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Stability Analysis of Nonlinear Mechanical Systems with Delay in Positional Forces

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Abstract: The paper is devoted to the problem of delay-independent stability for a class of nonlinear mechanical systems. Mechanical systems with linear velocity forces and essentially nonlinear positional ones are studied. It is assumed that there is a delay in the positional forces. With the aid of the decomposition method and original constructions of Lyapunov–Krasovskii functionals, conditions are found under which the trivial equilibrium positions of the considered systems are asymptotically stable for any constant nonnegative delay. An example is given to demonstrate the effectiveness of the obtained results.

Keywords: *mechanical system; delay; asymptotic stability, Lyapunov–Krasovskii functional, decomposition.*

Mathematics Subject Classification (2010): 34K20, 70K20, 93D30.

1 Introduction

An efficient approach to investigation of dynamical properties of complex systems is the decomposition method [15, 21]. The approach is successfully applied in various forms to the stability analysis of mechanical systems, see, for example, [15, 17, 20, 22, 24] and the bibliography therein.

An interesting and practically important result on the decomposition of mechanical system was obtained by V.I. Zubov [24]. He studied the stability of gyroscopic systems described by linear time-invariant second order systems and found conditions under which the stability problem for an original system can be reduced to that for two auxiliary independent first order subsystems. However, it should be noted that the Zubov approach

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is based on the Lyapunov first method, and it is inapplicable to nonstationary and nonlinear systems.

Another approach to derive the Zubov result has been proposed by A.A. Kosov [14]. He suggested to use a special transformation of variables and the Lyapunov direct method. This approach was further developed in [1, 4, 5], where it has been applied not only to linear time-invariant systems but also to switched systems and systems with nonlinear force fields. Furthermore, in [3, 7], with the aid of the Kosov approach and a special technique of using the Razumikhin theorem, new delay-independent stability conditions for some classes of mechanical systems were obtained.

In the present contribution, we consider mechanical systems with linear velocity forces and essentially nonlinear positional ones. It is assumed that there is a delay in the positional forces. We will look for conditions guaranteeing that the trivial equilibrium positions of the systems under consideration are asymptotically stable for any constant nonnegative delay.

Let us note that such conditions were derived in [7] with the aid of the decomposition method and Lyapunov–Razumikhin functions. In this paper, instead of Lyapunov–Razumikhin functions, we will use special constructions of Lyapunov–Krasovskii functionals. It will be shown that such an approach permits us to obtain less conservative delay-independent stability conditions than those in [7].

2 Notation

Throughout the paper the following notation is used:

- $\mathbb{R}$ is the field of real numbers and $\mathbb{R}^n$ denotes the n-dimensional Euclidean space.
- $\|\cdot\|$ is the Euclidean norm of a vector.
- $P > 0$ ($P < 0$) means that the matrix P is symmetric and positive (negative) definite.
- A^T is the transpose of a matrix A.
- A matrix C is called Metzler [13] if all its off-diagonal entries are nonnegative.
- $\mathrm{diag}\{\lambda_1, \ldots, \lambda_n\}$ is the diagonal matrix with the elements $\lambda_1, \ldots, \lambda_n$.
- A matrix C is called diagonally stable if there exists a diagonal matrix $\Lambda = \mathrm{diag}\{\lambda_1, \ldots, \lambda_n\} > 0$ such that $\Lambda C + C^T \Lambda < 0$.
- For a given positive number τ, let $C^1([-\tau, 0], \mathbb{R}^n)$ be the space of continuously differentiable functions $\varphi(\theta) : [-\tau, 0] \to \mathbb{R}^n$ with the uniform norm

$$\|\varphi\|_\tau = \max_{\theta \in [-\tau, 0]} \left(\|\varphi(\theta)\| + \|\dot{\varphi}(\theta)\| \right).$$

- Ω_Δ is the set of functions $\varphi(\theta) \in C^1([-\tau, 0], \mathbb{R}^n)$ satisfying the condition $\|\varphi\|_\tau < \Delta$, $0 < \Delta \leq +\infty$.

