Nonlinear Dynamics and Systems Theory, **18** (2) (2018) 182–190

A Novel Method for Solving Caputo-Time-Fractional Dispersive Long Wave Wu-Zhang System

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Received: January 11, 2018; Revised: March 21, 2018

Abstract: In this paper we presented a reliable efficient numerical scheme to find analytical supportive solution of Caputo-time-fractional Wu-Zhang system. A modified version of generalized Taylor power series method is used in this work. Graphical justifications of the reliability of the proposed method are provided. Finally, the effects of the fractional order on the solution of Wu-Zhang system is also discussed.

Keywords: *Caputo-time-fractional Wu-Zhang system; approximate solutions; generalized Taylor series.*

Mathematics Subject Classification (2010): 26A33, 35F25, 35C10.

1 Introduction

Wu-Zhang system is known also as (1+1)-dimensional dispersive long wave equations [25]. It is very helpful for coastal and civil engineers to apply the nonlinear water wave model in harbor and coastal design. Abundant soliton solutions are obtained to this model using the extended hyperbolic tangent expansion method. In [20], the Wu-Zhang system is considered to study dispersive long waves. The extended trial equation method is used and solitary wave solutions are obtained. Also, they used the mapping method to extract more solitonic solutions.

Finding analytical solution to fractional nonlinear differential equations is a difficult task. In the literature, different computational schemes were developed for either finding numerical solutions over a specific range or considering a few terms of an iterative computational series solution as an approximate. Such available methods are the variational

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iteration method [21], the iterative Laplace transform method [17], Adomian's decomposition method [23], homotopy analysis (perturbation) methods [12, 15, 16, 22], generalized iterative-monotone methods [11, 13, 24] and the fractional power series method [1–10, 14, 18, 19].

The motivation of this work is to study for the first time the fractional Wu-Zhang system

$$\begin{aligned} D_t^\alpha v(x,t) &= -v(x,t)v_x(x,t) - w_x(x,t), \\ D_t^\alpha w(x,t) &= -(v(x,t)w(x,t))_x - \frac{1}{3}v_{xxx}(x,t), \end{aligned} \tag{1}$$

where $0 < \alpha \leq 1$ in Caputo sense and $0 < t < R < 1$. Also, we desire to study the effect of the fractional derivative α on the solution of (1).

The generalized Taylor fractional series will be used as an alternative method to extract a reliable analytical supportive solution of the time-fractional Wu-Zhang system. The accuracy of the method will be provided and graphical analysis is conducted to study the effect of the fractional order α on the behavior of the obtained solution.

2 Analysis of the Proposed Method

In this section, we present in details the construction of the generalized Taylor fractional series. The suggested solutions of the problem are sought to have the form

$$v(x,t) = \sum_{j=0}^{\infty} c_j(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)}, \tag{2}$$

$$w(x,t) = \sum_{j=0}^{\infty} d_j(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)}. \tag{3}$$

The target of this study is obtaining a supportive approximate solution to the proposed model. Thus, we may write the suggested solution as

$$v(x,t) = \sum_{j=0}^{m} c_j(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)} = c_0(x) + \sum_{j=1}^{m} c_j(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)}, \tag{4}$$

$$w(x,t) = \sum_{j=0}^{m} d_j(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)} = d_0(x) + \sum_{j=1}^{m} d_j(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)}. \tag{5}$$

In Caputo sense, we recall the fact that

$$D_t^\alpha t^\beta = \begin{cases} \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}t^{\beta-\alpha}, & \beta \geq \alpha, \\ 0, & \beta < \alpha. \end{cases}$$

Therefore, applying the operator D_t^α on equations (4) and (5), will produce the formulas

$$D_t^\alpha v_m(x,t) = \sum_{j=0}^{m-1} c_{j+1}(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)}, \tag{6}$$

$$D_t^\alpha w_m(x,t) = \sum_{j=0}^{m-1} d_{j+1}(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)}. \tag{7}$$

