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Mild Solutions for Multi-Term Time-Fractional Impulsive Differential Systems

Vikram Singh [1*] and Dwijendra N. Pandey [1]

[1] *Department of Mathematics, Indian Institute of Technology Roorkee Roorkee-247667, India*

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Abstract: In this paper, we study the existence and uniqueness of mild solutions for multi-term time-fractional differential systems with non-instantaneous impulses and finite delay. We use the tools of the Banach fixed point theorem and condensing map along with generalization of the semigroup theory for linear operators and fractional calculus to come up with a new set of sufficient conditions for the existence and uniqueness of the mild solutions. An illustration is provided to demonstrate the established results.

Keywords: *fractional calculus, generalized semigroup theory, multi-term time-fractional differential system, (β, γ_j)-resolvent family, non-instantaneous impulses.*

Mathematics Subject Classification (2010): 34A08, 34G20, 35R12, 26A33, 34A12, 34A37.

1 Introduction

During the last few decades, the fractional differential equations (FDEs) including Riemann-Liouville and Caputo derivatives have attracted the interest of many researchers, motivated by demonstrated applications in widespread areas of science and engineering such as models of medicine (modeling of human tissue under mechanical loads), electrical engineering(transmission of ultrasound waves), biochemistry (modeling of proteins and polymers) etc. In addition, due to the memory and hereditary properties of the materials and processes, in some areas of science such as identification systems, signal processing, robotics or control theory, the fractional differential operators

* Corresponding author: mailto:vikramiitr1@gmail.com

seem more appropriate in modeling than the classical integer operators. For fundamental certainties regarding to fractional systems, one can make reference to the papers [6, 9, 14, 19–21, 25, 26], the monographs [10, 16, 24] and references therein. Moreover, fractional differential systems with delay are used frequently in many fields such as 3-D printing and oil drilling, modeling of equations, panorama of natural phenomena and porous media. For more details, see the cited papers [1, 3].

On the other hand, the theory of fractional impulsive differential equations (in short, FIDEs) also has generated a great interest among the researchers, because many real world processes and phenomena which are effected by abrupt changes in the state at certain moments are naturally described by FIDEs. These changes occur due to disturbances, changing operational conditions and component failures of the state. For example, mechanical and biological models subject to shocks. Generally, the abrupt changes in the state for instant period in evolution process are formulated by impulsive differential equations. However, it is not necessary that the dynamical systems with evolutionary processes always be characterized by instantaneous impulses. For example, pharmacotherapy [23], in which the hemodynamic equilibrium of a person is considered. The initiation of the drugs in the bloodstream and the resultant absorption for the body are gradual and continuous processes. Therefore, instantaneous impulses failed to describe such processes. To characterize these type of situations Hernández and O′Regan [8] introduce a new case of impulsive actions, which are triggered abruptly at an arbitrary instant and their action remains for a finite time interval. Meanwhile, Pierri et al. [22] extended the results of [8] with an α-normed Banach space. For the general theory of impulsive differential equations, we refer to the monographs [4, 12], research papers [5, 11, 13, 15, 17, 18, 28] and references therein.

Indeed, in [9, 14, 19, 27], the authors have obtained the existence and uniqueness results without impulsive conditions, and in [20], Pardo studied weighted pseudo almost automorphic mild solutions for two-term time-fractional order differential equations. In [21], Pardo and Lizama studied a nonlinear multi-term time-differential system of the form

$$ {}^cD_t^{\gamma} y(t) + \sum_{j=1}^{d} \mu_j {}^cD_t^{\beta_j} y(t) = Ay(t) + f(t, y(t)), \quad \beta_j > 0,\, t \in [0,1], 0 < \gamma \leq 2, \tag{1} $$

$$ y(0) = 0, \qquad y'(0) = g(y), \tag{2} $$

where $A : \mathcal{D}(A) \subset \mathbb{X} \to \mathbb{X}$ is a closed linear operator and f and g are suitable functions. In the foregoing cases, the initial value problems were considered, but the study of existence of mild solutions for the system modeled as $(1)-(2)$ involving non-instantaneous impulses and delay was left open. Anticipating a wide interest in the problems modeled as the system $(3) - (5)$, this paper contributes to fill this important gap.

This paper is organized as follows. Section 2 is devoted to recall basics of fractional calculus and mild solution which will be employed to attain our mains outcomes. In Section 3, the existence and uniqueness results for the system $(3) - (5)$ are analyzed under the Banach and condensing map fixed point theorems. In Section 4, as a final point, an example is provided to show the feasibility of the theory discussed in this paper.

2 Problem Formulation

Let $\mathbb{X}$ be a Banach space. Let $\mathcal{L}(\mathbb{X})$ be the space of all bounded and linear operators on $\mathbb{X}$ equipped with the norm $\|\cdot\|_{\mathcal{L}}$. Let $\mathbb{R}$ and $\mathbb{N}$ stand for real numbers and natural numbers, respectively. For a linear operator A on $\mathbb{X}$, $\mathcal{R}(A), \mathcal{D}(A)$ and $\varrho(A)$ represent the range, domain and resolvent of A respectively. To facilitate the discussion due to delay, we use the space $\mathcal{PC}_0 := \mathcal{C}([-\tau, 0], \mathbb{X})$ formed by the continuous functions from $[-\tau, 0]$ to $\mathbb{X}$ equipped with the norm $\|y\|_{\mathcal{PC}_0} = \sup_{t\in[-\tau,0]}\{\|y(t)\|_{\mathbb{X}} : y \in \mathcal{PC}_0\}$. To study the impulsive forces, we define a space $\mathcal{PC}_T := \mathcal{PC}([-\tau, T], \mathbb{X}), 0 \le t \le T$ of all functions $y : [-\tau, T] \to \mathbb{X}$, which are continuous everywhere except the points $t_k \in (0, T), k = 1, 2, ..., m$, at which $y(t_k^+)$ and $y(t_k^-)$ exist and $y(t_k^-) = y(t_k)$. Obviously, $\mathcal{PC}_T$ is a Banach space equipped with the norm $\|y\|_{\mathcal{PC}_T} = \sup_{t\in[-\tau,T]}\{\|y(t)\|_{\mathbb{X}} : y \in \mathcal{PC}_T\}$.