3 Problem Formulation

Consider the system

$$A\ddot{q}(t) + B\dot{q}(t) + Cf(q(t)) + Df(q(t - \tau)) = 0 \tag{1}$$

describing motions of a nonlinear mechanical system. Here $q(t), \dot{q}(t) \in \mathbb{R}^n$; A, B, C, D are constant matrices; vector function $f(q)$ is continuous for $\|q\| < \Delta$, $0 < \Delta \leq +\infty$; τ is a constant nonnegative delay.

Assume that $f(q)$ is a separable nonlinearity, i.e., $f(q) = (f_1(q_1), \ldots, f_n(q_n))^T$, and each scalar function $f_i(q_i)$ satisfies the sector-like condition $q_i f_i(q_i) > 0$ for $q_i \neq 0$, $i = 1, \ldots, n$. It is worth noting that such functions are widely used in models of automatic control systems and neural networks [2, 13, 16].

Hence, we consider a mechanical system with linear velocity forces and nonlinear positional ones. The term $-Df(q(t-\tau))$ can be treated as a control vector, and the presence of delay τ might be caused by a time lag between the moments of measuring of the state and the application of the corresponding control force, see [12, 19].

Let $q(t, t_0, \varphi, \dot{\varphi})$ stand for a solution of the system (1) with the initial conditions $t_0 \geq 0$, $\varphi(\theta) \in \Omega_\Delta$, and $q_t(t_0, \varphi, \dot{\varphi})$ denote the restriction of the solution to the segment $[t-\tau, t]$, i.e., $q_t(t_0, \varphi, \dot{\varphi}) : \theta \to q(t+\theta, t_0, \varphi, \dot{\varphi})$, $\theta \in [-\tau, 0]$.

In what follows, we will impose additional restrictions on the system (1).

Assumption 3.1 Let the matrices A and B be nonsingular.

Assumption 3.2 Let $f_i(q_i) = \alpha_i q_i^{\mu_i}$, where α_i are positive coefficients and $\mu_i > 1$ are rationals with odd numerators and denominators, $i = 1, \ldots, n$.

Remark 3.1 Without loss of generality, we will consider the case where $\alpha_i = 1$, $i = 1, \ldots, n$, and $\mu_1 \leq \ldots \leq \mu_n$.

Thus, the positional forces in (1) are essentially nonlinear ones. It should be noted that models with essentially nonlinear forces are widely used in contemporary mechanical and civil engineering, see, for instance, [8–10, 18].

The system (1) has the trivial equilibrium position

$$q = \dot{q} = 0. \tag{2}$$

We will look for conditions providing the asymptotic stability of the equilibrium position for an arbitrary constant nonnegative delay.

4 Main Results

According to the Zubov approach, consider two auxiliary isolated delay-free subsystems

$$\dot{y}(t) = Pf(y(t)), \tag{3}$$

$$\dot{z}(t) = -A^{-1}Bz(t). \tag{4}$$

Here $P = \{p_{ij}\}_{i,j=1}^n = -B^{-1}(C+D)$. It is worth mentioning that the subsystem (4) is linear, whereas the subsystem (3) belongs to the well-known class of Persidskii type systems [13].

Assumption 4.1 Let the subsystem (4) be asymptotically stable.

Define entries of the matrix $\bar{P} = \{\bar{p}_{ij}\}_{i,j=1}^n$ by the formulae $\bar{p}_{ii} = p_{ii}$, and $\bar{p}_{ij} = |p_{ij}|$ for $i \neq j$; $i, j = 1, \ldots, n$. The matrix $\bar{P}$ is Metzler, see [13].

In [7], with the aid of the decomposition method and Lyapunov–Razumikhin functions, it was proved that if Assumptions 3.1, 3.2, 4.1 are fulfilled, and the matrix $\bar{P}$ is Hurwitz, then the equilibrium position (2) of the system (1) is asymptotically stable for any $\tau \geq 0$.

To obtain less conservative stability conditions, we will use the original construction of Lyapunov–Krasovskii functionals for systems of the form (3) proposed in [6].

