral equation of second kind:*

$$x(t) = x_0\big(\log t\big)^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_1^t \big(\log\frac{t}{s}\big)^{\alpha-1} f(s, x(s))\frac{ds}{s}, \quad t > 1. \tag{11}$$

Proof. First we suppose that $x : I \to \mathbb{R}$ is the solution of IVP (5). Then $|\big(\log t\big)^{1-\gamma}x(t) - x_0| \le b$ for all $t \in I$. From (H1), there exist a $k > (\beta(1-\alpha) - 1)$ and $M \ge 0$ such that

$$|f(t, x(t))| = |f(t, (\log t)^{\gamma-1}(\log t)^{1-\gamma}x(t))| \le M(\log t)^k, \qquad \text{for all} \quad t \in I.$$

We have

$$\left|\frac{1}{\Gamma(\alpha)}\int_1^t \big(\log\frac{t}{s}\big)^{\alpha-1} f(s, x(s))\frac{ds}{s}\right| \le \frac{1}{\Gamma(\alpha)}\int_1^t \big(\log\frac{t}{s}\big)^{\alpha-1} M(\log s)^k\frac{ds}{s}$$
$$= M(\log t)^{\alpha+k}\frac{\Gamma(k+1)}{\Gamma(\alpha+k+1)}.$$

Clearly,

$$\lim_{t\to 1}\big(\log t\big)^{1-\gamma}\frac{1}{\Gamma(\alpha)}\int_1^t \big(\log\frac{t}{s}\big)^{\alpha-1} f(s, x(s))\frac{ds}{s} = 0.$$

It follows that

$$x(t) = x_0\big(\log t\big)^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_1^t \big(\log\frac{t}{s}\big)^{\alpha-1} f(s, x(s))\frac{ds}{s}, \quad t \in I.$$

Since $k > (\beta(1-\alpha) - 1)$, we see that $x \in C^0(I)$ is a solution of integral equation (11).

Conversely, it is easy to see the fact that $x : I \to \mathbb{R}$ is the solution of integral equation (11) implies that x is the solution of IVP (5) defined on J. This completes the proof.

To prove our main results, we choose a Picard function sequence as follows:

$$\begin{aligned}\phi_0(t) &= x_0(\log t)^{\gamma-1}, \qquad t \in I,\\ \phi_n(t) &= \phi_0(t) + \frac{1}{\Gamma(\alpha)}\int_1^t \big(\log\frac{t}{s}\big)^{\alpha-1} f(s, \phi_{n-1}(s))\frac{ds}{s}, \quad t \in I, \quad n = 1, 2, \cdots.\end{aligned} \tag{12}$$

Lemma 3.2 *Suppose that (H1) holds. Then ϕ_n is continuous on I and satisfies $|(\log t)^{1-\gamma}\phi_n(t) - x_0| \le b$.*

Proof. From (H1), clearly $|f(t, (\log t)^{\gamma-1}x)| \le M(\log t)^k$ for all $t \in D_h$ and $|x(\log t)^{1-\gamma} - x_0| \le b$. For $n = 1$, we have

$$\phi_1(t) = x_0(\log t)^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_1^t \big(\log\frac{t}{s}\big)^{\alpha-1} f(s, \phi_0(s))\frac{ds}{s}. \tag{13}$$

Then

$$\left|\frac{1}{\Gamma(\alpha)}\int_1^t \big(\log\frac{t}{s}\big)^{\alpha-1} f(s, \phi_0(s))\frac{ds}{s}\right| \le \frac{1}{\Gamma(\alpha)}\int_1^t \big(\log\frac{t}{s}\big)^{\alpha-1} M\big(\log s\big)^k\frac{ds}{s}$$
$$= M\big(\log t\big)^{\alpha+k}\frac{\Gamma(k+1)}{\Gamma(\alpha+k+1)}.$$

This implies $\phi_1 \in C^0(I)$ and from equation (13), we get

$$
\begin{aligned}
|(\log t)^{1-\gamma}\phi_1(t) - x_0| &\le (\log t)^{1-\gamma} M (\log t)^{\alpha+k} \frac{\Gamma(k+1)}{\Gamma(\alpha+k+1)} \\
&\le M l^{\alpha+k+1-\gamma} \frac{\Gamma(k+1)}{\Gamma(\alpha+k+1)}. \qquad (14)
\end{aligned}
$$

