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Existence of Solution for Nonlinear Anisotropic Degenerated Elliptic Unilateral Problems

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Abstract: In this paper, we prove the existence of entropy solutions of anisotropic elliptic equations $Au + \sum_{i=1}^{N} g_i(x, u, \nabla u) = f$, where the operator Au is a Leray-Lions anisotropic operator from $W_0^{1,\overrightarrow{p}}(\Omega, \overrightarrow{\omega})$ into its dual $W^{-1,\overrightarrow{p}'}(\Omega, \overrightarrow{\omega^*})$. The critical growth condition on g_i is with respect to ∇u and there is no the growth condition with respect to u and no the sign condition. The right-hand side f belongs to $L^1(\Omega)$.

Keywords: *nonlinear elliptic equations; quasilinear degenerated unilateral problems; non-variational inequalities.*

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1 Introduction

Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$ with Lipschitz continuous boundary and let $Au = -\sum_{i=1}^{N} \partial_i a_i(x, u, \nabla u)$ be a degenerate anisotropic operator of Leray-Lions type defined in the weighted anisotropic Sobolev space $W^{1,\overrightarrow{p}}(\Omega, \overrightarrow{\omega})$, where $\overrightarrow{\omega} = (\omega_0, \omega_1, ..., \omega_N)$ is a vector of weight functions defined on Ω and $\overrightarrow{p} = (p_0, ..., p_N)$ is a vector of real number such that $p_i > 1$ for $i = 0, ..., N$.

We consider the following nonlinear elliptic anisotropic problem

$$\begin{cases} -\sum_{i=1}^{N} \partial_i a_i(x, u, \nabla u) + \sum_{i=1}^{N} g_i(x, u, \nabla u) = f & \text{in} \quad \Omega, \\ u = 0, & \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

where $g_i(x, s, \xi)$ is a Carathéodory function satisfying only the following growth condition $|g_i(x, s, \xi)| \leq \gamma(x) + \rho(s)|\xi_i|^{p_i}$ and where the right-hand side f belongs to $L^1(\Omega)$. In the

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particular case, where $\sum_{i=1}^{N} g_i(x,s,\xi) = -C_0|u|^{p-2}u$, the following degenerated equation $-\text{div}(a(x,u,\nabla u)) - C_0|u|^{p-2}u = f(x,u,\nabla u)$ has been studied by Drabek-Nicolsi in [11] under more degeneracy and some additional assumptions on f and $a(x,u,\nabla u)$.

In the isotropic case, more precisely, when $p_0 = p_1 = ... = p_N = p$ and $\sum_{i=1}^{N} g_i(x,u,\nabla u) \equiv g(x,u,\nabla u)$, the existence result for the unilateral problem with $g(x,u,\nabla u)$ satisfying the following growth condition

$$|g(x,s,\xi)| \leq b(|s|)(C(x) + \sum_{i=1}^{N} \omega_i |\xi_i|^p) \tag{2}$$

and the sign condition

$$g(x,s,\xi)s \geq 0, \tag{3}$$

when f belongs to $W^{-1,p'}(\Omega,\omega^*)$, is studied by Akdim et al. in [7] under the following integrability condition

$$\sigma^{1-q'} \in L^1_{loc}(\Omega) \quad \text{with} \quad 1 < q < +\infty, \tag{4}$$

where σ is a weight function which is assumed satisfying the Hardy inequality

$$\int_\Omega |u|^q \sigma(x) dx \leq C\Big(\sum_{i=1}^{N} \int_\Omega |\partial_i u|^p \omega_i(x) dx\Big)^{\frac{1}{p}}. \tag{5}$$

Our aim in this paper is to prove the existence of entropy solution for the following weighted unilateral elliptic anisotropic problem

$$\begin{cases} u \geq \psi \ a.e. \text{ in } \Omega, \\ T_k(u) \in W_0^{1,\overrightarrow{p}}(\Omega, \overrightarrow{\omega}), \\ \displaystyle\sum_{i=1}^{N} \int_\Omega a_i(x,u,\nabla u)\partial_i T_k(u-v) + \sum_{i=1}^{N} \int_\Omega g_i(x,u,\nabla u) T_k(u-v) \leq \int_\Omega f T_k(u-v), \\ \forall v \in K_\psi(\Omega, \overrightarrow{\omega}) \cap L^\infty(\Omega) \text{ and } \forall k > 0, \end{cases} \tag{6}$$

without the conditions (3) and (4).

2 Preliminaries

Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$ with the Lipschitz continuous boundary and let $1 < p_0, p_1, ..., p_N < \infty$ be $N+1$ real numbers, $p^+ = \max\{p_1, ..., p_N\}$, $p^- = \min\{p_1, ..., p_N\}$. We denote $\partial_i = \dfrac{\partial}{\partial x_i}$, let ω_i be non negative functions on Ω such that $\omega_i > 0$ a.e. in Ω for all $i = 0, 1, ..., N$. We set $\overrightarrow{\omega} = (\omega_0, \omega_1, ..., \omega_N)$ and $\overrightarrow{p} = (p_0, p_1, ..., p_N)$. We suppose that for $i = 0, 1, ..., N$ and for $j = 0, 1, ..., N$

$$\omega_i \in L^1_{loc}(\Omega) \text{ and } \omega_i^{-\frac{1}{p_j - 1}} \in L^1_{loc}(\Omega). \tag{7}$$