Theorem 4.1 *Let Assumptions 3.1, 3.2, 4.1 be fulfilled, and the matrix P be diagonally stable. Then the equilibrium position* (2) *of the system* (1) *is asymptotically stable for an arbitrary nonnegative delay.*

Proof. Introduce new variables by the formulae

$$z(t) = \dot{q}(t), \qquad y(t) = B^{-1}A\dot{q}(t) + q(t). \tag{5}$$

Then

$$\begin{aligned} B\dot{y}(t) &= -(C+D)f(y(t)) + C\left(f(y(t)) - f\left(y(t) - B^{-1}Az(t)\right)\right) \\ &+ D\left(f(y(t)) - f\left(y(t-\tau) - B^{-1}Az(t-\tau)\right)\right), \\ A\dot{z}(t) &= -Bz(t) - Cf\left(y(t) - B^{-1}Az(t)\right) - Df\left(y(t-\tau) - B^{-1}Az(t-\tau)\right). \end{aligned} \tag{6}$$

Taking into account properties of the transformation (5), we obtain that the equilibrium position (2) of the system (1) is asymptotically stable if and only if the zero solution of (6) is asymptotically stable.

It is known, see [23], that under Assumption 4.1, for any number $\gamma > 1$, there exists a continuously differentiable for $z \in \mathbb{R}^n$ positive homogeneous of the order γ Lyapunov function $\widetilde{V}(z)$ such that the estimates

$$a_1\|z\|^\gamma \le \widetilde{V}(z) \le a_2\|z\|^\gamma, \qquad \left\|\frac{\partial \widetilde{V}(z)}{\partial z}\right\| \le a_3\|z\|^{\gamma-1}, \qquad \left(\frac{\partial \widetilde{V}(z)}{\partial z}\right)^T A^{-1}Bz \ge a_4\|z\|^\gamma$$

hold for $z \in \mathbb{R}^n$. Here $a_i > 0$, $i = 1,2,3,4$.

The matrix P is diagonally stable. Therefore, one can choose a matrix $\Lambda = \mathrm{diag}\{\lambda_1,\dots,\lambda_n\} > 0$ such that $\Lambda P + P^T\Lambda < 0$.

Using the approach proposed in [6], construct a Lyapunov–Krasovskii functional for the system (6) in the form

$$V(y_t, z_t) = \widetilde{V}(z) + \beta_1\int_{t-\tau}^t \|z(s)\|^\gamma ds + \sum_{i=1}^n \lambda_i \frac{y_i^{\mu_i+1}(t)}{\mu_i+1} + \beta_2\int_{t-\tau}^t \|f(y(s))\|^2 ds$$

$$+\beta_3\int_{t-\tau}^t (\tau + s - t)\|f(y(s))\|^2 ds - f^T(y(t))\Lambda B^{-1}D\int_{t-\tau}^t f(y(s))ds,$$

where $\beta_1, \beta_2, \beta_3$ are positive coefficients.

Differentiating functional $V(y_t, z_t)$ along the solutions of the system (6), we obtain

$$\dot{V} = -\left(\frac{\partial \widetilde{V}(z(t))}{\partial z}\right)^T A^{-1}Bz(t) + \beta_1\|z(t)\|^\gamma - \beta_1\|z(t-\tau)\|^\gamma$$

$$-\left(\frac{\partial \widetilde{V}(z(t))}{\partial z}\right)^T A^{-1}\left(Cf\left(y(t) - B^{-1}Az(t)\right) + Df\left(y(t-\tau) - B^{-1}Az(t-\tau)\right)\right)$$

$$+f^T(y(t))\Lambda Pf(y(t)) + (\beta_2 + \tau\beta_3)\|f(y(t))\|^2 - \beta_2\|f(y(t-\tau))\|^2 - \beta_3\int_{t-\tau}^t \|f(y(s))\|^2 ds$$

$$+f^T(y(t))\Lambda B^{-1}\Big(C\left(f(y(t)) - f\left(y(t) - B^{-1}Az(t)\right)\right)$$

$$+D\left(f(y(t-\tau))-f\left(y(t-\tau)-B^{-1}Az(t-\tau)\right)\right)\Big)$$

$$+\int_{t-\tau}^{t} f^T(y(s))dsD^T\left(B^{-1}\right)^T\Lambda\frac{\partial f(y(t))}{\partial y}B^{-1}\Big(Cf\left(y(t)-B^{-1}Az(t)\right)$$

$$+Df\left(y(t-\tau)-B^{-1}Az(t-\tau)\right)\Big).$$

If $\|z(\xi)\|<1$ for $\xi\in[t-\tau,t]$, then

$$\dot{V}\le(-a_4+\beta_1)\|z(t)\|^\gamma-\beta_1\|z(t-\tau)\|^\gamma+(\beta_2+\tau\beta_3-c_1)\|f(y(t))\|^2-\beta_2\|f(y(t-\tau))\|^2$$

