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On Multi-Switching Synchronization of Non-Identical Chaotic Systems via Active Backstepping Technique

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Abstract: An active backstepping scheme is proposed to attain three different types of synchronization between the chaotic Cai system and the Chen system. Complete synchronization, anti-synchronization and hybrid synchronization are accomplished by using the active backstepping method between different switches of the Cai and Chen systems, where the Cai system is considered as a master system and the Chen system is considered as a slave system. The goal is to design appropriate controllers by using the Lyapunov stability criteria and active backstepping method so that asymptotically stable synchronized state for different switches of the master and slave systems can be obtained. The results obtained by theoretical and graphical analysis are in agreement.

Keywords: *active backstepping method; multi-switching synchronization; chaotic systems; Lyapunov stability theory.*

Mathematics Subject Classification (2010): 34D06, 34H10, 93C10.

1 Introduction

In the area of applied sciences "chaos" is an important field as one of its beautiful features is its applications in several areas such as ecology, secure communication, medicine, biology etc. So many integer order chaotic and hyperchaotic systems have been obtained after the invention of the classical "Lorenz system" in 1963, and so many chaotic and hyperchaotic systems have also been developed in the field of fractional calculus. In the field of chaos, synchronization has been a fascinating branch for the last three decades and researchers have shown their interest to this branch.

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Since 1990, when the important phenomenon of synchronization was discovered by Pecora and Carroll [10], the field of synchronization has been growing day by day. Numerous new researches have been done theoretically and experimentally in the field of synchronization. Several researches have been done to extend the phenomenon of synchronization from complete synchronization [10] to a new range of synchronizations [2, 9, 17] and to develop the new techniques [6, 15, 16] to achieve synchronization. The active backstepping technique has been applied widely to achieve synchronization in different cases. In the last few years, outstanding work has been done on synchronization via active backstepping such as complete synchronization between identical systems [1], combination synchronization [11], reduced order synchronization [8], multi-switching synchronization for three chaotic systems [13] etc. The active backstepping method is found very effective for the cases given above. Some of these works have been done on multi-switching synchronization. Since 2008, when a new type of synchronization was achieved for two identical chaotic systems by Ucar [12], multi-switching synchronization has been a hot topic among researchers. Later, multi-switching synchronization between the Lorenz system and the Chen system with fully unknown parameters [14] has also been achieved. Inspite of all the work that has been done, there is a large scope of work in the field of multi-switching synchronization.

In this paper, for different switches of the chaotic Cai system and the Chen system three types of synchronizations are achieved by the active backstepping method. It is clear from numerical simulations that the active backstepping method is very fast, by which synchronization can be achieved very quickly. The proposed scheme has significant applications in the field of secure communications as the synchronization attained by any arbitrary pair increases the grade of security. Secure communication [5] is a field where synchronization is being used very widely. It was found by some researchers that because of arbitrary multiplying factor projective synchronization is an important tool to make communication more reliable [7]. Multi-switching synchronization is defined in such a manner that any pair of state variables may achieve synchronization, which increases the level of security. The advantage of the presented scheme is that by choosing different values of scaling factors different synchronizations can be achieved by a single approach.

This manuscript has been arranged in the following manner. Problem formulation is given in Section 2. In Section 3 dynamics of the Cai system and the Chen system is given. Section 4 contains the scheme for multi-switching synchronization achieved by the active backstepping method and Section 5 contains simulation results for three types of synchronization between the Cai system and the Chen system. In Section 6, main features of this work are highlighted.

2 Problem Formulation

Suppose an n-dimensional system is considered as the master system

$$\dot{v}_1 = h_{11}(v_1, v_2, \dots, v_n), \;\; \dot{v}_2 = h_{21}(v_1, v_2, \dots, v_n), \dots, \dot{v}_n = h_{n1}(v_1, v_2, \dots, v_n), \tag{1}$$

and the n-dimensional slave system is

$$\begin{aligned} &\dot{w}_1 = h_{12}(w_1, w_2, \dots, w_n) + u_1, \;\; \dot{w}_2 = h_{22}(w_1, w_2, \\ &\dots, w_n) + u_2, \dots, \dot{w}_n = h_{n2}(w_1, w_2, \dots, w_n) + u_n, \end{aligned} \tag{2}$$

where $u_1, u_2, \ldots, u_n \in R^n \to R$ are the controllers and $h_{i1}, h_{i2} \in R^n \to R$ for $i = 1, 2, \ldots, n$ are continuous functions. Suppose the errors are defined as

$$\dot{e}_1 = p_1 w_1 + q_1 v_1, \quad \dot{e}_2 = p_2 w_2 + q_2 v_2, \ldots, \dot{e}_n = p_n w_n + q_n v_n, \tag{3}$$

where, $p_i, q_i, i = 1, 2, ..., n$ are arbitrary scaling factors. By using equations (1) and (2) the error dynamical system can be expressed as