Now by the induction hypothesis for $n = m$, suppose that $\phi_m \in C^0(J)$ and for all $t \in J$, $|(\log t)^{1-\gamma}\phi_m(t) - x_0| \le b$. We have

$$
\phi_{m+1}(t) = x_0(\log t)^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_1^t \big(\log \frac{t}{s}\big)^{\alpha-1} f(s, \phi_m(s)) \frac{ds}{s}. \qquad (15)
$$

From the above discussion, we obtain $\phi_{m+1}(t) \in C^0(I)$ and from equation (15), we have

$$
\begin{aligned}
|(\log t)^{1-\gamma}\phi_{m+1}(t) - x_0| &\le (\log t)^{1-\gamma} \frac{1}{\Gamma(\alpha)} \int_1^t \big(\log \frac{t}{s}\big)^{\alpha-1} M (\log s)^k \frac{ds}{s} \\
&= M (\log t)^{\alpha+k+1-\gamma} \frac{\Gamma(k+1)}{\Gamma(\alpha+k+1)} \\
&\le M l^{\alpha+k+1-\gamma} \frac{\Gamma(k+1)}{\Gamma(\alpha+k+1)} \le b.
\end{aligned}
$$

Thus, the result is true for $n = m+1$. By the principle of mathematical induction, the result is true for all n. The proof is complete.

Theorem 3.1 *Suppose that (H1) and (H2) hold. Consider the Picard function ϕ_n given in* (12). *Then the sequence $\{(\log t)^{1-\gamma}\phi_n(t)\}$ is uniformly convergent on J.*

Proof. Consider the series

$$
(\log t)^{1-\gamma}\phi_0(t) + (\log t)^{1-\gamma}[\phi_1(t) - \phi_0(t)] + \cdots + (\log t)^{1-\gamma}[\phi_n(t) - \phi_{n-1}(t)] + \cdots, \quad t \in J.
$$

By relation (14) driven in the proof of Lemma 3.2 above, we get

$$
(\log t)^{1-\gamma}|\phi_1(t) - \phi_0(t)| \le M (\log t)^{\alpha+k+1-\gamma} \frac{\Gamma(k+1)}{\Gamma(\alpha+k+1)}, \qquad t \in J.
$$

From Lemma 3.2, we have

$$
\begin{aligned}
(\log t)^{1-\gamma}|\phi_2(t) - \phi_1(t)| &\le (\log t)^{1-\gamma} \frac{1}{\Gamma(\alpha)} \int_1^t \big(\log \frac{t}{s}\big)^{\alpha-1} |f(s, \phi_1(s)) - f(s, \phi_0(s))| \frac{ds}{s} \\
&= (\log t)^{1-\gamma} \frac{1}{\Gamma(\alpha)} \int_1^t \big(\log \frac{t}{s}\big)^{\alpha-1} \big| f\big(s, (\log s)^{\gamma-1}(\log s)^{1-\gamma}\phi_1(s)\big) - \\
&\qquad f\big(s, (\log s)^{\gamma-1}(\log s)^{1-\gamma}\phi_0(s)\big)\big| \frac{ds}{s} \\
&\le (\log t)^{1-\gamma} \frac{1}{\Gamma(\alpha)} \int_1^t \big(\log \frac{t}{s}\big)^{\alpha-1} A(\log s)^k \big|(\log s)^{1-\gamma}\phi_1(s) - \\
&\qquad (\log s)^{1-\gamma}\phi_0(s)\big| \frac{ds}{s} \\
&\le (\log t)^{1-\gamma} \frac{1}{\Gamma(\alpha)} \int_1^t \big(\log \frac{t}{s}\big)^{\alpha-1} A(\log s)^k \big[(\log s)^{1-\gamma}|\phi_1(s) - \phi_0(s)|\big] \frac{ds}{s} \\
&\le (\log t)^{1-\gamma} \frac{1}{\Gamma(\alpha)} \int_1^t \big(\log \frac{t}{s}\big)^{\alpha-1} A(\log s)^k \big[M(\log s)^{\alpha+k+1-\gamma} \frac{\Gamma(k+1)}{\Gamma(\alpha+k+1)}\big] \frac{ds}{s}.
\end{aligned}
$$