As the classical weighted Sobolev space in [10], we define the anisotropic weighted Sobolev space by

$$W^{1,\overrightarrow{p}}(\Omega, \overrightarrow{\omega}) = \Big\{u \in L^{p_0}(\Omega, \omega_0) : \partial_i u \in L^{p_i}(\Omega, \omega_i), i = 1, 2, ..., N\Big\}.$$

As in Theorem 1.11 in [13], by (7) the space $W^{1,\overrightarrow{p}}(\Omega,\overrightarrow{\omega})$ is a Banach space under the following norm

$$\|u\|_{W^{1,\overrightarrow{p}}(\Omega,\overrightarrow{\omega})} = \|u\|_{L^{p_0}(\Omega,\omega_0)} + \sum_{i=1}^{N} \|\partial_i u\|_{L^{p_i}(\Omega,\omega_i)}. \tag{8}$$

Since $\omega_i \in L^1_{loc}(\Omega)$, we have that $C_0^\infty(\Omega)$ is a subset of $W^{1,\overrightarrow{p}}(\Omega,\overrightarrow{\omega})$ and we can introduce the space $W_0^{1,\overrightarrow{p}}(\Omega,\overrightarrow{\omega})$ as the closure of $C_0^\infty(\Omega)$ with respect to norm (8). We recall that the dual space of weighted anisotropic Sobolev space $W_0^{1,\overrightarrow{p}}(\Omega,\overrightarrow{\omega})$ is equivalent to $W^{-1,\overrightarrow{p'}}(\Omega,\overrightarrow{\omega^*})$, where $\overrightarrow{\omega^*} = (\omega_1^*,...,\omega_N^*)$, $\omega_i^* = \omega_i^{1-p_i'}$, $\overrightarrow{p'} = (p_1',...,p_N')$ and $p_i' = \frac{p_i}{p_i-1}$, for all $i = 1,...,N$.

Now, we introduce the following assumptions:

Assumptions (H_1):

– The expression

$$\|u\|_{W_0^{1,\overrightarrow{p}}(\Omega,\overrightarrow{\omega})} = \sum_{i=1}^{N} \|\partial_i u\|_{L^{p_i}(\Omega,\omega_i)} \tag{9}$$

is a norm defined on $W_0^{1,\overrightarrow{p}}(\Omega,\overrightarrow{\omega})$ and it is equivalent to the norm (8).

– There exist a weight function σ on Ω and a parameter q, $1 < q < \infty$, such that the Hardy inequality

$$\Big(\int_\Omega |u|^q \sigma dx\Big)^{\frac{1}{q}} \le C \sum_{i=1}^{N} \Big(\int_\Omega \Big|\frac{\partial u}{\partial x_i}\Big|^{p_i} w_i\Big)^{\frac{1}{p_i}} \tag{10}$$

holds for every $u \in W_0^{1,\overrightarrow{p}}(\Omega,\overrightarrow{\omega})$, where C is a positive constant independent of u.

– The embedding

$$W_0^{1,\overrightarrow{p}}(\Omega,\overrightarrow{\omega}) \hookrightarrow L^q(\Omega,\sigma) \tag{11}$$

expressed by (10) is compact.

Remark 2.1 Let us take $p_0 = p_1 = p_2 = ... = p_N = p$, $\omega_0(x) = \omega_1(x) = \omega_2(x) = ... = \omega_N(x) = [dist(x,\partial\Omega)]^\lambda$ and $\sigma(x) = [dist(x,\partial\Omega)]^\gamma$, $\lambda,\gamma \in \mathbb{R}$. In this case, the Hardy inequality reads

$$\Big(\int_\Omega |u|^q [dist(x,\partial\Omega)]^\gamma\Big)^{\frac{1}{q}} dx \le \sum_{i=1}^{N} \Big(\int_\Omega |\partial_i u|^p [dist(x,\partial\Omega)]^\lambda dx\Big)^{\frac{1}{p}}.$$

The imbedding $W_0^{1,p}(\Omega, dist(x,\partial\Omega)) \hookrightarrow L^q(\Omega, dist(x,\partial\Omega))$ is compact (see Example 1.5 in [10]) if and only if either:

i) $1 < p \le q < +\infty$, $\lambda < p-1$, $\frac{N}{q} - \frac{N}{p} + 1 \ge 0$, $\gamma \ge \lambda\frac{q}{p} - N + N\frac{q}{p} - q$ or

ii) $1 \le q < p < +\infty$, $\lambda < p-1$, $\gamma \ge \lambda\frac{q}{p} - 1 + \frac{q}{p} - q$.

Similarly, in the isotropic case, see [1], we can construct an isometric from $W_0^{1,\overrightarrow{p}}(\Omega,\overrightarrow{\omega})$ in $\prod_{i=1}^{N} L^{p_i}(\Omega,\omega_i)$ which implies with (7) that the space $W_0^{1,\overrightarrow{p}}(\Omega,\overrightarrow{\omega})$ is a reflexive and separable Banach space. Moreover, we consider $\mathcal{T}_0^{1,\overrightarrow{p}}(\Omega,\overrightarrow{\omega}) = \{u \text{ measurable in } \Omega : T_k(u) \in W_0^{1,\overrightarrow{p}}(\Omega,\overrightarrow{\omega}), \forall k > 0\}$.