Next, we substitute both (4)-(7) in the fractional equation (1). Therefore, we arrive at the following recurrence relations

$$\begin{aligned} 0 &= \sum_{j=0}^{m-1} c_{j+1}(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)} + \sum_{j=0}^{m} d_j'(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)} \\ &+ \left(\sum_{j=0}^{m} c_j(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)}\right)\left(\sum_{j=0}^{m} c_j'(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)}\right) \end{aligned} \tag{8}$$

and

$$\begin{aligned} 0 &= \sum_{j=0}^{m-1} d_{j+1}(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)} + \frac{1}{3}\sum_{j=0}^{m} c_j'''(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)} \\ &+ \left(\sum_{j=0}^{m} c_j(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)}\right)\left(\sum_{j=0}^{m} d_j'(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)}\right) \\ &+ \left(\sum_{j=0}^{m} c_j'(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)}\right)\left(\sum_{j=0}^{m} d_j(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)}\right). \end{aligned} \tag{9}$$

We follow the same analogue used in obtaining the Taylor series coefficients. In particular, to determine the functions $c_n(x),\ d_n(x),\ n = 1, 2, 3, ...,$ we have to solve the following two systems simultaneously

$$\begin{aligned} &D_t^{(m-1)\alpha}\{L_1(x,t,\alpha,m)\} \downarrow_{t=0} = 0, \\ &D_t^{(m-1)\alpha}\{L_2(x,t,\alpha,m)\} \downarrow_{t=0} = 0, \end{aligned} \tag{10}$$

where

$$\begin{aligned} L_1(x,t,\alpha,m) &= \sum_{j=0}^{m-1} c_{j+1}(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)} + \sum_{j=0}^{m} d_j'(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)} \\ &+ \left(\sum_{j=0}^{m} c_j(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)}\right)\left(\sum_{j=0}^{m} c_j'(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)}\right) \end{aligned} \tag{11}$$

and

$$\begin{aligned} L_2(x,t,\alpha,m) &= \sum_{j=0}^{m-1} d_{j+1}(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)} + \frac{1}{3}\sum_{j=0}^{m} c_j'''(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)} \\ &+ \left(\sum_{j=0}^{m} c_j(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)}\right)\left(\sum_{j=0}^{m} d_j'(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)}\right) \\ &+ \left(\sum_{j=0}^{m} c_j'(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)}\right)\left(\sum_{j=0}^{m} d_j(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)}\right). \end{aligned} \tag{12}$$

Now, we explain the derivations of the first few terms of the sequence $\{c_m(x)\}_1^N$ and $\{d_m(x)\}_1^N$. We start with the index $m = 1$;

$$\begin{aligned}
L_1(x,t,\alpha,1) &= c_1(x) + d_0'(x) + d_1'(x)\frac{t^\alpha}{\Gamma(\alpha+1)} \\
&+ \left(c_0(x) + f_1(x)\frac{t^\alpha}{\Gamma(\alpha+1)}\right)\left(c_0'(x) + c_1'(x)\frac{t^\alpha}{\Gamma(\alpha+1)}\right), \\
L_2(x,t,\alpha,1) &= d_1(x) + \frac{1}{3}\left(c_0'''(x) + c_1'''(x)\frac{t^\alpha}{\Gamma(\alpha+1)}\right) \\
&+ \left(c_0(x) + c_1(x)\frac{t^\alpha}{\Gamma(\alpha+1)}\right)\left(d_0'(x) + d_1'(x)\frac{t^\alpha}{\Gamma(\alpha+1)}\right) \\
&+ \left(c_0'(x) + c_1'(x)\frac{t^\alpha}{\Gamma(\alpha+1)}\right)\left(d_0(x) + d_1(x)\frac{t^\alpha}{\Gamma(\alpha+1)}\right). \qquad (13)
\end{aligned}$$

Solving $L_1(x,0,\alpha,1) = 0$ and $L_2(x,0,\alpha,1) = 0$, yields

$$\begin{aligned}
c_1(x) &= -c_0(x)c_0'(x) - d_0'(x), \\
d_1(x) &= -\frac{1}{3}c_0'''(x) - \left(c_0(x)d_0'(x) + c_0'(x)d_0(x)\right). \qquad (14)
\end{aligned}$$