$$\dot{e}_1 = g_1 + f_1 + p_1 u_1, \quad \dot{e}_2 = g_2 + f_2 + p_2 u_2, \ldots, \dot{e}_n = g_n + f_n + p_n u_n, \tag{4}$$

where $e = (e_1, e_2, ...e_n)'$ is the error vector, $g_1, g_2, \ldots, g_n$ are the functions which contain only error components and $f_1, f_2, \ldots, f_n$ are the nonlinear functions which contain the terms of master and slave systems. First, put $l_1 = e_1$ and consider the l_1 subsystem so that $\dot{l}_1 = G_1(l_1, f_1, p_1 u_1)$, where a virtual controller $e_2 = \vartheta(l_1)$ is assumed. The aim is to design the virtual controller $\vartheta(l_1)$ and the controller $p_1 u_1$ by using the Lyapunov stability criteria so that the l_1 subsystem will be stabilized. The same procedure will be repeated in the next step to stabilize the (l_1, l_2) subsystem, where $l_2 = e_2 - \vartheta(l_1)$ and the virtual controller $e_3 = \vartheta(l_1, l_2)$. Thus, eventually an asymptotically stable $(l_1, l_2, \ldots, l_n)$ system will be achieved so that the master and slave systems will attain asymptotically stable synchronization state.

3 The Cai System and the Chen System

The Cai system [3] is considered as the master system which is given below

$$\begin{aligned}
\dot{v}_1 &= \zeta_1(v_2 - v_1), \\
\dot{v}_2 &= \eta_1 v_1 + \theta_1 v_2 - v_1 v_3, \\
\dot{v}_3 &= {v_1}^2 - \delta_1 v_3,
\end{aligned} \tag{5}$$

which shows chaotic behavior for the parameter values $\zeta_1 = 20, \eta_1 = 14, \theta_1 = 10.6, \delta_1 = 2.8$ and the well known Chen system [4] is considered as the slave system which is given below

$$\begin{aligned}
\dot{w}_1 &= \zeta_2(w_2 - w_1), \\
\dot{w}_2 &= (\theta_2 - \zeta_2) w_1 + \theta_2 w_2 - w_1 w_3, \\
\dot{w}_3 &= w_1 w_2 - \eta_2 w_3,
\end{aligned} \tag{6}$$

which exhibits chaotic behavior for the parameter values $\zeta_2 = 35, \eta_2 = 3, \theta_2 = 28$.

4 Multi-Switching Synchronization Methodology

The slave system with controller is

$$\begin{aligned}
\dot{w}_1 &= \zeta_2(w_2 - w_1) + u_{1j}, \\
\dot{w}_2 &= (\theta_2 - \zeta_2) w_1 + \theta_1 w_2 - w_1 w_3 + u_{2j}, \\
\dot{w}_3 &= w_1 w_2 - \eta_2 w_3 + u_{3j},
\end{aligned} \tag{7}$$

where u_{1j}, u_{2j}, u_{3j}, represent different controllers and $j = \overline{1, 6}$ represent different switching states.

First, we will define a general synchronization methodology using the active backstepping technique. In order to explain the method, for $j = 1$, the errors are defined as follows:

$$\begin{aligned} e_{11} &= p_1 w_1 + q_1 v_1, \\ e_{21} &= p_2 w_2 + q_2 v_2, \\ e_{31} &= p_3 w_3 + q_3 v_3, \end{aligned} \tag{8}$$

where $p_i, q_i, i = 1, 2, 3$ are arbitrary scaling factors. If $p_1 = p_2 = p_3 = 1$ and $q_1 = q_2 = q_3 = 1$, then anti-synchronization will be achieved for the pairs of state variables $(w_1, v_1), (w_2, v_2), (w_3, v_3)$. If $p_1 = p_2 = p_3 = 1$ and $q_1 = q_2 = q_3 = -1$, then complete synchronization will be achieved and if $p_1 = p_2 = p_3 = 1$ and $q_1 = 1, q_2 = -1, q_3 = 1$, then hybrid synchronization will be achieved.

Hybrid synchronization has been defined as the synchronization for which some state variables attain completely synchronized state and some state variables attain anti-synchronized state. But in this paper, since we have chosen the master-slave combination in multi-switching manner, we assume in the case of hybrid synchronization that any state variable which is taken with w_1 and w_3 will be completely synchronized with these state variables and the state variable which is taken with w_2 will be anti-synchronized. From (8) the error dynamics can be written as

$$\begin{cases} \dot{e}_{11} = p_1 \dot{w}_1 + q_1 \dot{v}_1, \\ \dot{e}_{21} = p_2 \dot{w}_2 + q_2 \dot{v}_2, \\ \dot{e}_{31} = p_3 \dot{w}_3 + q_3 \dot{v}_3. \end{cases} \tag{9}$$