In this paper, we study the existence and uniqueness of mild solutions for the following class of multi-term time-fractional differential equations with non-instantaneous impulses

$$
\begin{aligned}
{}^cD^{1+\beta}_{s_k} y(t) &+ \sum_{j=1}^{n}\alpha_j {}^cD^{\gamma_j}_{s_k} y(t) \\
&= Ay(t) + F\left(t, y_t, \int_0^t \mathfrak{K}(t,s)(y_s)ds\right), \quad t \in \cup_{k=0}^m (s_k, t_{k+1}], \qquad (3)
\end{aligned}
$$

$$y(t) = G_k(t, y_t), \quad y'(t) = H_k(t, y_t), \qquad t \in \cup_{k=1}^m (t_k, s_k], \qquad (4)$$

$$y(t) + g_1(y) = \phi(t), \quad y'(t) + g_2(y) = \varphi(t), \quad t \in [-\tau, 0], \qquad (5)$$

where $A : \mathcal{D}(A) \subset \mathbb{X} \to \mathbb{X}$ is a closed linear operator. ${}^cD^{\eta}_{s_k}$ stands for the Caputo derivative of order $\eta > 0$ and $\mathcal{I} = [0, T] = \{0\} \cup_{k=0}^m (s_k, t_{k+1}] \cup_{k=1}^m (t_k, s_k], T < \infty$ such that $0 = s_0 < t_1 \le s_1 \le t_2 < \cdots < t_m \le s_m \le t_{m+1} = T$ are prefix numbers. All γ_j, $j = 1, 2, 3...n$, are positive real numbers such that $0 < \beta \le \gamma_n \le \cdots \le \gamma_1 \le 1$. G_k and H_k are continuous functions from $\cup_{k=1}^m (t_k, s_k] \times \mathcal{PC}_0$ into $\mathbb{X}$ for all $k = 1, 2, ..., m$. $F : \mathcal{I} \times \mathcal{PC}_0 \times \mathcal{PC}_0 \to \mathbb{X}$ is a suitable function. The history function $y_t : [-\tau, 0] \to \mathbb{X}$ is the element of $\mathcal{PC}_0$ characterized by $y_t(\theta) = y(t + \theta), \theta \in [-\tau, 0]$ and also $\phi, \varphi \in \mathcal{PC}_0$. y' denotes the usual derivative of y with respect to t. $\mathfrak{K}$ is a positive and continuous operator on $\Omega := \{(t, s) \in \mathbb{R}^2 : 0 \le s \le t < T\}$ and $k^0 = \sup \int_0^t \mathfrak{K}(t, s)ds < \infty$. Here by non-instantaneous, we mean that the impulses start abruptly at t_k and their effect will continue on the interval $[t_k, s_k]$ for $k = 1, 2, 3, ..., m$.

Now, we recall some definitions and basic results on fractional calculus (for more details, see [24]). Define $g_\eta(t)$ for $\eta > 0$ by

$$g_\eta(t) = \begin{cases} \frac{1}{\Gamma(\eta)} t^{\eta-1}, & t > 0; \\ 0, & t \le 0, \end{cases}$$

where Γ denotes the gamma function. Let $(X * Y)(t)$ be the convolution of X and Y given by $(X * Y)(t) := \int_0^t X(t - s)Y(s)ds$.

Definition 2.1 The Riemann-Liouville fractional integral of a function $f \in L^1_{loc}(\mathbb{R}^+, \mathbb{X})$ of order $\eta > 0$ with the lower limit $a \ge 0$ is defined as follows

$$I_a^\eta f(t) = \int_a^t g_\eta(t - s)f(s)ds, \quad t > 0,$$

and $I_a^0 f(t) = f(t)$. This fractional integral satisfies the properties $I_0^\eta \circ I_0^b = I_0^{\eta+b}$ for $b > 0$ and $I_0^\eta f(t) = (g_\eta * f)(t)$.

Definition 2.2 [21] Let $\eta > 0$ be given and denote $m = \lceil \eta \rceil$. The Caputo fractional derivative of order $\eta > 0$ of a function $f \in \mathcal{C}^m([0,\infty), \mathbb{R})$ with the lower limit $a \geq 0$ is given by

$$^cD_a^\eta f(t) = I_a^{m-\eta} D_a^m f(t) = \int_a^t g_{m-\eta}(t-s)\frac{d^m}{dt^m} f(s)ds,$$

and $^cD_a^0 f(t) = f(t)$. In addition, we have $^cD_0^\eta f(t) = (g_{m-\eta} * D^m f)(t)$.

To give an appropriate representation of mild solution in terms of certain family of bounded and linear operators, we define the following family of operators.

Definition 2.3 [21] Let A be a closed linear operator on a Banach space $\mathbb{X}$ with the domain $\mathcal{D}(A)$ and $\beta > 0, \gamma_j, \alpha_j$ be the real positive numbers. Then A is called the generator of a $(\beta, \gamma_j)-$ resolvent family if there exists $\omega > 0$ and a strongly continuous function $\mathcal{S}_{\beta,\gamma_j} : \mathbb{R}^+ \to \mathcal{L}(\mathbb{X})$ such that $\{\lambda^{\beta+1} + \sum_{j=1}^n \alpha_j \lambda^{\gamma_j} : \operatorname{Re}\lambda > \omega\} \subset \varrho(A)$ and

$$\lambda^\beta\bigg(\lambda^{\beta+1} + \sum_{j=1}^n \alpha_j\lambda^{\gamma_j} - A\bigg)^{-1} y = \int_0^\infty e^{-\lambda t}\mathcal{S}_{\beta,\gamma_j}(t)ydt, \quad \operatorname{Re}\lambda > \omega, y \in \mathbb{X}. \tag{6}$$

The following result provides the existence of $(\beta, \gamma_j)-$ resolvent family under some suitable conditions.