To determine $c_2(x)$ and $d_2(x)$, we consider $L_1(x,t,\alpha,2)$ & $L_2(x,t,\alpha,2)$ and we solve $D_t^\alpha\{L_1(x,t,\alpha,2)\}\downarrow_{t=0}= 0$ and $D_t^\alpha\{L_2(x,t,\alpha,2)\}\downarrow_{t=0}= 0$. Therefore

$$\begin{aligned}
c_2(x) &= -(c_1(x)c_0'(x) + c_1'(x)c_0(x)) - d_1'(x), \\
d_2(x) &= -\frac{1}{3}c_1'''(x) - (c_0(x)d_1'(x) + c_0'(x)d_1(x)) \\
&- \left(c_1(x)d_0'(x) + c_1'(x)d_0(x)\right). \qquad (15)
\end{aligned}$$

We should point here that chain rule differentiation is not applicable when using Caputo sense. Thus, in the preceding step "as well as the forthcoming steps" we had to expand all the terms involved in both $L_1(x,t,\alpha,2)$, $L_2(x,t,\alpha,2)$ "in general $L_1(x,t,\alpha,n)$, $L_2(x,t,\alpha,n)$" and use the following fact

$$D_t^\alpha t^\beta \downarrow_{t=0}= \begin{cases} 0, \ \ \beta < \alpha, \\ \\ \Gamma(\alpha+1), \ \ \beta = \alpha, \\ \\ 0, \ \ \beta > \alpha. \end{cases}$$

To determine $c_3(x)$ and $d_3(x)$, we consider $L_1(x,t,\alpha,3)$ and $L_2(x,t,\alpha,3)$ and we solve $D_t^{2\alpha}\{L_1(x,t,\alpha,3)\}\downarrow_{t=0}= 0$ and $D_t^{2\alpha}\{L_2(x,t,\alpha,3)\}\downarrow_{t=0}= 0$. Therefore

$$\begin{aligned}
c_3(x) &= -(c_2(x)c_0'(x) + c_2'(x)c_0(x)) - \frac{\Gamma(1+2\alpha)}{\Gamma^2(1+\alpha)}c_1(x)c_1'(x) - d_2'(x), \\
d_3(x) &= -\frac{1}{3}c_2'''(x) - (c_0(x)d_2'(x) + c_0'(x)d_2(x)) - (c_2(x)d_0'(x) + c_2'(x)d_0(x)) \\
&- \frac{\Gamma(1+2\alpha)}{\Gamma^2(1+\alpha)}\left(c_1(x)d_1'(x) + c_1'(x)d_1(x)\right). \qquad (16)
\end{aligned}$$

Finally, we proceed as above to obtain the other coefficient functions $c_k(x)$ and $d_k(x)$ by solving $D_t^{(k-1)\alpha}\{L_1(x,t,\alpha,k)\}\downarrow_{t=0}= 0$ and $D_t^{(k-1)\alpha}\{L_2(x,t,\alpha,k)\}\downarrow_{t=0}= 0$.

3 Discussion and Concluding Remarks

The purpose of this section is to test the validity of the proposed scheme and to study the effect of the fractional order α on the solution of the time-fractional Wu-Zhang system. To achieve these goals, we solve (1) subject to the initial conditions

$$\begin{aligned} v(x,0) &= \frac{2}{3}\left(1-\tanh\left(\sqrt{\frac{1}{3}}x\right)\right), \\ w(x,0) &= \frac{2}{9}\left(1-\tanh^2\left(\sqrt{\frac{1}{3}}x\right)\right). \end{aligned} \tag{17}$$

Provided that the exact solution of (1) when $\alpha = 1$ is [25]

$$\begin{aligned} v(x,t) &= \frac{2}{3}\left(1-\tanh\left(\sqrt{\frac{1}{3}}\left(x-\frac{2}{3}t\right)\right)\right), \\ w(x,t) &= \frac{2}{9}\left(1-\tanh^2\left(\sqrt{\frac{1}{3}}\left(x-\frac{2}{3}t\right)\right)\right). \end{aligned} \tag{18}$$