By using (5) and (7) in (9), we get

$$\begin{aligned} \dot{e}_{11} &= p_1 \left\{ \zeta_2 (w_2 - w_1) + u_{11} \right\} + q_1 \left\{ \zeta_1 (v_2 - v_1) \right\}, \\ \dot{e}_{21} &= p_2 \left\{ (\theta_2 - \zeta_2) w_1 + \theta_2 w_2 - w_1 w_3 + u_{21} \right\} + q_2 \left(\eta_1 v_1 + \theta_1 v_2 - v_1 v_3 \right), \\ \dot{e}_{31} &= p_3 \left(w_1 w_2 - \eta_2 w_3 + u_{31} \right) + q_3 \left({v_1}^2 - \delta_1 v_3 \right). \end{aligned} \tag{10}$$

Hence the error dynamical system can be written as

$$\begin{aligned} \dot{e}_{11} &= \zeta_2 (p_1 w_2 - p_1 w_1) + \zeta_1 (q_1 v_2 - q_1 v_1) + p_1 u_{11} \\ &= \frac{p_1 \zeta_2}{p_2} (e_{21} - q_2 v_2) - \zeta_2 (e_{11} - q_1 v_1) + \zeta_1 (q_1 v_2 - q_1 v_1) + p_1 u_{11}, \\ &= \frac{p_1 \zeta_2}{p_2} e_{21} - \zeta_2 e_{11} + f_1 + p_1 u_{11}. \end{aligned} \tag{11}$$

Similarly

$$\begin{aligned} \dot{e}_{21} &= p_2 \left\{ (\theta_2 - \zeta_2) w_1 + \theta_2 w_2 - w_1 w_3 \right\} + q_2 (\eta_1 v_1 + \theta_1 v_2 - v_1 v_3) + p_2 u_{21} \\ &= \frac{p_2 (\theta_2 - \zeta_2)}{p_1} (e_{11} - q_1 v_1) + \theta_2 (e_{21} - q_2 v_2) - \frac{p_2}{p_1 p_3} (e_{11} - q_1 v_1)(e_{31} - q_3 v_3) \\ &+ q_2 (\eta_1 v_1 - \theta_1 v_2 - v_1 v_3) + p_2 u_{21} = \frac{p_2 (\theta_2 - \zeta_2)}{p_1} e_{11} - \frac{p_2}{p_1 p_3} e_{11} e_{31} + \frac{p_2 q_3}{p_1 p_3} e_{11} v_3 \\ &+ \frac{p_2 q_1}{p_1 p_3} e_{31} v_1 + \theta_2 e_{21} + f_2 + p_2 u_{21} \end{aligned} \tag{12}$$

and

$$\begin{aligned}\dot{e}_{31} &= p_3(w_1w_2 - \eta_2 w_3) + q_3({v_1}^2 - \delta_1 v_3) + p_3 u_{31}\\ &= \frac{p_3}{p_1p_2}(e_{11} - q_1v_1)(e_{21} - q_2v_2) - \eta_2(e_{31} - q_3v_3) + q_3({v_1}^2 - \delta_2 v_3) + p_3u_{33}\\ &= \frac{p_3}{p_1p_2}e_{11}e_{21} - \frac{p_3q_2}{p_1p_2}e_{11}v_2 - \frac{p_3q_1v_1}{p_1p_2}e_{21} - \eta_2 e_{31} + f_3 + p_3u_{31},\end{aligned} \tag{13}$$

where

$$\begin{aligned}f_1 &= \frac{-p_1\zeta_2}{p_2}q_2v_2 + \zeta_2q_1v_1 + \zeta_1q_1v_2 - \zeta_1q_1v_1,\\ f_2 &= -\frac{q_1p_2}{p_1}(\theta_2 - \zeta_2)v_1 - \frac{p_2q_3q_1}{p_1p_3}v_1v_3 - \theta_2q_2v_2 + q_2(\eta_1v_1 - \theta_1v_2 - v_1v_3),\\ f_3 &= \frac{p_3}{p_1p_2}q_1q_2v_1v_2 + \eta_2q_3v_3 + q_3({v_1}^2 - \delta_2v_3).\end{aligned} \tag{14}$$

Let $l_1 = e_{11}$. Then its derivative will be

$$\dot{l_1} = \dot{e_{11}} = \frac{p_1\zeta_2}{p_2}e_{21} - \zeta_2 l_1 + f_1 + p_1u_{11}, \tag{15}$$

where $e_{21} = \vartheta_1(l_1)$ is considered as a virtual controller. Our aim is to design $\vartheta_1(l_1)$ so that the l_1 subsystem (15) could be stabilized. Consider the following Lyapunov function

$$K_1 = 0.5{l_1}^2. \tag{16}$$

Then the derivative of K_1 will be

$$\dot{K_1} = l_1\dot{l_1} = l_1\left(\frac{p_1\zeta_2}{p_2}\vartheta_1(l_1) - \zeta_2 e_{11} + f_1 + p_1u_{11}\right). \tag{17}$$