3 Mains Results

Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$ with the Lipschitz continuous boundary $\partial\Omega$. The functions a: $\Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}^N$ such that $a(x,s,\xi) = \Big(a_1(x,s,\xi), ..., a_N(x,s,\xi)\Big)$ and g_i: $\Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ with a_i and g_i are Carathéodory functions satisfying the following assumptions for all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N, \xi^{'} \in \mathbb{R}^N$ and a. e. in Ω :

Assumptions H_2:

$$\sum_{i=1}^{N} a_i(x,s,\xi)\xi_i \geq \alpha \sum_{i=1}^{N} \omega_i |\xi_i|^{p_i}, \tag{12}$$

$$|a_i(x,s,\xi)| \leq \beta\omega_i^{\frac{1}{p_i}}[j_i(x) + \sigma^{\frac{1}{p_i'}}|s|^{\frac{q}{p_i'}} + \omega^{\frac{1}{p_i'}}|\xi_i|^{p_i-1}], \tag{13}$$

$$(a_i(x,s,\xi) - a_i(x,s,\xi^{'}))(\xi_i - \xi_i^{'}) > 0 \quad \text{for } \xi_i \neq \xi_i^{'}, \tag{14}$$

where α, β are some positive constants, j_i is a positive function in $L^{p_i^{'}}(\Omega)$.

Assumptions H_3:

$$|g_i(x,s,\xi)| \leq \gamma(x) + \rho(s)\omega_i|\xi_i|^{p_i} \ \forall i = 1, ..., N, \tag{15}$$

where γ is a positive function in $L^1(\Omega)$ and $\rho : \mathbb{R} \to \mathbb{R}^+$ is a continuous positive function in $L^1(\mathbb{R})$.

Moreover, we suppose that

$$f \in L^1(\Omega). \tag{16}$$

Let us define the convex set $K_\psi(\Omega, \overrightarrow{\omega}) = \{u \in W_0^{1,\overrightarrow{p}}(\Omega, \overrightarrow{\omega}),\ u \geq \psi \ a.e. \text{ in } \Omega\}$, where ψ is a measurable function with values in $\overline{\mathbb{R}}$ such that

$$\psi^+ \in W_0^{1,\overrightarrow{p}}(\Omega, \overrightarrow{\omega}) \cap L^\infty(\Omega). \tag{17}$$

3.1 Some technical lemmas

The following lemma generalizes to the anisotropic case the analogous Lemma 5 in [9]. We use the method of [7] and [9].

Lemma 3.1 *Assume that (12)-(14) hold and let $(u_n)_n$ be a sequence in $W_0^{1,\overrightarrow{p}}(\Omega, \overrightarrow{\omega})$ such that $u_n \rightharpoonup u$ in $W_0^{1,\overrightarrow{p}}(\Omega, \overrightarrow{\omega})$ and $\lim\limits_{n\to+\infty} \int_\Omega \Big(a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)\Big)\nabla(u_n - u)$ $= 0$. Then $u_n \to u$ strongly in $W_0^{1,\overrightarrow{p}}(\Omega, \overrightarrow{\omega})$ for a subsequence.*

Definition 3.1 A function u is an entropy solution for problem (1) if it satisfies (6).

Theorem 3.1 *Assume that (12)-(17) hold. Then there exists at least one entropy solution in the sense of the definition (3.1) of problem (1).*

Proof of Theorem 3.1.
The proof of this theorem is done in four steps.
Step 1 : Approximate problems.
We consider the following approximate problems

$$\begin{cases} u_n \in K_\psi(\Omega, \overrightarrow{\omega}). \\ \displaystyle\int_\Omega a(x,u_n,\nabla u_n)\nabla(u_n - v) + \sum_{i=1}^N \int_\Omega g_i^n(x,u_n,\nabla u_n)(u_n - v) \le \int_\Omega f_n(u_n - v), \\ \forall v \in K_\psi(\Omega, \overrightarrow{\omega}), \end{cases} \tag{18}$$

where $g_i^n(x,s,\xi) = \frac{g_i(x,s,\xi)}{1+\frac{1}{n}|g_i(x,s,\xi)|} T_{\frac{1}{n}}(\sigma^{\frac{1}{q}}(x))$ and $f_n(x) = \frac{f(x)}{1+\frac{1}{n}|f(x)|}$. We have $|g_i^n(x,s,\xi)| \le |g_i(x,s,\xi)|$, $|g_i^n(x,s,\xi)| \le n$, $|g_i^n(x,u,\nabla u)| \le n^2 \sigma^{\frac{1}{q}}(x)$, $|f_n(x)| \le |f(x)|$ and $|f_n(x)| \le n$.
For all u and v in $W_0^{1,\overrightarrow{p}}(\Omega, \overrightarrow{\omega})$, we have

$$\begin{aligned} \Big|\int_\Omega g_i^n(x,u,\nabla u) v dx\Big| &\le \Big(\int_\Omega |g_i^n(x,u,\nabla u)|^{q'} \sigma^{\frac{-q'}{q}} dx\Big)^{\frac{1}{q'}} \Big(\int_\Omega |v|^q \sigma dx\Big)^{\frac{1}{q}} \\ &\le n^2 \Big(\int_\Omega \sigma^{\frac{q'}{q}} \sigma^{\frac{-q'}{q}} dx\Big)^{\frac{1}{q'}} \|v\|_{L^q(\Omega,\sigma)} \\ &\le C_n \|v\|_{W_0^{1,\overrightarrow{p}}(\Omega,\overrightarrow{\omega})}. \end{aligned}$$