Thus

$$(\log t)^{1-\gamma}|\phi_2(t)-\phi_1(t)| \le AM\frac{\Gamma(k+1)}{\Gamma(\alpha+k+1)}\frac{\Gamma(\alpha+2k+2-\gamma)}{\Gamma(2\alpha+2k+2-\gamma)}(\log t)^{2(\alpha+k+1-\gamma)}.$$

Now suppose that for $n=m$

$$(\log t)^{1-\gamma}|\phi_{m+1}(t)-\phi_m(t)| \le$$
$$A^m M(\log t)^{(m+1)(\alpha+k+1-\gamma)}\prod_{i=0}^{m}\frac{\Gamma((i+1)k+i(\alpha+1-\gamma)+1)}{\Gamma((i+1)(\alpha+k)+i(1-\gamma)+1)}.$$

We have

$$\begin{aligned}
(\log t)^{1-\gamma}|\phi_{m+2}(t)-\phi_{m+1}(t)| &\le \frac{(\log t)^{1-\gamma}}{\Gamma(\alpha)}\int_1^t \big(\log\frac{t}{s}\big)^{\alpha-1}|f(s,\phi_{m+1}(s))-f(s,\phi_m(s))|\frac{ds}{s}\\
&= (\log t)^{1-\gamma}\frac{1}{\Gamma(\alpha)}\int_1^t\big(\log\frac{t}{s}\big)^{\alpha-1}\big|f\big(s,(\log s)^{\gamma-1}(\log s)^{1-\gamma}\phi_{m+1}(s)\big)-\\
&\qquad f\big(s,(\log s)^{\gamma-1}(\log s)^{1-\gamma}\phi_m(s)\big)\big|\frac{ds}{s}\\
&\le (\log t)^{1-\gamma}\frac{1}{(\Gamma(\alpha))}\int_1^t\big(\log\frac{t}{s}\big)^{\alpha-1}A(\log s)^k\big[(\log s)^{1-\gamma}|\phi_{m+1}(s)-\phi_m(s)|\big]\frac{ds}{s}\\
&\le (\log t)^{1-\gamma}\frac{1}{\Gamma(\alpha)}\int_1^t\big(\log\frac{t}{s}\big)^{\alpha-1}A(\log s)^k\Big[A^mM(\log s)^{(m+1)(\alpha+k+1-\gamma)}\\
&\qquad\times\prod_{i=0}^{m}\frac{\Gamma((i+1)k+i(\alpha+1-\gamma)+1)}{\Gamma((i+1)(\alpha+k)+i(1-\gamma)-1)}\Big]\frac{ds}{s}\\
&= A^{m+1}M(\log t)^{(m+2)(\alpha+k+1-\gamma)}\prod_{i=0}^{m+1}\frac{\Gamma((i+1)k+i(\alpha+1-\gamma)+1)}{\Gamma((i+1)(\alpha+k)+i(1-\gamma)+1)}.
\end{aligned}$$

Thus

$$(\log t)^{1-\gamma}|\phi_{m+2}(t)-\phi_{m+1}(t)| \le$$
$$A^{m+1}Ml^{(m+2)(\alpha+k+1-\gamma)}\prod_{i=0}^{m+1}\frac{\Gamma((i+1)k+i(\alpha+1-\gamma)+1)}{\Gamma((i+1)(\alpha+k)+i(1-\gamma)+1)}.$$

The result is true for $n=m+1$. By the principle of mathematical induction the result is true for all n.

Consider

$$\sum_{n=1}^{\infty}u_n=\sum_{n=1}^{\infty}MA^{n+1}l^{(n+2)(\alpha+k+1-\gamma)}\prod_{i=0}^{n+1}\frac{\Gamma((i+1)k+i(\alpha+1-\gamma)+1)}{\Gamma((i+1)(\alpha+k)+i(1-\gamma)+1)}.$$