If $\vartheta_1(l_1) = 0$ and $u_{11} = -\frac{1}{p_1}(f_1)$, then $\dot{K_1} = -\zeta_2 e_{11}^2$ which is negative definite. Hence by the Lyapunov stability criteria the l_1 subsystem is asymptotically stable. Suppose the error between e_{21} and $\vartheta_1(l_1)$ is denoted by $l_2 = e_{21} - \vartheta_1(l_1)$. Then we have the (l_1, l_2) subsystem given below

$$\begin{aligned}\dot{l_1} &= \frac{p_1\zeta_2}{p_2}l_2 - \zeta_2 l_1,\\ \dot{l_2} &= \left(\frac{p_2(\theta_2 - \zeta_2)}{p_1} - \frac{p_2}{p_1p_3}e_{31} + \frac{p_2q_3}{p_1p_3}v_3\right)l_1 + \theta_2 l_2 + \frac{p_2q_1}{p_1p_3}e_{31}v_1 + f_2 + p_2u_{21}.\end{aligned} \tag{18}$$

In order to make the (l_1, l_2) subsystem stable, $e_{31} = \vartheta_2(l_1, l_2)$ is taken as a virtual controller. Now, we take the Lyapunov function and its derivatives as

$$\begin{aligned}K_2 &= K_1 + (0.5){l_2}^2,\\ \dot{K_2} &= -\zeta_2{l_1}^2 - \theta_2{l_2}^2 + l_2\Bigg[\left(\frac{p_2(\theta_2 - \zeta_2)}{p_1} - \frac{p_2}{p_1p_3}\vartheta_2(l_1, l_2) + \frac{p_2q_3}{p_1p_3}v_3\right)l_1 + 2\theta_2 l_2\\ &\quad + \frac{p_2q_1}{p_1p_3}v_1\vartheta_2(l_1, l_2) + \frac{p_1\zeta_2}{p_2}l_1 + f_2 + p_2u_{21}\Bigg].\end{aligned} \tag{19}$$

Hence by choosing the controller u_{21} in the following way

$$u_{21} = -\frac{1}{p_2}\left(\frac{p_2}{p_1}(\theta_2 - \zeta_2)l_1 + \frac{p_2 q_3}{p_1 p_3} l_1 v_3 + 2\theta_2 l_2 + \frac{p_1\zeta_2}{p_2} l_1 + f_2\right) \tag{20}$$

and the virtual controller $\vartheta_2(l_1, l_2) = 0$, we get $\dot{K}_2 = -\zeta_2 {l_1}^2 - \theta_2 {l_2}^2$ which is negative definite. Hence the (l_1, l_2) subsystem is asymptotically stable. Now, suppose the error between e_{31} and $\vartheta_2(l_1, l_2)$ is $l_3 = e_{31} - \vartheta_2(l_1, l_2)$. Then

$$\dot{l_3} = \frac{p_3}{p_1 p_2} l_1 l_2 - \frac{p_3 q_2}{p_1 p_2} l_1 v_2 - \frac{p_3}{p_1 p_2} l_2 q_1 v_1 - \eta_2 l_3 + f_3 + p_3 u_{31}. \tag{21}$$

Now to stabilize the (l_1, l_2, l_3) system, the controller u_{31} is defined as

$$u_{31} = -\frac{1}{p_3}\left[\left\{\left(\frac{p_3}{p_1 p_2} - \frac{p_2}{p_1 p_3}\right) l_2 - \frac{p_3 q_2}{p_1 p_2} v_2\right\} l_1 - \left(\frac{p_3}{p_1 p_2} - \frac{p_2}{p_1 p_3}\right) q_1 l_2 v_1 + f_3\right] \tag{22}$$

and the Lyapunov function K_3 as

$$K_3 = K_2 + 0.5{l_3}^2. \tag{23}$$

Its derivative will be

$$\dot{K_3} = -\zeta_2 {l_1}^2 - \theta_2 {l_2}^2 - \eta_2 {l_3}^2 \tag{24}$$

which is negative definite. Hence according to the Lyapunov stability theory $(0, 0, 0)$ equilibrium point of (l_1, l_2, l_3) system is now asymptotically stable. The (l_1, l_2, l_3) system is given by

$$\begin{cases} \dot{l_1} = \dfrac{p_1\zeta_2}{p_2} l_2 - \zeta_2 l_1, \\ \dot{l_2} = -\dfrac{p_2}{p_1 p_3} l_1 l_3 + \dfrac{p_2}{p_1 p_3} q_1 v_1 l_3 - \theta_2 l_2 - \dfrac{p_1 \zeta_2}{p_2} l_1, \\ \dot{l_3} = \dfrac{p_2}{p_1 p_3} l_2 l_1 - \eta_2 l_3 - \dfrac{p_2 q_1}{p_1 p_3} l_2 v_1. \end{cases} \tag{25}$$