Theorem 2.1 *[21] Let $0 < \beta \leq \gamma_i \leq, \cdots, \leq \gamma_1 \leq 1$ and $\alpha_j \geq 0$ be given and let A be a generator of a bounded and strongly continuous cosine family $\{C(t)\}_{t\in\mathbb{R}}$. Then A generates a bounded $(\beta, \gamma_j)-$ resolvent family $\{\mathcal{S}_{\beta,\gamma_j}(t)\}_{t\geq 0}$.*

Motivated by [21], we define a mild solution for the system $(3)-(5)$ as follows.

Definition 2.4 A function $y \in \mathcal{PC}_T$ is called a mild solution of the system $(3)-(5)$, if $y(t) = \phi(t) - g_1(y), y'(t) = \varphi(t) - g_2(y)$ for $[-\tau, 0]$ and $y(t) = G_k(t, y_t), y'(t) = H_k(t, y_t)$ for $t \in \cup_{k=1}^m (t_k, s_k]$ and satisfy the following integral equations

$$y(t) = \begin{cases} \mathcal{S}_{\beta,\gamma_j}(t)[\phi(0) - g_1(y)] + \int_0^t \mathcal{S}_{\beta,\gamma_j}(s)[\varphi(0) - g_2(y)]ds \\ +\displaystyle\sum_{j=1}^n \alpha_j \int_0^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)}\mathcal{S}_{\beta,\gamma_j}(s)[\phi(0) - g_1(y)]ds \\ + \int_0^t (g_\beta * \mathcal{S}_{\beta,\gamma_j})(t-s)F(s, y_s, K(y_s))ds, \qquad t \in [0, t_1]; \\ \mathcal{S}_{\beta,\gamma_j}(t-s_k)G_k(s_k, y_{s_k}) + \int_{s_k}^t \mathcal{S}_{\beta,\gamma_j}(s-s_k)H_k(s_k, y_{s_k})ds \\ +\displaystyle\sum_{j=1}^n \alpha_j \int_{s_k}^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)}\mathcal{S}_{\beta,\gamma_j}(s-s_k)G_k(s_k, y_{s_k})ds \\ + \int_{s_k}^t (g_\beta * \mathcal{S}_{\beta,\gamma_j})(t-s)F(s, y_s, K(y_s))ds, \qquad t \in \cup_{k=1}^m (s_k, t_{k+1}], \end{cases} \tag{7}$$

where $K(y_s) = \int_0^s \mathfrak{K}(s,\xi)(y_\xi)d\xi$.

Theorem 2.2 *[7, Condensing theorem] Let $\mathcal{M}$ be a closed, bounded and convex subset of a Banach space $\mathbb{X}$ and assume that $Q : \mathcal{M} \to \mathcal{M}$ is a condensing map. Then Q admits a fixed point in $\mathcal{M}$.*

3 Main Results

In this section, we establish the existence and uniqueness of mild solution for the system (3) − (5). We denote $S_0 = \sup_{t\in[0,T]} \|\mathcal{S}_{\beta,\gamma_j}(t)\|_{\mathcal{L}}$. In order to establish the existence and uniqueness result by the Banach fixed point theorem, we consider the following assumptions:

(A_1) There exist positive constants μ_F and μ_F^0 such that

$$\|F(t,\psi_1,\chi_1) - F(t,\psi_2,\chi_2)\|_{\mathbb{X}} \leq \mu_F\|\psi_1-\psi_2\|_{\mathcal{PC}_0} + \mu_F^0\|\chi_1-\chi_2\|_{\mathcal{PC}_0},$$

where $\psi_i, \chi_i \in \mathcal{PC}_0$, $i = 1, 2$.

(A_2) There exist positive constants μ_G, μ_{g_i} and μ_H such that

$$\|G_k(t,\psi) - G_k(t,\chi)\|_{\mathbb{X}} \leq \mu_G\|\psi-\chi\|_{\mathcal{PC}_0},\ \|H_k(t,\psi) - H_k(t,\chi)\|_{\mathbb{X}} \leq \mu_H\|\psi-\chi\|_{\mathcal{PC}_0},$$
$$\|g_i(x) - g_i(y)\|_{\mathbb{X}} \leq \mu_{g_i}\|x-y\|_{\mathbb{X}},$$

for all $\psi, \chi \in \mathcal{PC}_0$, $x, y \in \mathbb{X}$, $i = 1, 2$ and $k = 1, 2, 3, \ldots, m$.

Theorem 3.1 *Assume that the assumptions $(A_1)-(A_2)$ are fulfilled, then the system* (3) − (5) *has a unique mild solution in $\mathcal{I}$ if $\Theta < 1$, where*

$$\Theta = \max\left[S_0 d + T_0 S_0 e + \sum_{j=1}^{n} \frac{\alpha_j S_0 d T_0^{1+\beta-\gamma_j}}{\Gamma(2+\beta-\gamma_j)} + \frac{S_0 T_0^{1+\beta}}{\Gamma(2+\beta)}[\mu_F + \mu_F^0 k^0], \mu_G\right],$$

where $d = \max\{\mu_{g_1}, \mu_G\}$, $e = \max\{\mu_{g_2}, \mu_H\}$ and $T_0 = \max\limits_{0\leq k\leq m} |t_{k+1} - s_k|$.