$$+c_2\left\|\frac{\partial f(y(t))}{\partial y}\right\|\int_{t-\tau}^{t}\|f(y(s))\|ds\left(\|f(y(t))\|+\|z(t)\|^{\mu_1}+\|f(y(t-\tau))\|+\|z(t-\tau)\|^{\mu_1}\right)$$

$$+c_3\|z(t)\|^{\gamma-1}\left(\|f(y(t))\|+\|z(t)\|^{\mu_1}+\|f(y(t-\tau))\|+\|z(t-\tau)\|^{\mu_1}\right)$$

$$-\beta_3\int_{t-\tau}^{t}\|f(y(s))\|^2ds+c_4\|f(y(t))\|\left\|f(y(t))-f\left(y(t)-B^{-1}Az(t)\right)\right\|$$

$$+c_5\|f(y(t))\|\left\|f(y(t-\tau))-f\left(y(t-\tau)-B^{-1}Az(t-\tau)\right)\right\|,$$

where c_1,c_2,c_3,c_4,c_5 are positive constants.

Let $2<\gamma<2\mu_1$. Using homogeneous functions properties, see [23], it is easy to show that, for sufficiently small values of parameters β_1,β_2,β_3, there exist positive numbers $\tilde{c}_1,\tilde{c}_2,\tilde{c}_3,\delta$ such that if $\|y(\xi)\|+\|z(\xi)\|<\delta$ for $\xi\in[t-\tau,t]$, then

$$\tilde{c}_1\left(\|z(t)\|^\gamma+\sum_{i=1}^{n}y_i^{\mu_i+1}(t)\right)\le V(y_t,z_t)$$

$$\le\tilde{c}_2\left(\|z(t)\|^\gamma+\int_{t-\tau}^{t}\|z(s)\|^\gamma ds+\sum_{i=1}^{n}y_i^{\mu_i+1}(t)+\int_{t-\tau}^{t}\|f(y(s))\|^2ds\right),$$

$$\dot{V}\le-\tilde{c}_3\left(\|z(t)\|^\gamma+\|z(t-\tau)\|^\gamma+\|f(y(t))\|^2+\|f(y(t-\tau))\|^2+\int_{t-\tau}^{t}\|f(y(s))\|^2ds\right).$$

From the obtained estimates it follows [11] that the zero solution of the system (6) is asymptotically stable. This implies that the equilibrium position (2) of the original system (1) is asymptotically stable as well. □

Remark 4.1 On the one hand, it is well known, see [13], that if the matrix $\bar{P}$ is Hurwitz, then the matrix P is diagonally stable. On the other hand, the matrix

$$P=\begin{pmatrix}-1 & 10\\ -10 & -1\end{pmatrix}$$

is diagonally stable, but the corresponding matrix

$$\bar{P}=\begin{pmatrix}-1 & 10\\ 10 & -1\end{pmatrix}$$

is not Hurwitz. Hence, conditions of Theorem 4.1 are less conservative than those obtained in [7].

Next, together with (1), consider the perturbed system

$$A\ddot{q}(t) + B\dot{q}(t) + Cf(q(t)) + Df(q(t-\tau)) = G(t, q(t), q(t-\tau)). \tag{7}$$

Here vector function $G(t,q,u)$ is continuous for $t \geq 0$, $\|q\| < \Delta$, $\|u\| < \Delta$.

Assumption 4.2 The estimate $\|G(t,q,u)\| \leq \tilde{a}\,(\|f(q)\| + \|f(u)\|)^{\sigma}$ is valid for $t \geq 0$, $\|q\| < \Delta$, $\|u\| < \Delta$, where $\tilde{a}$ and σ are positive constants.

If Assumption 4.2 is fulfilled, then the system (7) admits the equilibrium position (2). We will look for conditions under which perturbations do not disturb the asymptotic stability of the equilibrium position.

Theorem 4.2 *Let Assumptions 3.1, 3.2, 4.1, 4.2 be fulfilled, and the matrix P be diagonally stable. If $\sigma > 1$, then the equilibrium position* (2) *of the system* (7) *is asymptotically stable for an arbitrary nonnegative delay.*

The proof of the theorem is similar to that of Theorem 4.1.