Now, for the second switch the errors are defined as follows

$$e_{12} = p_1 w_1 + q_1 v_2, e_{22} = p_2 w_2 + q_2 v_3, e_{32} = p_3 w_3 + q_3 v_1. \tag{26}$$

Then, the controllers are

$$\begin{cases} u_{12} = -\dfrac{1}{p_1}(f_1), \\ u_{22} = -\dfrac{1}{p_2}\left(\dfrac{p_2}{p_1}(\theta_2 - \zeta_2)l_1 + \dfrac{p_2 q_3}{p_1 p_3} l_1 v_1 + 2\theta_2 l_2 + \dfrac{p_1 \zeta_2}{p_2} l_1 + f_2\right), \\ u_{32} = -\dfrac{1}{p_3}\left[\left(\left(\dfrac{p_3}{p_1 p_2} - \dfrac{p_2}{p_1 p_3}\right) l_2 - \dfrac{p_3 q_2}{p_1 p_2} v_3\right) l_1 - \left(\dfrac{p_3}{p_1 p_2} - \dfrac{p_2}{p_1 p_3}\right) q_1 l_2 v_2 + f_3\right], \end{cases} \tag{27}$$

where $l_1 = e_{12}, l_2 = e_{22}, l_3 = e_{32}$. For the third switch the errors are

$$e_{13} = p_1 w_1 + q_1 v_3, e_{22} = p_2 w_2 + q_2 v_1, e_{32} = p_3 w_3 + q_3 v_2. \tag{28}$$

The controllers defined by using the above procedure are

$$\begin{cases} u_{13} = -\dfrac{1}{p_1}(f_1), \\ u_{23} = -\dfrac{1}{p_2}\left(\dfrac{p_2}{p_1}(\theta_2 - \zeta_2)l_1 + \dfrac{p_2 q_3}{p_1 p_3} l_1 v_2 + 2\theta_2 l_2 + \dfrac{p_1 \zeta_2}{p_2} l_1 + f_2\right), \\ u_{33} = -\dfrac{1}{p_3}\left[\left(\left(\dfrac{p_3}{p_1 p_2} - \dfrac{p_2}{p_1 p_3}\right) l_2 - \dfrac{p_3 q_2}{p_1 p_2} v_1\right) l_1 - \left(\dfrac{p_3}{p_1 p_2} - \dfrac{p_2}{p_1 p_3}\right) q_1 l_2 v_3 + f_3\right], \end{cases} \tag{29}$$

where $l_1 = e_{13}, l_2 = e_{23}, l_3 = e_{33}$. In case of switch four the errors are defined as

$$e_{14} = p_1 w_1 + q_1 v_1, e_{24} = p_2 w_2 + q_2 v_3, e_{34} = p_3 w_3 + q_3 v_2 \tag{30}$$

and the controllers are

$$\begin{cases} u_{14} = -\dfrac{1}{p_1}(f_1), \\ u_{24} = -\dfrac{1}{p_2}\left(\dfrac{p_2}{p_1}(\theta_2 - \zeta_2)l_1 + \dfrac{p_2 q_3}{p_1 p_3} l_1 v_2 + 2\theta_2 l_2 + \dfrac{p_1 \zeta_2}{p_2} l_1 + f_2\right), \\ u_{34} = -\dfrac{1}{p_3}\left[\left(\left(\dfrac{p_3}{p_1 p_2} - \dfrac{p_2}{p_1 p_3}\right) l_2 - \dfrac{p_3 q_2}{p_1 p_2} v_3\right) l_1 - \left(\dfrac{p_3}{p_1 p_2} - \dfrac{p_2}{p_1 p_3}\right) q_1 l_2 v_1 + f_3\right), \end{cases} \tag{31}$$

where $l_1 = e_{14}, l_2 = e_{24}, l_3 = e_{34}$. For switch five the errors are taken as

$$e_{15} = p_1 w_1 + q_1 v_3, e_{25} = p_2 w_2 + q_2 v_2, e_{35} = p_3 w_3 + q_3 v_1. \tag{32}$$