Proof. To transform the problem into a fixed point problem, we define an operator $Q : \mathcal{PC}_T \to \mathcal{PC}_T$ by $Qy(t) = \phi(t)$ for $t \in [-\tau, 0]$ and $Qy(t) = G_k(t, y_t)$ for all $t \in \cup_{k=1}^m (t_k, s_k]$, and

$$Qy(t) = \begin{cases} \mathcal{S}_{\beta,\gamma_j}(t)[\phi(0) - g_1(y)] + \int_0^t \mathcal{S}_{\beta,\gamma_j}(s)[\varphi(0) - g_2(y)]ds \\ + \sum\limits_{j=1}^{n} \alpha_j \displaystyle\int_0^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} \mathcal{S}_{\beta,\gamma_j}(s)[\phi(0) - g_1(y)]ds \\ + \int_0^t (g_\beta * \mathcal{S}_{\beta,\gamma_j})(t-s)F(s, y_s, K(y_s))ds, \qquad t \in [0, t_1]; \\ \mathcal{S}_{\beta,\gamma_j}(t-s_k)G_k(s_k, y_{s_k}) \\ + \int_{s_k}^t \mathcal{S}_{\beta,\gamma_j}(s-s_k)H_k(s_k, y_{s_k})ds \\ + \sum\limits_{j=1}^{n} \alpha_j \displaystyle\int_{s_k}^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} \mathcal{S}_{\beta,\gamma_j}(s-s_k)G_k(s_k, y_{s_k})ds \\ + \int_{s_k}^t (g_\beta * \mathcal{S}_{\beta,\gamma_j})(t-s)F(s, y_s, K(y_s))ds, \qquad t \in \cup_{k=1}^m (s_k, t_{k+1}]. \end{cases} \tag{8}$$

Let $x, y \in \mathcal{PC}_T$. For $t \in [0, t_1]$, we have

$$
\begin{aligned}
&\|Qx(t) - Qy(t)\|_{\mathbb{X}} \\
&\leq \|\mathcal{S}_{\beta,\gamma_j}(t)\|_{\mathcal{L}} \|g_1(x) - g_1(y)\|_{\mathbb{X}} + \int_0^t \|\mathcal{S}_{\beta,\gamma_j}(s)\|_{\mathcal{L}} \|g_2(x) - g_2(y)\|_{\mathbb{X}} ds \\
&\quad + \sum_{j=1}^{n} \alpha_j \int_0^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} \|\mathcal{S}_{\beta,\gamma_j}(s)\|_{\mathcal{L}} \|g_1(x) - g_1(y)\|_{\mathbb{X}} ds \\
&\quad + \int_0^t \|(g_\beta * \mathcal{S}_{\beta,\gamma_j})(t-s)\|_{\mathcal{L}} \|F(s, x_s, K(x_s)(s)) - F(s, y_s, K(y_s))\|_{\mathbb{X}} ds \\
&\leq \left[S_0 \mu_{g_1} + T_0 S_0 \mu_{g_2} + \sum_{j=1}^{n} \frac{\alpha_j S_0 \mu_{g_1} T_0^{1+\beta-\gamma_j}}{\Gamma(2+\beta-\gamma_j)} + \frac{S_0 T_0^{1+\beta}}{\Gamma(2+\beta)} [\mu_F + \mu_F^0 k^0] \right] \|x - y\|_{\mathcal{PC}_T}.
\end{aligned}
$$

For $t \in \cup_{k=1}^m (t_k, s_k]$, we get

$$
\|Qx(t) - Qy(t)\|_{\mathbb{X}} \leq \|G_k(s_k, x_{s_k}) - G_k(s_k, y_{s_k})\|_{\mathbb{X}} \leq \mu_G \|x - y\|_{\mathcal{PC}_T}, \quad k = 1, 2, 3, \ldots, m.
$$

Similarly, for $t \in \cup_{k=1}^m (s_k, t_{k+1}]$ we get

$$
\begin{aligned}
&\|Qx(t) - Qy(t)\|_{\mathbb{X}} \\
&\leq \|\mathcal{S}_{\beta,\gamma_j}(t - s_k)\|_{\mathcal{L}} \|G_k(s_k, x_{s_k}) - G_k(s_k, y_{s_k})\|_{\mathbb{X}} \\
&\quad + \int_{s_k}^t \|\mathcal{S}_{\beta,\gamma_j}(s - s_k)\|_{\mathcal{L}} \|H_k(s_k, x_{s_k}) - H_k(s_k, y_{s_k})\|_{\mathbb{X}} ds \\
&\quad + \sum_{j=1}^{n} \alpha_j \int_{s_k}^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} \|\mathcal{S}_{\beta,\gamma_j}(s - s_k)\|_{\mathcal{L}} \|G_k(s_k, x_{s_k}) - G_k(s_k, y_{s_k})\|_{\mathbb{X}} ds \\
&\quad + \int_{s_k}^t \|(g_\beta * \mathcal{S}_{\beta,\gamma_j})(t-s)\|_{\mathcal{L}} \|F(s, x_s, K(x_s)(s)) - F(s, y_s, K(y_s))\|_{\mathbb{X}} ds \\
&\leq \left[S_0 \mu_G + T_0 S_0 \mu_H + \sum_{j=1}^{n} \frac{\alpha_j S_0 \mu_G T_0^{1+\beta-\gamma_j}}{\Gamma(2+\beta-\gamma_j)} + \frac{S_0 T_0^{1+\beta}}{\Gamma(2+\beta)} [\mu_F + \mu_F^0 k^0] \right] \|x - y\|_{\mathcal{PC}_T}.
\end{aligned}
$$

Gathering the above results, we have $\|Qx - Qy\|_{\mathcal{PC}_T} \leq \Theta \|x - y\|_{\mathcal{PC}_T}$. Now, by the Banach contraction theorem the system $(3) - (5)$ has a unique mild solution.
In order to establish the existence results by virtue of the condensing map, we consider the following assumptions:

(A_3) The functions G_k, H_k, g_1 and g_2 are continuous functions and F is compact and continuous, and there exist positive constants ν_F, ν_F^0, ν_G, ν_H, ν_{g_1}, ν_{g_2} such that

$$
\begin{aligned}
\|F(t, \psi, \chi)\|_{\mathbb{X}} \leq \nu_F \|\psi\|_{\mathcal{PC}_0} + \nu_F^0 \|\chi\|_{\mathcal{PC}_0}, \quad \|g_i(x)\|_{\mathbb{X}} \leq \nu_{g_i} \|x\|_{\mathbb{X}}, \\
\|G_k(t, \psi)\|_{\mathbb{X}} \leq \nu_G \|\psi\|_{\mathcal{PC}_0}, \quad \|H_k(t, \psi)\|_{\mathbb{X}} \leq \nu_H \|\psi\|_{\mathcal{PC}_0}
\end{aligned}
$$

for all $x \in \mathbb{X}, \psi, \chi \in \mathcal{PC}_0$.