We have

$$\begin{aligned}
\frac{u_{n+1}}{u_n} &= \frac{MA^{n+2}l^{(n+3)(\alpha+k+1-\gamma)}\prod_{i=0}^{n+2}\frac{\Gamma((i+1)k+i(\alpha+1-\gamma)+1)}{\Gamma((i+1)(\alpha+k)+i(1-\gamma)+1)}}{MA^{n+1}l^{(n+2)(\alpha+k+1-\gamma)}\prod_{i=0}^{n+1}\frac{\Gamma((i+1)k+i(\alpha+1-\gamma)+1)}{\Gamma((i+1)(\alpha+k)+i(1-\gamma)+1)}}\\
&= Al^{\alpha+k+1-\gamma}\frac{\Gamma((n+3)k+(n+2)(\alpha+1-\gamma)+1)}{\Gamma((n+3)(k+\alpha)+(n+2)(1-\gamma)+1)}.
\end{aligned}$$

Using Lemma 2.2, we have

$$\frac{u_{n+1}}{u_n} = Al^{\alpha+k+1-\gamma} \frac{\lim_{m\to\infty} \frac{m^{(n+3)k+(n+2)(\alpha+1-\gamma)+1} m!}{((n+3)k+(n+2)(\alpha+1-\gamma)+1)\cdots((n+3)k+(n+2)(\alpha+1-\gamma)+m+1)}}{\lim_{m\to\infty} \frac{m^{(n+3)(k+\alpha)+(n+2)(1-\gamma)+1} m!}{((n+3)(k+\alpha)+(n+2)(1-\gamma)+1)\cdots((n+3)(k+\alpha)+(n+2)(1-\gamma)+m+1)}}$$

$$= Al^{\alpha+k+1-\gamma}[\lim_{m\to\infty} m^{-\alpha} \tfrac{((n+3)(k+\alpha)+(n+2)(1-\gamma)+1)\cdots((n+3)(k+\alpha)+(n+2)(1-\gamma)+m+1)}{((n+3)k+(n+2)(\alpha+1-\gamma)+1)\cdots((n+3)k+(n+2)(\alpha+1-\gamma)+m+1)}].$$

It is easy to see that

$$\frac{((n+3)(k+\alpha)+(n+2)(1-\gamma)+1)\cdots((n+3)(k+\alpha)+(n+2)(1-\gamma)+m+1)}{((n+3)k+(n+2)(\alpha+1-\gamma)+1)\cdots((n+3)k+(n+2)(\alpha+1-\gamma)+m+1)}$$

is bounded for all m, n. Thus $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 0$ implies $\sum_{n=1}^{\infty} u_n$ is convergent. Hence

$$(\log t)^{1-\gamma}\phi_0(t) + (\log t)^{1-\gamma}[\phi_1(t) - \phi_0(t)] + \cdots + (\log t)^{1-\gamma}[\phi_n(t) - \phi_{n-1}(t)] + \cdots$$

is uniformly convergent for $t \in J$. Hence $\{(\log t)^{1-\gamma}\phi_n(t)\}$ is uniformly convergent on J.

Theorem 3.2 *Suppose that (H1) and (H2) hold. Then the solution*

$$\phi(t) = (\log t)^{\gamma-1} \lim_{n\to\infty} (\log t)^{1-\gamma}\phi_n(t)$$

is a unique continuous solution of the integral equation (11) *defined on J.*

Proof. Since $\phi(t) = (\log t)^{\gamma-1} \lim_{n\to\infty} (\log t)^{1-\gamma}\phi_n(t)$ on J, and by Lemma 3.2, we have $(\log t)^{1-\gamma}|\phi(t) - x_0| \le b$. Then

$$|f(t, \phi_n(t)) - f(t, \phi(t))| \le A(\log t)^k |\phi_n(t) - \phi(t)|, \quad t \in I.$$

Clearly, $(\log t)^{-k}|f(t, \phi_n(t)) - f(t, \phi(t))| \le A|\phi_n(t) - \phi(t)| \to 0$ uniformly as $n \to \infty$ on I. Therefore

$$\begin{aligned}
(\log t)^{1-\gamma}\phi(t) &= \lim_{n\to\infty} \phi_n(t) \\
&= x_0 + (\log t)^{1-\gamma} \frac{1}{\Gamma(\alpha)} \int_1^t \big(\log \frac{t}{s}\big)^{\alpha-1} (\log s)^k \lim_{n\to\infty} \big((\log s)^{-k} f(s, \phi_{n-1}(s))\big) \frac{ds}{s} \\
&= x_0 + (\log t)^{1-\gamma} \frac{1}{\Gamma(\alpha)} \int_1^t \big(\log \frac{t}{s}\big)^{\alpha-1} f(s, \phi(s)) \frac{ds}{s}.
\end{aligned}$$

Then $\phi(t)$ is a continuous solution of integral equation (11) defined on J.