For switch five the controllers are

$$\begin{cases} u_{15} = -\dfrac{1}{p_1}(f_1), \\ u_{25} = -\dfrac{1}{p_2}\left(\dfrac{p_2}{p_1}(\theta_2 - \zeta_2)l_1 + \dfrac{p_2 q_3}{p_1 p_3} l_1 v_1 + 2\theta_2 l_2 + \dfrac{p_1 \zeta_2}{p_2} l_1 + f_2\right), \\ u_{35} = -\dfrac{1}{p_3}\left[\left(\left(\dfrac{p_3}{p_1 p_2} - \dfrac{p_2}{p_1 p_3}\right) l_2 - \dfrac{p_3 q_2}{p_1 p_2} v_2\right) l_1 - \left(\dfrac{p_3}{p_1 p_2} - \dfrac{p_2}{p_1 p_3}\right) q_1 l_2 v_3 + f_3\right], \end{cases} \tag{33}$$

where $l_1 = e_{15}, l_2 = e_{25}, l_3 = e_{35}$. For switch six the errors are

$$e_{16} = p_1 w_1 + q_1 v_2, e_{26} = p_2 w_2 + q_2 v_1, e_{36} = p_3 w_3 + q_3 v_3 \tag{34}$$

and the controllers are

$$\begin{cases} u_{16} = -\dfrac{1}{p_1}(f_1), \\ u_{26} = -\dfrac{1}{p_2}\left(\dfrac{p_2}{p_1}(\theta_2 - \zeta_2)l_1 + \dfrac{p_2 q_3}{p_1 p_3} l_1 v_3 + 2\theta_2 l_2 + \dfrac{p_1 \zeta_2}{p_2} l_1 + f_2\right), \\ u_{36} = -\dfrac{1}{p_3}\left[\left(\left(\dfrac{p_3}{p_1 p_2} - \dfrac{p_2}{p_1 p_3}\right) l_2 - \dfrac{p_3 q_2}{p_1 p_2} v_1\right) l_1 - \left(\dfrac{p_3}{p_1 p_2} - \dfrac{p_2}{p_1 p_3}\right) q_1 l_2 v_2 + f_3\right], \end{cases} \tag{35}$$

where $l_1 = e_{16}, l_2 = e_{26}, l_3 = e_{36}$. It is obvious that the values of f_1, f_2, f_3 will be different in all the switches, since the values of f_1, f_2, f_3 will be changed according to

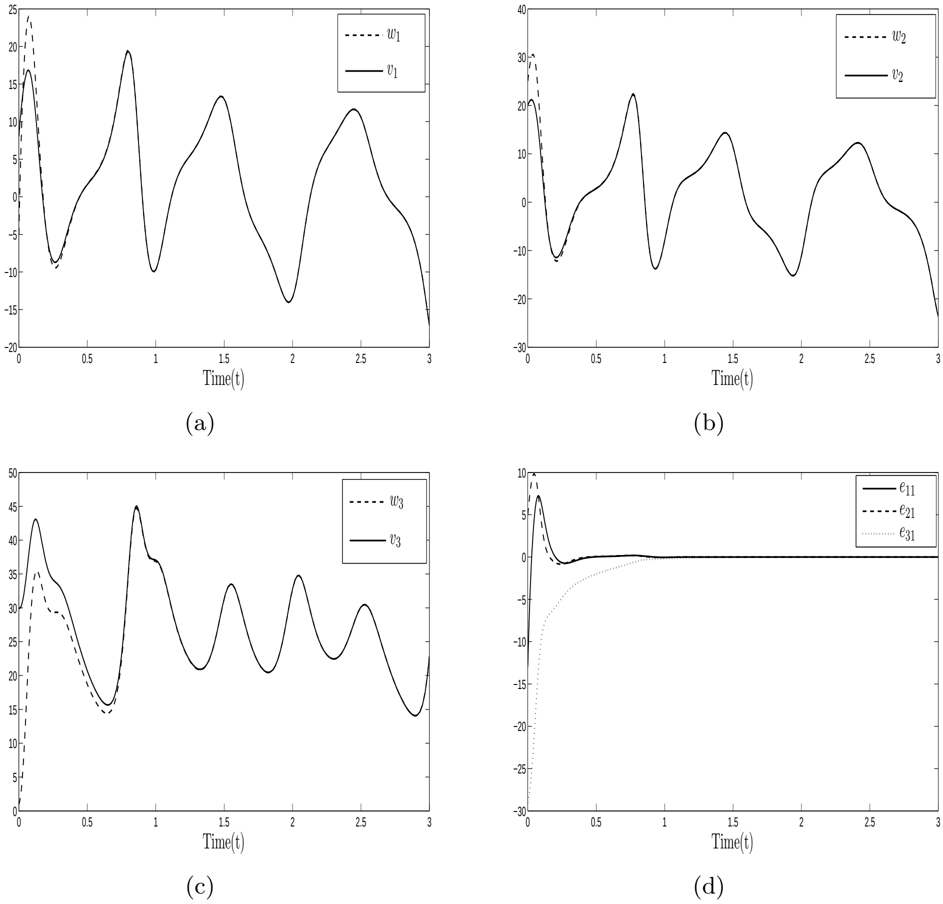


Figure 1: (a) Synchronization between the state variables w_1, v_1 in switch one; (b) Synchronization between the state variables w_2, v_2 in switch one; (c) Synchronization between the state variables w_3, v_3 in switch one; (d) Convergence of e_{11}, e_{21}, e_{31} to zero for switch one.

the error defined. If q_1, q_2, q_3 are chosen as any arbitrary scalars but not equal and all $p_1 = p_2 = p_3 = 1$, then this will become a case of modified projective synchronization. If $q_1 = q_2 = q_3$ are chosen as any arbitrary scalars and all $p_1 = p_2 = p_3 = 1$, then the problem will be reduced to projective synchronization which is a particular case of modified projective synchronization. The method described above is easy to apply for the dynamical systems having dimension greater than three also.