Theorem 3.2 *Assume that the assumptions $(A_2) - (A_3)$ are fulfilled, then the system $(3) - (5)$ has a mild solution in $\mathcal{I}$ if $\Delta < 1$, where*

$$
\Delta = \max \left[S_0 d + T_0 S_0 e + \sum_{j=1}^{n} \frac{\alpha_j S_0 d T_0^{1+\beta-\gamma_j}}{\Gamma(2+\beta-\gamma_j)}, \mu_G \right],
$$

where $d = \max\{\mu_{g_1}, \mu_G\}$, $e = \max\{\mu_{g_2}, \mu_H\}$.

Proof. Consider the operator $Q : \mathcal{PC}_T \to \mathcal{PC}_T$ defined in Theorem 3.1. We show that Q has a fixed point. It is easy to see that $Qy(t) \in \mathcal{PC}_T$. Let $\mathcal{B}_{r_0} := \{y \in \mathcal{PC}_T : \|y\|_{\mathcal{PC}_T} \leq r_0\}$, where

$$r_0 \geq \max\Bigg[S_0Y_1 + T_0S_0Z_1 + \sum_{j=1}^{n} \frac{\alpha_j S_0 Y_1 T_0^{1+\beta-\gamma_j}}{\Gamma(2+\beta-\gamma_j)},\ \nu_G r_0,\ S_0\nu_G r_0 + T_0 S_0 \nu_H r_0 + \sum_{j=1}^{n} \frac{\alpha_j S_0 \nu_G r_0 T_0^{1+\beta-\gamma_j}}{\Gamma(2+\beta-\gamma_j)}\Bigg] + \frac{S_0 T_0^{1+\beta}}{\Gamma(2+\beta)}[\nu_F + \nu_F^0 k^0] r_0, \tag{9}$$

where $Y_1 = \|\phi(0)\| + \nu_{g_1} r_0$, $Z_1 = \|\varphi(0)\| + \nu_{g_2} r_0$. It is clear that $\mathcal{B}_{r_0}$ is a closed, bounded and convex subset of $\mathcal{PC}_T$. Let $y \in \mathcal{B}_{r_0}$, then for $t \in [0, t_1]$, we have

$$\begin{aligned}
\|Qy(t)\|_{\mathbb{X}} \leq & \|\mathcal{S}_{\beta,\gamma_j}(t)\|_{\mathcal{L}}(\|\phi(0)\| + \|g_1(y)\|_{\mathbb{X}}) + \int_0^t \|\mathcal{S}_{\beta,\gamma_j}(s)\|_{\mathcal{L}}(\|\varphi(0)\| + \|g_2(y)\|_{\mathbb{X}})ds \\
& + \sum_{j=1}^{n} \alpha_j \int_0^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} \|\mathcal{S}_{\beta,\gamma_j}(s)\|_{\mathcal{L}}(\|\phi(0)\| + \|g(y)\|_{\mathbb{X}})ds \\
& + \int_0^t \|(g_\beta * \mathcal{S}_{\beta,\gamma_j})(t-s)\|_{\mathcal{L}} \|F(s, y_s, K(y_s))\|_{\mathbb{X}} ds \\
\leq & S_0Y_1 + T_0S_0Z_1 + \sum_{j=1}^{n} \frac{\alpha_j S_0 Y_1 T_0^{1+\beta-\gamma_j}}{\Gamma(2+\beta-\gamma_j)} + \frac{S_0 T_0^{1+\beta}}{\Gamma(2+\beta)}[\nu_F + \nu_F^0 k^0] r_0.
\end{aligned}$$

For $t \in \cup_{k=1}^m (t_k, s_k]$, we get

$$\|Qy(t)\|_{\mathbb{X}} \leq \|G_k(t, y_t)\|_{\mathbb{X}} \leq \nu_G r_0, \quad k = 1, 2, 3, \ldots, m.$$

Similarly, for $t \in \cup_{k=1}^m (s_k, t_{k+1}]$, we get

$$\begin{aligned}
\|Qy(t)\|_{\mathbb{X}} \leq & \|\mathcal{S}_{\beta,\gamma_j}(t-s_k)\|_{\mathcal{L}} \|G_k(s_k, y_{s_k})\|_{\mathbb{X}} + \int_{s_k}^t \|\mathcal{S}_{\beta,\gamma_j}(s-s_k)\|_{\mathcal{L}} \|H_k(s_k, y_{s_k})\|_{\mathbb{X}} ds \\
& + \sum_{j=1}^{n} \alpha_j \int_{s_k}^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} \|\mathcal{S}_{\beta,\gamma_j}(s-s_k)\|_{\mathcal{L}} \|G_k(s_k, y_{s_k})\|_{\mathbb{X}} ds \\
& + \int_{s_k}^t \|(g_\beta * \mathcal{S}_{\beta,\gamma_j})(t-s)\|_{\mathcal{L}} \|F(s, y_s, K(y_s))\|_{\mathbb{X}} ds \\
\leq & S_0 \nu_G r_0 + T_0 S_0 \nu_H r_0 + \sum_{j=1}^{n} \frac{\alpha_j S_0 \nu_G r_0 T_0^{1+\beta-\gamma_j}}{\Gamma(2+\beta-\gamma_j)} + \frac{S_0 T_0^{1+\beta}}{\Gamma(2+\beta)}[\nu_F + \nu_F^0 k^0] r_0.
\end{aligned}$$