Proposition 3.1 *Under the conditions (12)-(17), there exists at least one solution of the problem (18).*

Proof of Proposition 3.1.
Thanks to the Leray-Lions theorem and Theorem 8.2 from Chapter 2 in [14], there exists at least one solution to problem (18).
Step 2 : A priori estimate.

Proposition 3.2 *Assume that (12)- (17) hold and if u_n is a solution of the approximate problem (18), then there exists a constant C such that*

$$\sum_{i=1}^N \int_\Omega |\partial_i T_k(u_n)|^{p_i} \omega_i \le Ck \quad \forall k > 0.$$

Proof: Let $v = u_n - \eta \exp(G(u_n)) T_k(u_n^+ - \psi^+)$, where $G(s) = \int_0^s \frac{\rho(t)}{\alpha} dt$ and $\eta \ge 0$. Since $v \in W_0^{1,\overrightarrow{p}}(\Omega, \overrightarrow{\omega})$ and for all η small enough, we have $v \in K_\psi(\Omega, \overrightarrow{\omega})$. We take v as a test function in problem (18), thanks to (12) and (15), we obtain

$$\begin{aligned} \sum_{i=1}^N \int_\Omega a_i(x,u_n,\nabla u_n) \exp(G(u_n)) \partial_i T_k(u_n^+ - \psi^+) &\le (\|f\|_{L^1(\Omega)} + \|\gamma\|_{L^1(\Omega)}) \exp(\frac{\|\rho\|_{L^1(\mathbb{R})}}{\alpha}) k \\ &\le Ck. \end{aligned}$$

By (12) and Young's inequality, we have

$$\sum_{i=1}^N \int_{\{|u_n^+ - \psi^+| \le k\}} |\partial_i u_n^+|^{p_i} \omega_i dx \le C'k \quad \forall k > 0. \tag{19}$$

Since $\{x \in \Omega, \ |u_n^+| \le k\} \subset \{x \in \Omega, \ |u_n^+ - \psi^+| \le k + \|\psi^+\|_\infty\}$, we have

$$\sum_{i=1}^N \int_\Omega |\partial_i T_k(u_n^+)|^{p_i} \omega_i dx = \sum_{i=1}^N \int_{\{|u_n^+| \le k\}} |\partial_i u_n^+|^{p_i} \omega_i dx$$

$$\leq \sum_{i=1}^{N} \int_{\{|u_n^+ - \psi^+| \leq k + \|\psi^+\|_\infty\}} |\partial_i u_n^+|^{p_i} \omega_i dx.$$

This implies, by (19), that

$$\sum_{i=1}^{N} \int_{\Omega} |\partial_i T_k(u_n^+)|^{p_i} \omega_i dx \leq C' k, \ \forall k > 0. \tag{20}$$

Similarly, taking $v = u_n + \exp(-G(u_n)) T_k(u_n^-)$ as a test function in approximate problem (18), thanks to (12) and (15), we obtain

$$\sum_{i=1}^{N} \int_{\{u_n \leq 0\}} a_i(x, u_n, \nabla u_n) \exp(-G(u_n)) \partial_i T_k(u_n) \leq Ck.$$

By (12), we deduce that

$$\sum_{i=1}^{N} \int_{\{u_n \leq 0\}} |\partial_i T_k(u_n)|^{p_i} \omega_i \leq Ck. \tag{21}$$

Combining (20) and (21), we obtain $\displaystyle\sum_{i=1}^{N} \int_{\Omega} |\partial_i T_k(u_n)|^{p_i} \omega_i \leq Ck$. It yields

$$\|T_k(u_n)\|_{W_0^{1,\overrightarrow{p}}(\Omega, \overrightarrow{\omega})} \leq C k^{\frac{1}{p_-}}, \qquad \forall k > 1. \tag{22}$$

Step 3: Strong convergence of truncations.

Lemma 3.2 *There exist a measurable function u and a subsequence of u_n such that*

$$T_k(u_n) \to T_k(u) \text{ strongly in } W_0^{1,\overrightarrow{p}}(\Omega, \overrightarrow{\omega}).$$

Proof: By (22), the sequence $(T_k(u_n))_n$ is bounded in $W_0^{1,\overrightarrow{p}}(\Omega, \overrightarrow{\omega})$, there exists a subsequence $(T_k(u_n))_n$ such that $T_k(u_n)$ converges to v_k a. e. in Ω, weakly in $W_0^{1,\overrightarrow{p}}(\Omega, \overrightarrow{\omega})$ and strongly in $L^q(\Omega, \sigma)$ as n tends to $+\infty$. Since $(u_n)_n$ is a Cauchy sequence in measure in Ω, there exists a subsequence denoted by $(u_n)_n$ such that u_n converges to a measurable function u a. e. in Ω and

$$T_k(u_n) \rightharpoonup T_k(u) \text{ weakly in } W_0^{1,\overrightarrow{p}}(\Omega, \overrightarrow{\omega}) \text{ and a. e. in } \Omega, \quad \forall k > 0. \tag{23}$$