Now we prove uniqueness of solution $\phi(t)$. Suppose that $\psi(t)$ is a solution of integral equation (11). Then $(\log t)^{1-\gamma}|\psi(t)| \le b$ for all $t \in I$ and

$$\psi(t) = x_0(\log t)^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_1^t \big(\log \frac{t}{s}\big)^{\alpha-1} f(s, \phi(s)) \frac{ds}{s}, \quad t \in I.$$

We prove $\phi(t) \equiv \psi(t)$ on I. From (H1), there exist a $k > (\beta(1-\alpha) - 1)$ and $M \ge 0$ such that

$$|f(t, \psi(t))| = \big|f\big(t, (\log t)^{\gamma-1}(\log t)^{1-\gamma}\psi(t)\big)\big| \le M(\log t)^k, \qquad \text{for all } t \in I.$$

Therefore

$$\begin{aligned}(\log t)^{1-\gamma}|\phi_0(t)-\psi(t)| =&(\log t)^{1-\gamma}\left|\frac{1}{\Gamma(\alpha)}\int_1^t \big(\log\frac{t}{s}\big)^{\alpha-1} f(s,\psi(s))\frac{ds}{s}\right|\\ &\le (\log t)^{1-\gamma}\frac{1}{\Gamma(\alpha)}\int_1^t \big(\log\frac{t}{s}\big)^{\alpha-1} M(\log s)^k\frac{ds}{s}\\ &= M(\log t)^{\alpha+k+1-\gamma}\frac{\Gamma(k+1)}{\Gamma(\alpha+k+1)}.\end{aligned}$$

Furthermore

$$\begin{aligned}(\log t)^{1-\gamma}|\phi_1(t)-\psi(t)| =&(\log t)^{1-\gamma}\left|\frac{1}{\Gamma(\alpha)}\int_1^t \big(\log\frac{t}{s}\big)^{\alpha-1} [f(s,\phi_0(s))-f(s,\psi(s))]\frac{ds}{s}\right|\\ &\le AM\frac{\Gamma(k+1)}{\Gamma(\alpha+k+1)}\frac{\Gamma(\alpha+2k+2-\gamma)}{\Gamma(2\alpha+2k+2-\gamma)}(\log t)^{2(\alpha+k+1-\gamma)}.\end{aligned}$$

By the induction hypothesis, we suppose that

$$(\log t)^{1-\gamma}|\phi_n(t)-\psi(t)| \le A^n M(\log t)^{(n+1)(\alpha+k+1-\gamma)}\prod_{i=0}^{n}\frac{\Gamma((i+1)k+i(\alpha+1-\gamma)+1)}{\Gamma((i+1)(\alpha+k)+i(1-\gamma)+1)}.$$

Then

$$\begin{aligned}(\log t)^{1-\gamma}|\phi_{n+1}(t)-\psi(t)| \le& (\log t)^{1-\gamma}\left|\frac{1}{\Gamma(\alpha)}\int_1^t \big(\log\frac{t}{s}\big)^{\alpha-1} [f(s,\phi_n(s))-f(s,\psi(s))]\frac{ds}{s}\right|\\ \le& A^{n+1}M(\log t)^{(n+2)(\alpha+k+1-\gamma)}\prod_{i=0}^{n+1}\frac{\Gamma((i+1)k+i(\alpha+1-\gamma)+1)}{\Gamma((i+1)(\alpha+k)+i(1-\gamma)+1)}\\ \le& A^{n+1}Ml^{(n+2)(\alpha+k+1-\gamma)}\prod_{i=0}^{n+1}\frac{\Gamma((i+1)k+i(\alpha+1-\gamma)+1)}{\Gamma((i+1)(\alpha+k)+i(1-\gamma)+1)}.\end{aligned}$$