For a reliability verification, we consider the 4-th order approximation

$$\begin{aligned} v_4(x,t) &= v(x,0)+\sum_{k=1}^{4} c_k(x)\frac{t^{k\alpha}}{\Gamma(k\alpha+1)}, \\ w_4(x,t) &= w(x,0)+\sum_{k=1}^{4} d_k(x)\frac{t^{k\alpha}}{\Gamma(k\alpha+1)}, \end{aligned} \tag{19}$$

as the supportive solution of the time-fractional Wu-Zhang system. We present here the plots of this obtained approximate solution against the exact solution (18) when $\alpha = 1$, see Figure 1 (i) and (ii) and Figure 4 (a) and (b). For the accuracy of the used method, we provide Figures 2 and 5 which represent respectively $|v(x,t) - v_4(x,t)|$ and $|w(x,t) - w_4(x,t)|$.

Figure 3 provides profile solutions of the function $v(x,t)$ for different values of the fractional order α, the plot on the left when t is fixed, $t = 0.2$, and the plot on the right when x is fixed, $x = 0.5$. Figure 6 provides profile solutions of the function $w(x,t)$ for different values of the fractional order α, the plot on the left when t is fixed, $t = 0.2$, and the plot on the right when x is fixed, $x = 0.5$.

We point here that the proposed method is effective for all nonlinear equations. If the order of the nonlinear terms involved in the equation is small, then a few terms of the fractional power series provide a high accuracy approximation. But, if the order of the nonlinear term is big, it is required to add more terms to reach the desired reasonable approximation.

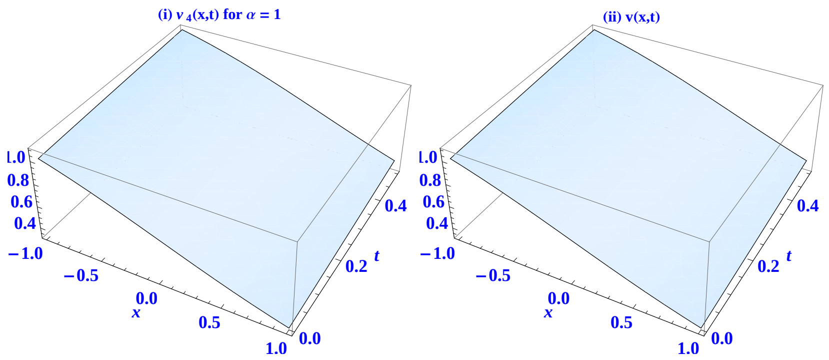

Figure 1: The approximate $v_4(x,t)$ and exact $v(x,t)$ solutions, respectively, when $-1 < x < 15$ and $0 < t < 0.5$ and $\alpha = 1$.