Now, we prove that

$$\lim_{n \to \infty} \sum_{i=1}^{N} \int_{\Omega} \Big(a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u)) \Big) \Big(\partial_i T_k(u_n) - \partial_i T_k(u) \Big) = 0. \tag{24}$$

Let us take $v = u_n + \exp(-G(u_n)) T_1(u_n - T_m(u_n))^-$ in approximate problem (18), by (12) and (15), we have

$$\sum_{i=1}^{N} \int_{\{-(m+1) \leq u_n \leq -m\}} a_i(x, u_n, \nabla u_n) \exp(-G(u_n)) \partial_i u_n$$

$$\leq -\int_\Omega f_n \exp(-G(u_n))T_1(u_n - T_m(u_n))^- + \sum_{i=1}^N \int_\Omega \gamma(x)\exp(-G(u_n))T_1(u_n - T_m(u_n))^-. \tag{25}$$

By Lebesgue's theorem, we have the right-hand side in (25) tends to zero as n and m tend to ∞. Then, we get

$$\lim_{m\to\infty}\limsup_{n\to\infty}\sum_{i=1}^N \int_{\{-(m+1)\leq u_n\leq -m\}} a_i(x,u_n,\nabla u_n)\partial_i u_n = 0. \tag{26}$$

Similarly, taking $v = u_n - \eta\exp(G(u_n))T_1(u_n - T_m(u_n))^+$ as a test function in approximate problem (18), we get

$$\lim_{m\to\infty}\limsup_{n\to\infty}\sum_{i=1}^N \int_{\{m\leq u_n\leq m+1\}} a_i(x,u_n,\nabla u_n)\partial_i u_n = 0. \tag{27}$$

We consider the following function of one real variable:

$$h_m(s) = \begin{cases} 1, & \text{if } |s|\leq m, \\ 0, & \text{if } |s|\geq m+1, \\ m+1-|s|, & \text{if } m\leq |s|\leq m+1, \end{cases}$$

where $m > k$. Let $\varphi = u_n - \eta\exp(G(u_n))(T_k(u_n) - T_k(u))^+ h_m(u_n)$ be a test function in approximate problem (18), using (12) and (15), we get

$$\begin{aligned}\sum_{i=1}^N \int_\Omega a_i(x,u_n,\nabla u_n)\exp(G(u_n))\partial_i(T_k(u_n) - T_k(u))^+ h_m(u_n)&\\ +\sum_{i=1}^N \int_\Omega a_i(x,u_n,\nabla u_n)\exp(G(u_n))(T_k(u_n) - T_k(u))^+\partial_i u_n h_m'(u_n)&\\ \leq \sum_{i=1}^N \int_\Omega \gamma(x)\exp(G(u_n))(T_k(u_n) - T_k(u))^+ h_m(u_n)&\\ +\int_\Omega f_n\exp(G(u_n))(T_k(u_n) - T_k(u))^+ h_m(u_n).&\end{aligned}$$

This implies that

$$\begin{aligned}\sum_{i=1}^N \int_\Omega a_i(x,u_n,\nabla u_n)\exp(G(u_n))\partial_i(T_k(u_n) - T_k(u))^+ h_m(u_n)&\\ \leq \sum_{i=1}^N \int_{\{m\leq u_n\leq m+1\}} a_i(x,u_n,\nabla u_n)\exp(G(u_n))(T_k(u_n) - T_k(u))^+\partial_i u_n&\\ +\sum_{i=1}^N \int_\Omega \gamma(x)\exp(G(u_n))(T_k(u_n) - T_k(u))^+ h_m(u_n)&\\ +\int_\Omega f_n\exp(G(u_n))(T_k(u_n) - T_k(u))^+ h_m(u_n).&\end{aligned}$$

Thanks to Lebesgue's theorem and (27), we obtain

$$\lim_{m\to\infty}\lim_{n\to\infty}\sum_{i=1}^N \int_\Omega a_i(x,u_n,\nabla u_n)\exp(G(u_n))\partial_i(T_k(u_n) - T_k(u))^+ h_m(u_n) \leq 0,$$

which implies that

$$\lim_{m\to\infty}\lim_{n\to\infty}\sum_{i=1}^{N}\int_{\{T_k(u_n)-T_k(u)\geq 0,|u_n|\leq k\}} a_i(x,u_n,\nabla u_n)\exp(G(u_n))\partial_i(T_k(u_n)-T_k(u))h_m(u_n)$$

$$-\lim_{m\to\infty}\lim_{n\to\infty}\sum_{i=1}^{N}\int_{\{T_k(u_n)-T_k(u)\geq 0,|u_n|>k\}} a_i(x,u_n,\nabla u_n)\exp(G(u_n))\partial_i(T_k(u))^{+}h_m(u_n)\ \leq 0,$$

since $h_m(u_n)=0$ if $|u_n|>m+1$, we have

$$\sum_{i=1}^{N}\int_{\{T_k(u_n)-T_k(u)\geq 0,|u_n|>k\}} a_i(x,u_n,\nabla u_n)\exp(G(u_n))\partial_i(T_k(u))^{+}h_m(u_n)$$

$$=\sum_{i=1}^{N}\int_{\{T_k(u_n)-T_k(u)\geq 0,|u_n|>k\}} a_i(x,T_{m+1}(u_n),\nabla T_{m+1}(u_n))\exp(G(u_n))\partial_i(T_k(u))^{+}h_m(u_n).$$