We conclude by (9) that $\|Qy\|_{\mathcal{PC}_T} \leq r_0$. Thus we conclude that $Q(\mathcal{B}_{r_0}) \subseteq \mathcal{B}_{r_0}$. Next, we show that Q is a condensing operator. Let us decompose Q by $Q = Q_1 + Q_2$, where $Q_1 y(t) = G_k(t, y_t)$ for all $t \in \cup_{k=1}^m (t_k, s_k]$ and

$$Q_1 y(t) = \begin{cases}
\mathcal{S}_{\beta,\gamma_j}(t)[\phi(0) - g_1(y)] + \int_0^t \mathcal{S}_{\beta,\gamma_j}(s)[\varphi(0) - g_2(y)]ds \\
\quad + \sum_{j=1}^{n} \alpha_j \int_0^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} \mathcal{S}_{\beta,\gamma_j}(s)[\phi(0) - g_1(y)]ds, \quad t \in [0, t_1]; \\
\mathcal{S}_{\beta,\gamma_j}(t-s_k) G_k(s_k, y_{s_k}) + \int_{s_k}^t \mathcal{S}_{\beta,\gamma_j}(s-s_k) H_k(s_k, y_{s_k}) ds \\
\quad + \sum_{j=1}^{n} \alpha_j \int_{s_k}^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} \mathcal{S}_{\beta,\gamma_j}(s-s_k) G_k(s_k, y_{s_k}) ds,\ t \in \cup_{k=1}^m (s_k, t_{k+1}],
\end{cases} \tag{10}$$

and

$$Q_2y(t) = \begin{cases} \int_0^t (g_\beta * \mathcal{S}_{\beta,\gamma_j})(t-s)F(s, y_s, K(y_s))ds, & t \in [0, t_1]; \\ \int_{s_k}^t (g_\beta * \mathcal{S}_{\beta,\gamma_j})(t-s)F(s, y_s, K(y_s))ds, & t \in \cup_{k=1}^m (s_k, t_{k+1}]. \end{cases} \tag{11}$$

First, we show that Q_1 is continuous, so consider a sequence in $\mathcal{B}_{r_0}$ such that $y^n \to y \in \mathcal{B}_{r_0}$, then for $t \in [0, t_1]$, we get

$$\begin{aligned} \|Q_1y^n(t) - Q_1y(t)\|_{\mathbb{X}} \leq & S_0\|g_1(y^n) - g_1(y)\|_{\mathbb{X}} + S_0T_0\|g_2(y^n) - g_2(y)\|_{\mathbb{X}} \\ & + \sum_{j=1}^n \frac{\alpha_j S_0 T_0^{1+\beta-\gamma_j}}{\Gamma(2+\beta-\gamma_j)}\|g_1(y^n) - g_1(y)\|_{\mathbb{X}}. \end{aligned}$$

For $t \in \cup_{k=1}^m (s_k, t_{k+1}]$, we obtain

$$\begin{aligned} \|Q_1y^n(t) - Q_1y(t)\|_{\mathbb{X}} \leq & S_0\|G_k(s_k, y_{s_k}^n) - G_k(s_k, y_{s_k})\|_{\mathbb{X}} \\ & + S_0T_0\|H_k(s_k, y_{s_k}^n) - H_k(s_k, y_{s_k})\|_{\mathbb{X}} \\ & + \sum_{j=1}^n \frac{\alpha_j S_0 T_0^{1+\beta-\gamma_j}}{\Gamma(2+\beta-\gamma_j)}\|G_k(s_k, y_{s_k}^n) - G_k(s_k, y_{s_k})\|_{\mathbb{X}}. \end{aligned}$$

By continuity of G_k, H_k, g_1 and g_2, we have $\|Q_1y^n - Q_1y\|_{\mathcal{PC}_T} \to 0$ as $n \to \infty$. Hence Q_1 is continuous. Let $x, y \in \mathcal{PC}_T$. As we have done in Theorem 3.1 for $t \in [0, t_1]$, we have

$$\|Q_1x(t) - Q_1y(t)\|_{\mathbb{X}} \leq \left[S_0\mu_{g_1} + T_0S_0\mu_{g_2} + \sum_{j=1}^n \frac{\alpha_j S_0\mu_{g_1} T_0^{1+\beta-\gamma_j}}{\Gamma(2+\beta-\gamma_j)}\right]\|x - y\|_{\mathcal{PC}_T}.$$

For $t \in \cup_{k=1}^m (t_k, s_k]$, we get

$$\|Q_1x(t) - Q_1y(t)\|_{\mathbb{X}} \leq \|G_k(s_k, x_{s_k}) - G_k(s_k, y_{s_k})\|_{\mathbb{X}} \leq \mu_G\|x - y\|_{\mathcal{PC}_T}, \quad k = 1, 2, \ldots, m,$$

and for $t \in \cup_{k=1}^m (s_k, t_{k+1}]$, we obtain

$$\|Q_1x(t) - Q_1y(t)\|_{\mathbb{X}} \leq \left[S_0\mu_G + T_0S_0\mu_H + \sum_{j=1}^n \frac{\alpha_j S_0\mu_G T_0^{1+\beta-\gamma_j}}{\Gamma(2+\beta-\gamma_j)}\right]\|x - y\|_{\mathcal{PC}_T}.$$

Gathering the above results, we have $\|Q_1x - Q_1y\|_{\mathcal{PC}_T} \leq \Delta\|x - y\|_{\mathcal{PC}_T}$. Hence, Q_1 is a contraction mapping.