Using the same arguments as in Theorem 3.1, we obtain the series

$$\sum_{n=1}^{\infty}A^{n+1}Ml^{(n+2)(\alpha+k+1-\gamma)}\prod_{i=0}^{n+1}\frac{\Gamma((i+1)k+i(\alpha+1-\gamma)+1)}{\Gamma((i+1)(\alpha+k)+i(1-\gamma)+1)},$$

which is convergent. Therefore

$$A^{n+1}Ml^{(n+2)(\alpha+k+1-\gamma)}\prod_{i=0}^{n+1}\frac{\Gamma((i+1)k+i(\alpha+1-\gamma)+1)}{\Gamma((i+1)(\alpha+k)+i(1-\gamma)+1)} \to 0 \quad \text{as} \quad n\to\infty.$$

Also we observe that $\lim_{n\to\infty}(\log t)^{1-\gamma}\phi_n(t) = (\log t)^{1-\gamma}\psi(t)$ uniformly on J. Thus $\phi(t)\equiv\psi(t)$ on I. The proof is complete.

Theorem 3.3 *Suppose that (H1) and (H2) hold. Then the IVP* (5) *has a unique continuous solution* $\phi(t)=(\log t)^{\gamma-1}\lim_{n\to\infty}(\log t)^{1-\gamma}\phi_n(t)$ *on* I.

Proof. From Lemma 3.1 and Theorem 3.1, we can easily obtain that the solution

$$\phi(t)=(\log t)^{\gamma-1}\lim_{n\to\infty}(\log t)^{1-\gamma}\phi_n(t)$$

is a unique continuous solution of IVP (5) defined on I. The proof is complete.

Example 3.1 We consider the Hilfer-Hadamard fractional differential problem

$$\begin{cases} {}_H\mathcal{D}_1^{\frac{1}{2},\frac{1}{2}} x(t) = f(t,x), & \alpha = \frac{1}{2}, \beta = \frac{1}{2}, \\ \lim\limits_{t\to 1} \big(\log t\big)^{\frac{1}{4}} x(t) = x_0, & \gamma = \dfrac{3}{4}, \end{cases} \tag{16}$$

where

$$\begin{cases} f(t,x(t)) = \frac{(\log t)^{-\frac{1}{4}} \sin(\log t)}{8(1+\sqrt{(\log t)})(1+|\sin(\log t)|)}, & \text{for} \quad t \in (1,e], \ x \in \mathbb{R}, \\ f(1,x(1)) = 0, & \text{for} \quad x \in \mathbb{R}. \end{cases}$$

It is easy to see that f is singular at $t = 1$, and is a continuous function for $t \in (1, e]$. We choose $\mu = \frac{3}{4}$, $b = 4$, $k = -\frac{1}{4} > -\frac{3}{4}$. Thus $l = \min\big\{1.7182, \big(\frac{4}{M}\frac{\Gamma(\frac{5}{4})}{\Gamma(\frac{3}{4})}\big)^2\big\}$, where

$$M = \max_{t\in[1,e]} \frac{\sin(\log t)}{8(1+\sqrt{\log t})(1+|\sin(\log t)|)} \approx 32$$

with

$$\phi_0(t) = x_0(\log t)^{-\frac{1}{4}}, \qquad t \in (1,e],$$
$$\phi_n(t) = \phi_0(t) + \frac{1}{\Gamma(\frac{1}{2})} \int_1^t \big(\log\frac{t}{s}\big)^{-\frac{1}{2}} f(s, \phi_{n-1}(s)) \frac{ds}{s}, \quad n = 1, 2, \cdots.$$

Clearly, all the conditions of Theorem 3.3 hold. Therefore IVP (16) has the unique continuous solution

$$\phi(t) = (\log t)^{-\frac{1}{4}} \lim_{n\to\infty} (\log t)^{\frac{1}{4}} \phi_n(t) \quad \text{on } [1,e].$$

Remark 3.1 The initial value considered in IVP (5) is more suitable than that considered in IVP (1) and nonlinear function f may be singular at $t = 1$.

Remark 3.2 In hypothesis (H1), if $(\log t)^{-k} f(t, (\log t)^{\gamma-1}x(t))$ is continuous on $D \times E$, one may choose $M = \max_{t\in J}(\log t)^{-k} f(t, (\log t)^{\gamma-1}x(t))$ continuous on $D_h \times E$ for all $x \in E$.

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