By (13) and (22), we have $a_i(x,T_{m+1}(u_n),\nabla T_{m+1}(u_n))\rightharpoonup X_m^i$ in $L^{p_i'}(\Omega,\omega_i^*)$. It yields

$$\lim_{m,n\to\infty}\sum_{i=1}^{N}\int_{\{T_k(u_n)-T_k(u)\geq 0,|u_n|>k\}} a_i(x,T_{m+1}(u_n),\nabla T_{m+1}(u_n))\exp(G(u_n))\partial_i(T_k(u))^{+}h_m(u_n)$$

$$=\lim_{m\to\infty}\sum_{i=1}^{N}\int_{\{|u|>k\}} X_m^i\exp(G(u))\partial_i T_k(u)h_m(u)=0.$$

Using $a_i(x,T_k(u_n),\nabla T_k(u_n))h_m(u_n)\to a_i(x,T_k(u),\nabla T_k(u))h_m(u)$ a. e. in Ω, we see that the sequence $\Big(a_i(x,T_k(u_n),\nabla T_k(u_n))h_m(u_n)\Big)_n$ is equi-integrable in $L^{p_i'}(\Omega,\omega_i^*)$ and Vitali's theorem implies that

$$a_i(x,T_k(u_n),\nabla T_k(u_n))h_m(u_n)\to a_i(x,T_k(u),\nabla T_k(u))h_m(u)\ in\ L^{p_i'}(\Omega,\omega_i^*).$$

Since $\partial_i T_k(u_n)\rightharpoonup\partial_i T_k(u))$ weakly in $L^{p_i}(\Omega,\omega_i)$, we get

$$\lim_{n\to\infty}\sum_{i=1}^{N}\int_{\{T_k(u_n)-T_k(u)\geq 0\}} a_i(x,T_k(u_n),\nabla T_k(u))\exp(G(u_n))\partial_i(T_k(u_n)-T_k(u))h_m(u_n)=0,$$

thus we conclude that

$$\lim_{m\to\infty}\lim_{n\to\infty}\sum_{i=1}^{N}\int_{\{T_k(u_n)-T_k(u)\geq 0\}}\Big(a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u))\Big)\times$$
$$\partial_i(T_k(u_n)-T_k(u))h_m(u_n)=0. \tag{28}$$

Similarly, we take $\varphi=u_n+\exp(-G(u_n))(T_k(u_n)-T_k(u))^{-}h_m(u_n)$ as a test function in approximating problem (18), we obtain

$$\lim_{m\to\infty}\lim_{n\to\infty}\sum_{i=1}^{N}\int_{\{T_k(u_n)-T_k(u)\leq 0\}}\Big(a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u))\Big)\times$$
$$\partial_i(T_k(u_n)-T_k(u))h_m(u_n)=0. \tag{29}$$

Combining (28) and (29), we deduce that

$$\lim_{m\to\infty}\lim_{n\to\infty}\sum_{i=1}^{N}\int_{\Omega}\Big(a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u))\Big)$$
$$\partial_i(T_k(u_n)-T_k(u))h_m(u_n)=0. \tag{30}$$

Let $\varphi = u_n + \exp(-G(u_n))T_k(u_n)^-(1-h_m(u_n))$ be a test function in approximate problem (18) and using (13) and (15), we get

$$\sum_{i=1}^{N}\int_{\{u_n\le 0\}} a_i(x,u_n,\nabla u_n)\exp(-G(u_n))\partial_i T_k(u_n)(1-h_m(u_n))$$
$$\le -\sum_{i=1}^{N}\int_{\{-(j+1)\le u_n\le -j\}} a_i(x,u_n,\nabla u_n)\exp(-G(u_n))T_k(u_n)\partial_i u_n$$
$$+\sum_{i=1}^{N}\int_{\Omega}\gamma(x)\exp(-G(u_n))T_k(u_n)^-(1-h_m(u_n))$$
$$-\sum_{i=1}^{N}\int_{\Omega} f_n(x)\exp(-G(u_n))T_k(u_n)^-(1-h_m(u_n)).$$

In view of (26) and Lebesgue's theorem, the integrals in the righthand side converge to zero as n and m tend to infinity. Then

$$\lim_{m\to\infty}\lim_{n\to\infty}\sum_{i=1}^{N}\int_{\{u_n\le 0\}} a_i(x,u_n,\nabla u_n)\partial_i T_k(u_n)(1-h_m(u_n)) = 0. \tag{31}$$