Next, we show that Q_2 is completely continuous. First, we verify that Q_2 is continuous, so we consider a sequence in $\mathcal{B}_{r_0}$ such that $y^n \to y \in \mathcal{B}_{r_0}$ as $n \to \infty$, then for $t \in [0, t_1]$, we get

$$\begin{aligned} \|Q_2y^n(t) - & Q_2y(t)\|_{\mathbb{X}} \\ & \leq \int_0^t \|(g_\beta * \mathcal{S}_{\beta,\gamma_j})(t-s)\|_{\mathcal{L}}\|F(s, y_s^n, K(y_s^n)) - F(s, y_s, K(y_s))\|_{\mathbb{X}}ds, \end{aligned}$$

for $t \in \cup_{k=1}^m (s_k, t_{k+1}]$, we obtain

$$\begin{aligned} \|Q_2y^n(t) - & Q_2y(t)\|_{\mathbb{X}} \\ & \leq \int_{s_k}^t \|(g_\beta * \mathcal{S}_{\beta,\gamma_j})(t-s)\|_{\mathcal{L}}\|F(s, y_s^n, K(y_s^n)) - F(s, y_s, K(y_s))\|_{\mathbb{X}}ds. \end{aligned}$$

By continuity of F, we get $\|Q_2y^n - Q_2y\|_{\mathcal{PC}_T} \to 0$ as $n \to \infty$. Hence Q_2 is continuous. Further, we show that Q_2 is a family of equi-continuous functions. Let $l_2, l_1 \in [0, t_1]$ such that $0 \le l_1 < l_2 \le t_1$, we have

$$
\begin{aligned}
\|Q_2y(l_2)-&Q_2y(l_1)\|_{\mathbb{X}}\\
&\le \int_0^{l_1} \|(g_\beta * \mathcal{S}_{\beta,\gamma_j})(l_2-s) - (g_\beta * \mathcal{S}_{\beta,\gamma_j})(l_1-s)\|_{\mathcal{L}} \|F(s, y_s, K(y_s))\|_{\mathbb{X}} ds\\
&\quad + \int_{l_1}^{l_2} \|(g_\beta * \mathcal{S}_{\beta,\gamma_j})(l_2-s)\|_{\mathcal{L}} \|F(s, y_s, K(y_s))\|_{\mathbb{X}} ds\\
&\le S_0 \left[\int_0^{l_1} \left(\frac{(l_2-s)^\beta}{\Gamma(1+\beta)} - \frac{(l_1-s)^\beta}{\Gamma(1+\beta)} \right) ds + \frac{(l_2-l_1)^{1+\beta}}{\Gamma(2+\beta)} \right] [\nu_F + \nu_F^0 k^0] r_0\\
&\le \frac{S_0}{\Gamma(2+\beta)} \left[\left| (l_2^{1+\beta} - l_1^{1+\beta}) - (l_2-l_1)^{1+\beta} \right| + \frac{(l_2-l_1)^{1+\beta}}{\Gamma(2+\beta)} \right] [\nu_F + \nu_F^0 k^0] r_0.
\end{aligned}
$$

For $l_2, l_1 \in \cup_{k=1}^m (s_k, t_{k+1}]$ such that $s_k \le l_1 < l_2 \le t_{k+1}$, we have

$$
\begin{aligned}
&\|Q_2y(l_2) - Q_2y(l_1)\|_{\mathbb{X}}\\
&\le \int_{s_k}^{l_1} \|(g_\beta * \mathcal{S}_{\beta,\gamma_j})(l_2-s) - (g_\beta * \mathcal{S}_{\beta,\gamma_j})(l_1-s)\|_{\mathcal{L}} \|F(s, y_s, K(y_s))\|_{\mathbb{X}} ds\\
&\quad + \int_{l_1}^{l_2} \|(g_\beta * \mathcal{S}_{\beta,\gamma_j})(l_2-s)\|_{\mathcal{L}} \|F(s, y_s, K(y_s))\|_{\mathbb{X}} ds\\
&\le S_0 \left[\int_{s_k}^{l_1} \left(\frac{(l_2-s)^\beta}{\Gamma(1+\beta)} - \frac{(l_1-s)^\beta}{\Gamma(1+\beta)} \right) ds + \frac{(l_2-l_1)^{1+\beta}}{\Gamma(2+\beta)} \right] [\nu_F + \nu_F^0 k^0] r_0\\
&\le \frac{S_0}{\Gamma(2+\beta)} \left[\left| ((l_2-s_k)^{1+\beta} - (l_1-s_k)^{1+\beta}) - (l_2-l_1)^{1+\beta} \right| + \frac{(l_2-l_1)^{1+\beta}}{\Gamma(2+\beta)} \right] [\nu_F + \nu_F^0 k^0] r_0,
\end{aligned}
$$

from aforemention inequalities we conclude that $\|Q_2y(l_2) - Q_2y(l_1)\|_{\mathcal{PC}_T} \to 0$ as $l_2 \to l_1$ for $t \in [0, T]$. This shows that Q_2 is a family of equi-continuous functions.

Finally, we will show that $\mathbb{Y} = \{Q_2y(t) : y \in \mathbb{B}_{r_0}\}$ is precompact in $\mathbb{X}$. Let $t > 0$ be fixed and let $y^n \in \mathbb{B}_{r_0}$, $\{y^n\}$ be a bounded sequence in $\mathcal{PC}_T$. Let $\mathbb{Y} = \{Q_2y^n(t) : y^n \in \mathbb{B}_{r_0}\}$ be a bounded sequence in $\mathbb{B}_{r_0}$. Hence, for any $t^* \in \cup_{k=0}^m (s_k, t_{k+1}]$, the sequence $\{y^n(t^*)\}$ is bounded in $\mathbb{B}_{r_0}$. Since F is compact, it has a convergent subsequence such that

$$F(t^*, y^n_{t^*}, K(y^n_{t^*})) \to F(t^*, y_{t^*}, K(y_{t^*})),$$

or

$$\|F(t^*, y^n_{t^*}, K(y^n_{t^*})) - F(t^*, y_{t^*}, K(y_{t^*}))\|_{\mathbb{X}} \to 0 \text{ as } n \to \infty.$$

Using the bounded convergence theorem, we can conclude that

$$(Q_2y^n)(t) \to (Q_2y)(t), \text{ in } \mathbb{B}_{r_0}.$$

This proves that Q_2 is a compact operator. Therefore Q_1 is a continuous and contraction operator and Q_2 is a completely continuous operator, hence $Q = Q_1 + Q_2$ is a condensing map on $\mathcal{B}_{r_0}$. Finally, by Theorem 2.2, we infer that there exists a mild solution of the system $(3) - (5)$ in $\mathcal{B}_{r_0}$.