5 Numerical Simulations

5.1 Complete synchronization

Numerical simulations are shown only for three switches as the remaining ones can be achieved in a similar manner. The values of $p_i's$ and $q_i's$ are chosen in such a manner which lead to complete synchronization, anti-synchronization and hybrid synchronization

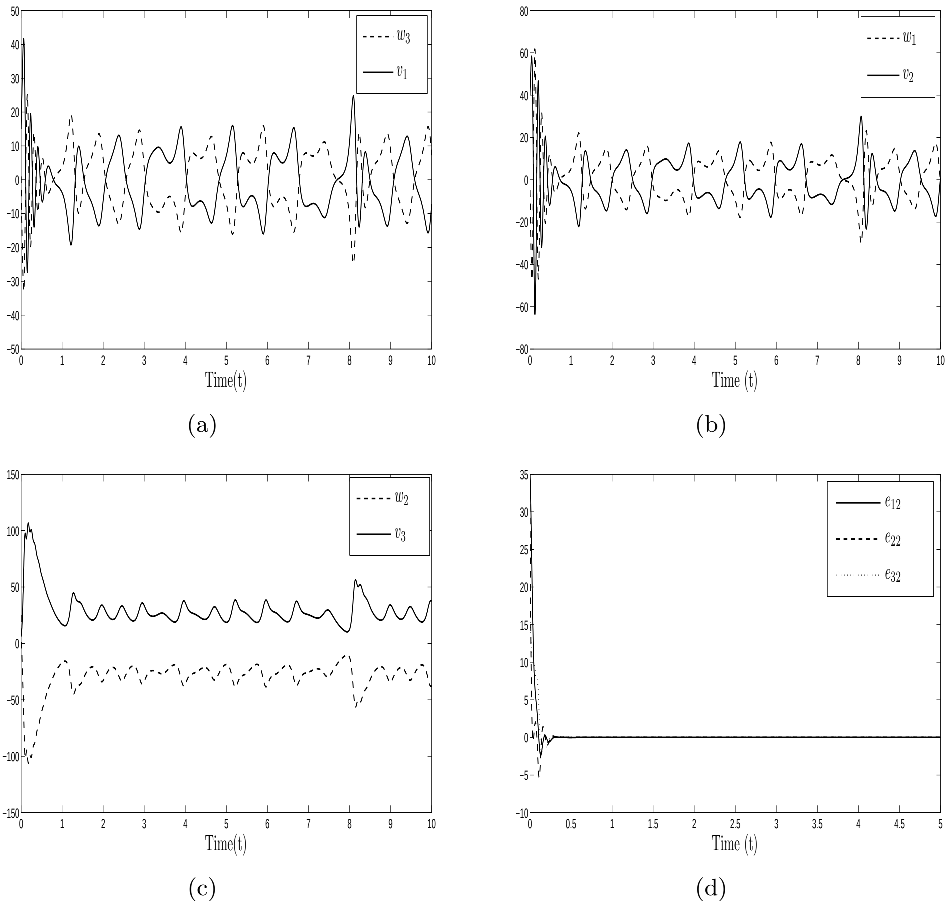
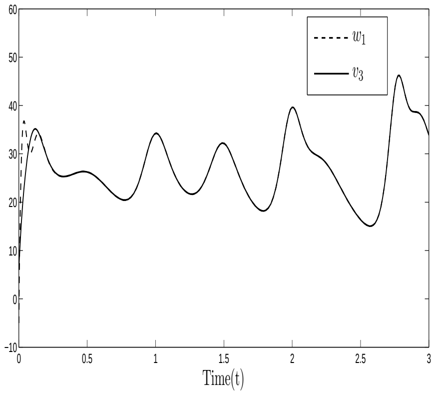


Figure 2: (a) Anti-synchronization between the state variables w_1, v_2 in switch two; (b) Anti-synchronization between the state variables w_2, v_3 for switch two; (c) Anti-synchronization between the state variables w_3, v_1 in switch two; (d) Convergence of the errors e_{12}, e_{22}, e_{32} to zero for switch two.

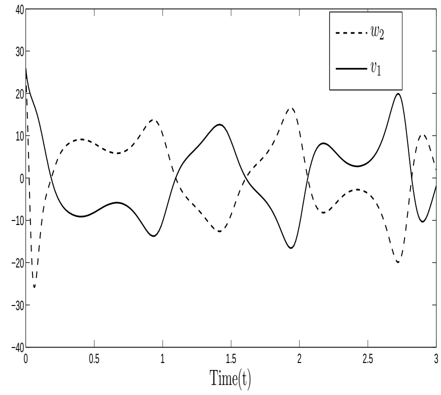
between different state variables of the drive and response systems.