On the other hand, we take $\varphi = u_n - \eta\exp(G(u_n))T_k(u_n^+ - \psi^+)(1-h_m(u_n))$ as a test function in approximate problem (18) and using (13) and (15), we get

$$\sum_{i=1}^{N}\int_{\Omega} a_i(x,u_n,\nabla u_n)\exp(G(u_n))\partial_i T_k(u_n^+ - \psi^+)(1-h_m(u_n))$$
$$\le \sum_{i=1}^{N}\int_{\{-(j+1)\le u_n\le -j\}} a_i(x,u_n,\nabla u_n)\exp(G(u_n))T_k((u_n)^+ - \psi^+)\partial_i u_n$$
$$+\sum_{i=1}^{N}\int_{\Omega}\gamma(x)\exp(G(u_n))T_k(u_n^+ - \psi^+)(1-h_m(u_n))$$
$$+\sum_{i=1}^{N}\int_{\Omega} f_n(x)\exp(G(u_n))T_k(u_n^+ - \psi^+)(1-h_m(u_n)).$$

By Lebesgue's theorem and (26), we deduce that

$$\sum_{i=1}^{N}\int_{\{|u_n^+ - \psi^+|\le k\}} a_i(x,u_n,\nabla u_n)\exp(G(u_n))\partial_i u_n^+(1-h_m(u_n))$$
$$\le \sum_{i=1}^{N}\int_{\{|u_n^+ - \psi^+|\le k\}} a_i(x,u_n,\nabla u_n)\partial_i\psi^+(1-h_m(u_n)) + \varepsilon_1(n,m). \tag{32}$$

Thanks to (13) and Young's inequality, we have

$$\sum_{i=1}^{N}\int_{\{|u_n^+ - \psi^+|\le k\}} a_i(x,u_n,\nabla u_n)\exp(G(u_n))\partial_i u_n^+(1-h_m(u_n)) \le \varepsilon_2(n,m),$$

where $\varepsilon_1(n,m)$ and $\varepsilon_2(n,m)$ converge to zero as n and m tend to infinity. Since $\rho\in L^1(\mathbb{R})$, we have $\exp(G(u_n))$ is bounded. It yields

$$\sum_{i=1}^{N}\int_{\{|u_n^+ - \psi^+|\le k\}} a_i(x,u_n,\nabla u_n)\partial_i u_n^+(1-h_m(u_n)) \le \varepsilon_3(n,m).$$

Since $\{x \in \Omega,\ |u_n^+| \le k\} \subset \{x \in \Omega,\ |u_n^+ - \psi^+| \le k + \|\psi^+\|_{L^\infty(\Omega)}\}$, hence $\displaystyle\sum_{i=1}^N \int_{\{|u_n^+| \le k\}} a_i(x, u_n, \nabla u_n)\partial_i u_n^+(1-h_m(u_n)) \le \varepsilon_3(n,m)$, which implies that, for all $k > 0$,

$$\lim_{m\to\infty}\lim_{n\to\infty}\sum_{i=1}^N \int_{\{u_n \ge 0\}} a_i(x, T_k(u_n), \nabla T_k(u_n))\partial_i T_k(u_n)(1-h_m(u_n)) = 0. \tag{33}$$

Combining (31) and (33), we obtain

$$\lim_{m\to\infty}\lim_{n\to\infty}\sum_{i=1}^N \int_{\Omega} a_i(x, T_k(u_n), \nabla T_k(u_n))\partial_i T_k(u_n)(1-h_m(u_n)) = 0. \tag{34}$$

Moreover, we have

$$\begin{aligned}
&\sum_{i=1}^N \int_\Omega \Big(a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u))\Big)\Big(\partial_i T_k(u_n) - \partial_i T_k(u)\Big) = \\
&\ \sum_{i=1}^N \int_\Omega \Big(a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u))\Big)\Big(\partial_i T_k(u_n) - \partial_i T_k(u)\Big)h_m(u_n) \\
&\ + \sum_{i=1}^N \int_\Omega a_i(x, T_k(u_n), \nabla T_k(u_n))\partial_i T_k(u_n)(1-h_m(u_n)) \\
&\ - \sum_{i=1}^N \int_\Omega a_i(x, T_k(u_n), \nabla T_k(u_n))\partial_i T_k(u)(1-h_m(u_n)) \\
&\ - \sum_{i=1}^N \int_\Omega a_i(x, T_k(u_n), \nabla T_k(u))(\partial_i T_k(u_n) - \partial_i T_k(u))(1-h_m(u_n)).
\end{aligned}$$

By (30) and (33), the first and the second integrals of the right-hand side converge to zero as $n, m \to +\infty$. Since $\Big(a_i(x, T_k(u_n), \nabla T_k(u_n))\Big)_n$ is bounded in $L^{p_i'}(\Omega, \omega_i^*)$ and $\partial_i T_k(u)(1-h_m(u_n))$ converges to zero in $L^{p_i}(\Omega, \omega_i)$, the third integral converges to zero. So the fourth integral converges to zero while $\partial_i T_k(u_n) \rightharpoonup \partial_i T_k(u)$ weakly in $L^{p_i}(\Omega, \omega_i)$ and $a_i(x, T_k(u_n), \nabla T_k(u_n))(1-h_m(u_n))$ converges to $a_i(x, T_k(u), \nabla T_k(u))(1-h_m(u))$ strongly in $L^{p_i'}(\Omega, \omega_i^*)$. We conclude the proof of (24).
Using (23), (24) and Lemma 3.1, we deduce

$$T_k(u_n) \to T_k(u) \text{ strongly in } W_0^{1,\overrightarrow{p}}(\Omega, \overrightarrow{\omega}) \text{ and a. e. in } \Omega, \quad \forall k > 0. \tag{35}$$