4 Example

In this section, we provide an example to illustrate the feasibility of the established results. Set $\mathbb{X} = L^2(\mathbb{R}^n)$, then $\mathcal{PC}_0 := \mathcal{C}([-\tau, 0], L^2(\mathbb{R}^n))$. Let $\beta, \gamma_J > 0$ for $j = 1, 2, 3, \ldots, n$ be given, satisfying $0 < \beta \leq \gamma_n \leq \cdots \leq \gamma_1 \leq 1$ and $\tau \in \mathbb{R}$ such that $\tau > 0$. We consider the following system

$$\begin{aligned}\partial_t^{1+\beta} u(t,x) + \sum_{j=1}^{n} \alpha_j \partial_t^{\gamma_j} u(t,x) =& \Delta u(t,x) + \frac{u_t(\theta,x)}{50} \\ &+ \int_{-\tau}^{t} \cos(t-\xi) \frac{u_t(\theta,x)}{25} d\xi, \qquad (12)\end{aligned}$$

for all $(t,x) \in \cup_{k=0}^{m} (s_k, t_{k+1}] \times [0,1]$,

$$G_k(t, u_t(\theta,x)) = \int_{-\tau}^{t} \frac{\sin(t-\xi)}{(k+1)} \frac{u_t(\theta,x)}{25} d\xi,$$

$$H_k(t, u_t(\theta,x)) = \int_{-\tau}^{t} \frac{\cos(t-\xi)}{(k+1)} \frac{u_t(\theta,x)}{25} d\xi, \qquad t \in \cup_{k=1}^{m} (t_k, s_k], \qquad (13)$$

$$u(\theta,x) + \sum_{r=1}^{q} a_r y(t_r) = \phi(\theta,x), \quad u'(\theta,x) + \sum_{r=1}^{q} b_r y(t_r) = \varphi(\theta,x), \qquad (14)$$

where $a_r, b_r \in \mathbb{R}, \theta \in [-\tau, 0]$. The points $0 = t_0 = s_0 < t_1 \leq s_1 \leq t_2 < \cdots < t_m \leq s_m \leq t_{m+1} = 1$ are prefix numbers, $\partial_t^{1+\beta}$ denotes the Caputo derivative of order $(1+\beta)$ and Δ is the Laplacian with a maximal domain $\{v \in \mathbb{X} : v \in H^2(\mathbb{R}^n)\}$. The history function $u_t(\theta,x) : [-\tau, 0] \to \mathbb{X}$ is the element of $\mathcal{PC}_0$ characterized by $u_t(\theta,x) = u(t+\theta,x), \theta \in [-\tau,0]$. Set $y(t)(x) = u(t,x)$, $g_1(x) = \sum_{r=1}^{p} a_r x(t_r)$, $g_2(x) = \sum_{r=1}^{p} b_r x(t_r)$ and $\phi(\theta)(x) = \phi(\theta,x)$, $(\theta,x) \in [-\tau,0] \times [0,1]$. Now, we have $F(t,\psi,K(\psi)) = \frac{\psi}{50} + \int_{-\tau}^{t} \cos(t-\xi) \frac{\psi}{5^2} d\xi$, $G_k(t,\psi) = \int_{-\tau}^{t} \frac{\sin(t-\xi)}{(k+1)} \frac{\psi}{25} d\xi$, $H_k(t,\psi) = \int_{-\tau}^{t} \frac{\cos(t-\xi)}{(k+1)} \frac{\psi}{25} d\xi$. Now, we observe that the system (12) – (14) has the abstract form of the system (3) – (5). Moreover, for $t \in [0,1], \psi_i, \chi_i \in \mathcal{PC}_0, i = 1,2$ and $x, y \in \mathbb{X}$, we have

$$\|F(t,\psi_1,K(\chi_1)) - F(t,\psi_2,K(\chi_2))\| \leq \frac{1}{50}\|\psi_1 - \psi_2\| + \frac{1}{25}\|\chi_1 - \chi_2\|,$$

$$\|G_k(t,\chi_1) - G_k(t,\chi_2)\| \leq \frac{2}{25}\|\chi_1 - \chi_2\|; \ \|H_k(t,\chi_1) - H_k(t,\chi_2)\| \leq \frac{1}{25}\|\chi_1 - \chi_2\|,$$

$$\|g_1(x) - g_1(y)\|_{\mathbb{X}} \leq qa\|x - y\|_{\mathbb{X}}; \ \|g_2(x) - g_2(y)\|_{\mathbb{X}} \leq qb\|x - y\|_{\mathbb{X}},$$

where $a = \max_{1 \leq r \leq q} |a_r|$ and $b = \max_{1 \leq r \leq q} |b_r|$. Thus the assumptions (A_1) and (A_2) are satisfied. On the other hand, it follows from the theory of cosine families that Δ generates a bounded cosine function $\{C(t)\}_{t \geq 0}$ on $L^2(\mathbb{R}^n)$. Moreover, by Theorem 2.1 the operator Δ in equation (12) generates a bounded $\{\mathcal{S}_{\beta,\gamma_j}(t)\}_{t \geq 0}$-resolvent family. Let $S_0 = \sup_{t \in [0,1]} \|\mathcal{S}_{\beta,\gamma_j}(t)\|_{\mathcal{L}}$. Now, by Theorem 3.1 if

$$\max\left[S_0 d + S_0 e + \sum_{j=1}^{n} \frac{\alpha_j S_0 d}{\Gamma(2+\beta-\gamma_j)} + \frac{3S_0}{50\Gamma(2+\beta)}, \frac{1}{25}\right] < 1,$$

where $d = \max\{qa, \frac{2}{25}\}$, $e = \max\{qb, \frac{2}{25}\}$, then the system (12) – (14) admits a unique mild solution.

5 Conclusion

In this paper, an approach has been developed concerning the existence and uniqueness of mild solutions for the system $(3)-(5)$ using the Banach fixed point theorem and condensing map theorem. The system $(3)-(5)$ involves abrupt forces(impulsive effects), hence our results generalize the results of Pardo and Lizama studied in [21]. Thus, our results are more general and interesting.

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