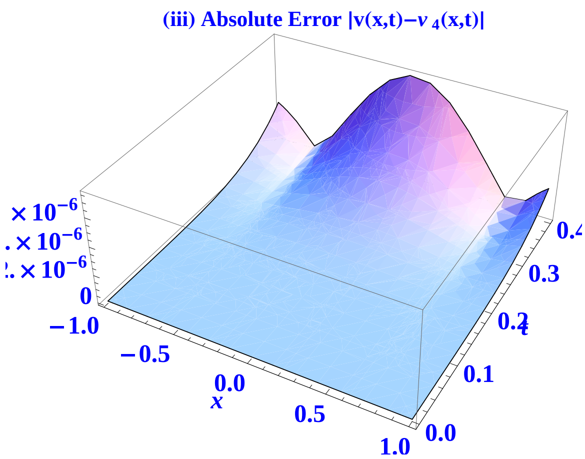

Figure 2: Absolute error $|v(x,t) - v_4(x,t)|$

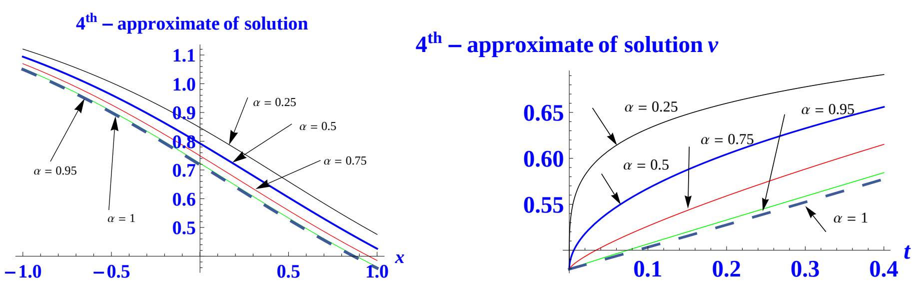

Figure 3: Profile solutions of $v_4(x, 0.2)$ on the left and $v_4(0.5, t)$ on the right for different values of the fractional order α.

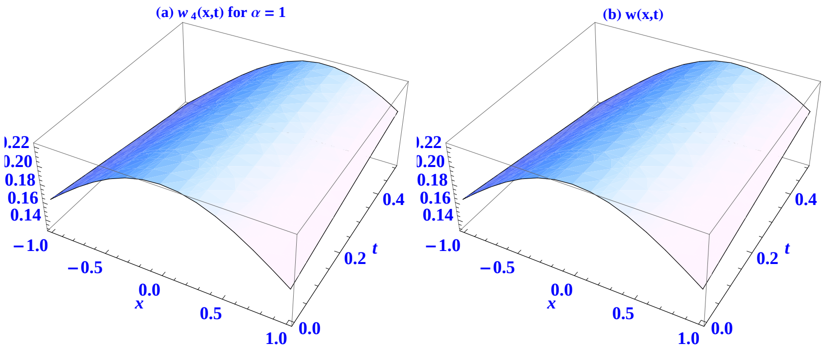

Figure 4: The approximate $w_4(x,t)$ and exact $w(x,t)$ solutions, respectively, when $-1 < x < 15$ and $0 < t < 0.5$ and $\alpha = 1$.

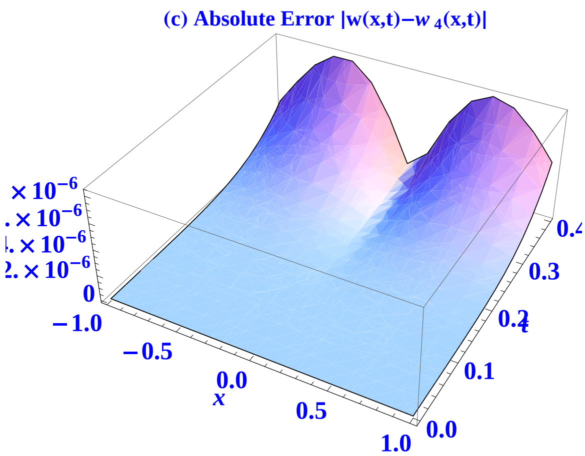

Figure 5: Absolute error $|w(x,t) - w_4(x,t)|$.

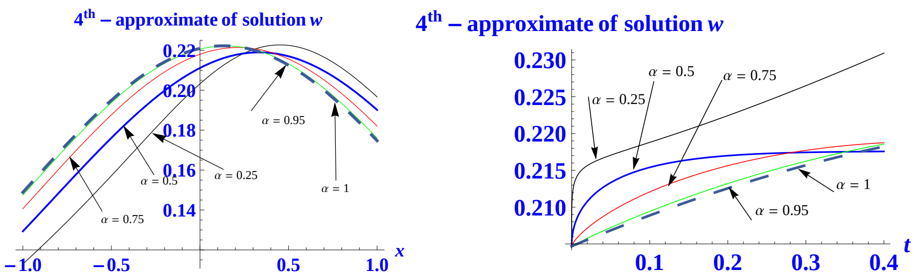

Figure 6: Profile solutions of $w_4(x, 0.2)$ on the left and $w_4(0.5, t)$ on the right for different values of the fractional order α.

Acknowledgment

This work is financially supported by UKM Grant DIP-2017-011 and Ministry of Education Malaysia Grant FRGS/1/2017/STG06/UKM/01/1.

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