This implies that

$$\nabla u_n \to \nabla u \text{ a. e. in } \Omega, \tag{36}$$

which gives

$$a_i(x, u_n, \nabla u_n) \rightharpoonup a_i(x, u, \nabla u) \text{ in } L^{p_i'}(\Omega, \omega_i^*). \tag{37}$$

Step 4: Equi integrability of the non linearity sequence.
We shall prove that $g_i^n(x, u_n, \nabla u_n) \to g_i(x, u, \nabla u)$ in $L^1(\Omega)$.
We have $g_i^n(x, u_n, \nabla u_n) \to g_i(x, u, \nabla u)$ a. e. in Ω.
Let $v = u_n + \exp(-G(u_n))\displaystyle\int_{u_n}^0 \rho(\nu)\chi_{\{\nu < -h\}}d\nu$. Since $v \in K_\psi(\Omega, \overrightarrow{\omega})$, we take v as a test

function in approximate problem (18). Then, by (12) and (15), we have

$$\sum_{i=1}^{N}\int_\Omega a_i(x,u_n,\nabla u_n)\exp(-G(u_n))\partial_i u_n\rho(u_n)\chi_{\{u_n<-h\}}$$
$$\leq \sum_{i=1}^{N}\int_\Omega \gamma(x)\exp(-G(u_n))\int_{u_n}^{0}\rho(\nu)\chi_{\{\nu<-h\}}d\nu - \int_\Omega f_n\exp(-G(u_n))\int_{u_n}^{0}\rho(\nu)\chi_{\{\nu<-h\}}d\nu$$
$$\leq \exp\Big(\frac{\|\rho\|_{L^1(\mathbb{R})}}{\alpha}\Big)\Big(\int_{-\infty}^{-h}\rho(s)ds\Big)\Big(N\|\gamma\|_{L^1(\Omega)}+\|f\|_{L^1(\Omega)}\Big).$$

Using again (12), we obtain $\sum_{i=1}^{N}\int_\Omega \alpha|\partial_i u_n|^{p_i}\omega_i\rho(u_n)\chi_{\{u_n<-h\}} \leq c\int_{-\infty}^{-h}\rho(s)ds.$

Since $\rho\in L^1(\mathbb{R})$, we have

$$\lim_{h\to\infty}\sup_{n\in\mathbb{N}}\sum_{i=1}^{N}\int_{\{u_n<-h\}}\alpha|\partial_i u_n|^{p_i}\omega_i\rho(u_n)=0. \tag{38}$$

Let h be such that $h\geq \exp(G(u_n))\displaystyle\int_0^{+\infty}\rho(\nu)d\nu+\|\psi^+\|_{L^\infty(\Omega)}$ and we take $v=u_n-\exp(G(u_n))\displaystyle\int_0^{u_n}\rho(\nu)\chi_{\{\nu>h\}}d\nu$ as a test function in approximate problem (18). Then, similarly as in (38), we deduce that

$$\lim_{h\to\infty}\sup_{n\in\mathbb{N}}\sum_{i=1}^{N}\int_{\{u_n>h\}}\alpha|\partial_i u_n|^{p_i}\omega_i\rho(u_n)=0. \tag{39}$$

Combining (38) and (39), we deduce

$$\lim_{h\to\infty}\sup_{n\in\mathbb{N}}\sum_{i=1}^{N}\int_{\{|u_n|>h\}}\alpha|\partial_i u_n|^{p_i}\omega_i\rho(u_n)=0. \tag{40}$$

Using (35), (36), (40) and Vitali's theorem, we obtain

$$g_i^n(x,u_n,\nabla u_n)\to g_i(x,u,\nabla u) \text{ in } L^1(\Omega). \tag{41}$$

On the other hand, let $\varphi\in K_\psi\cap L^\infty(\Omega)$ and $v=u_n-T_k(u_n-\varphi)$ be a test function in approximate problem (18). We get

$$\begin{cases} u_n\in K_\psi. \\ \displaystyle\sum_{i=1}^{N}\int_\Omega a_i(x,u_n,\nabla u_n)\partial_i T_k(u_n-\varphi)+\sum_{i=1}^{N}\int_\Omega g_i^n(x,u_n,\nabla u_n)T_k(u_n-\varphi) \\ \qquad\qquad\displaystyle\leq \int_\Omega f_n T_k(u_n-\varphi), \\ \forall\varphi\in K_\psi\cap L^\infty(\Omega) \text{ and } \forall k>0, \end{cases} \tag{42}$$

Using (35), (37) and (41), we can pass to the limit in (42).

4 Example

Let us consider the following case:

$$a_i(x,s,\xi)=\omega_i|\xi_i|^{p_i-1}sign(\xi_i) \text{ and } g_i(x,s,\xi)=\frac{1}{1+s^2}\omega_i|\xi_i|^{p_i}.$$

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