The case of complete synchronization is considered for the first switch and the values of scaling factors are $p_1 = p_2 = p_3 = 1$ and $q_1 = q_2 = q_3 = -1$.

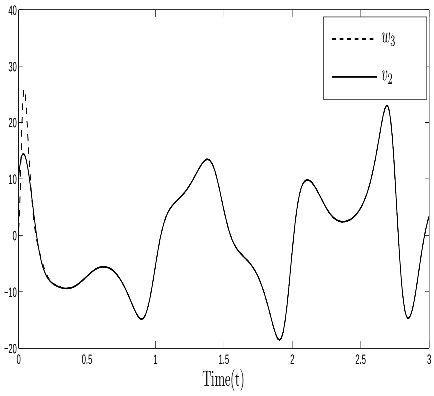
The initial conditions are kept fixed for the slave system throughout the paper, which are $(-5, 25, 1)$, but in each type of synchronization the initial conditions for the master system are different. In the case of complete synchronization the initial conditions for the master system are $(8, 20, 30)$. Hence for the first switch the initial conditions for the errors are $(-13, 5, -29)$. Complete synchronization between w_1, v_1 and w_2, v_2 is shown in Figure 1a-b. Figure 1c-d show synchronization between w_3, v_3 and the errors e_{11}, e_{21}, e_{31} converging to zero.



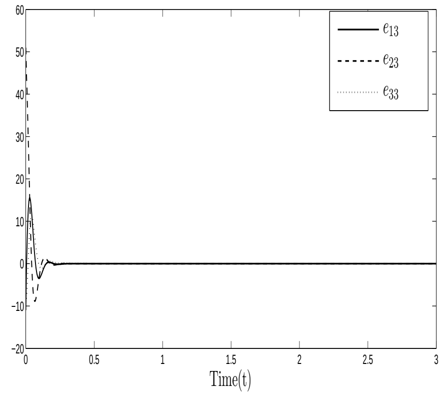
(a)



(b)



(c)



(d)

Figure 3: (a) Complete synchronization between the state variables w_1, v_3; (b) Anti-synchronization between w_2, v_1 for switch three; (c) Complete synchronization between the state variables w_3, v_2; (d) Convergence of e_{13}, e_{23}, e_{33} to zero for switch three.

5.2 Anti-synchronization

Anti-synchronization is shown for the second switch. In order to achieve anti-synchronization, the values of $p_1 = p_2 = p_3 = 1$ and $q_1 = q_2 = q_3 = 1$ are chosen. Since the initial conditions for the master and slave systems are $(15, 40, 6)$ and $(-5, 25, 1)$, the initial conditions for the errors are $(35, 31, 16)$. Figure 2 a-b show anti-synchronization between w_1, v_2 and w_2, v_3, and Figure 2 c-d show anti-synchronization between the state variables w_3, v_1 and the errors e_{12}, e_{22}, e_{32} converging to zero.

5.3 Hybrid synchronization

In this subsection the case of hybrid synchronization is considered for the third switch. In order to attain hybrid synchronization, the values of $p_1 = p_2 = p_3 = 1$ and $q_1 = 1, q_2 = -1, q_3 = -1$ are chosen. In the case of hybrid synchronization the initial conditions for the master system are $(26, 10, 6)$. According to the initial conditions for the master and slave systems $(26, 10, 6)$ and $(-5, 25, 1)$ respectively, for the third switch the initial conditions for the errors are $(-11, 51, -9)$. Figure 3 a-b show complete synchronization between w_1, v_3 and anti-synchronization for w_2, v_1. Figure 3 c-d exhibit the state variables w_3, v_2 in complete synchronized states and the errors e_{13}, e_{23}, e_{33} converging to zero.

The numerical results presented in this paper are obtained by using Matlab software. In numerical simulations, complete synchronization, anti-synchronization and hybrid synchronization are shown and other types of synchronization can be achieved by choosing different scaling factors.

6 Conclusion

In this manuscript, we have investigated multi-switching synchronization between the Cai system and the Chen system by using the active backstepping method. An efficient and easy method is proposed to design suitable controllers and fruitful results are obtained. Both theoretical and graphical analysis lead to the same conclusion. The controllers designed by this approach are very effective as synchronizations are achieved very rapidly by this method. Since the chaos synchronization has its applications in secure communications and multi-switching increases the grade of security as it is very difficult to guess which pair of state variables will attain synchronization, the proposed method has significant applications in the field of secure communication. The approach is also significant in the sense that by simply taking different scaling factors, various types of synchronization can be achieved. This work can be extended to fractional order systems and various higher dimensional systems.

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