5 Example

Let system (1) be of the form

$$\begin{aligned} &\ddot{q}_1(t) + b\dot{q}_1(t) + g\dot{q}_2(t) - cq_1^3(t) = u_1, \\ &\ddot{q}_2(t) + b\dot{q}_2(t) - g\dot{q}_1(t) - cq_2^5(t) = u_2. \end{aligned} \tag{8}$$

Here $q_1(t), q_2(t) \in \mathbb{R}$, b, g, c are positive constants, u_1, u_2 are control variables.

If $u_1 = u_2 = 0$, then the equilibrium position

$$q_1 = q_2 = \dot{q}_1 = \dot{q}_2 = 0 \tag{9}$$

of the system (8) is unstable, see [17]. We are going to design a feedback control providing the asymptotic stability of the equilibrium position.

Assume that the control law depends on q_1 and q_2, and is independent of $\dot{q}_1$ and $\dot{q}_2$. Moreover, we consider the case where there exists a delay τ in the control scheme.

It should be noted that for the linear control law

$$u_1 = a_{11}q_1(t-\tau) + a_{12}q_2(t-\tau), \qquad u_2 = a_{21}q_1(t-\tau) + a_{22}q_2(t-\tau),$$

where $a_{11}, a_{12}, a_{21}, a_{22}$ are constants, the presence of delay might result in instability of the equilibrium position. Therefore, we choose a nonlinear control in the form

$$u_1 = -dq_2^5(t-\tau), \qquad u_2 = dq_1^3(t-\tau), \qquad d = \text{const} > 0.$$

Verifying the conditions of Theorem 4.1, it is easy to show that if $d > bc/g$, then the equilibrium position (9) of the corresponding closed-loop system is asymptotically stable for an arbitrary constant nonnegative delay.

6 Conclusion

In this paper, new delay-independent conditions of the asymptotic stability are found for a class of nonlinear mechanical systems. Compared with the results of [7], these conditions are less conservative. However, it is worth mentioning that in [7] it was assumed that the delay may be a continuous nonnegative and bounded function of time, whereas the results of the present paper are valid only for systems with constant delays.

It should be noted that the approach to construction of Lyapunov–Krasovskii functionals proposed in the paper not only permits us to prove the asymptotic stability but also can be used to derive estimates of the convergence rate of solutions to the equilibrium position.

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References

[1] Aleksandrov, A.Y. and Aleksandrova, E.B. Asymptotic stability conditions for a class of hybrid mechanical systems with switched nonlinear positional forces. *Nonlinear Dynamics* **83** (4) (2016) 2427–2434.

[2] Aleksandrov, A.Yu., Aleksandrova, E.B. and Zhabko, A.P. Stability analysis for a class of nonlinear nonstationary systems via averaging. *Nonlinear Dynamics and Systems Theory* **13** (4) (2013) 332–343.

[3] Aleksandrov, A.Yu., Aleksandrova, E.B. and Zhabko, A.P. Asymptotic stability conditions for certain classes of mechanical systems with time delay. *WSEAS Transactions on Systems and Control* **9** (2014) 388–397.

[4] Aleksandrov, A.Yu., Chen, Y., Kosov, A.A. and Zhang, L. Stability of hybrid mechanical systems with switching linear force fields. *Nonlinear Dynamics and Systems Theory* **11** (1) (2011) 53–64.

[5] Aleksandrov, A.Yu., Kosov, A.A. and Chen, Y. Stability and stabilization of mechanical systems with switching. *Automation and Remote Control* **72** (6) (2011) 1143–1154.

[6] Aleksandrov, A.Yu. and Zhabko, A.P. Estimates of solutions of nonlinear systems with delay on the basis of the Lyapunov direct method. In: *Proc. of the Intern. Conf. "Conterporary Methods of Applied Mathematics, Control Theory and Computer Science" (AMCTCS'2014), Voronezh, Russia* (2014) 3–5. [Russian]

[7] Aleksandrov, A.Yu. and Zhabko, A.P. Stability analysis of equilibrium positions of mechanical systems with delay via decomposition. In: *Proc. of the IEEE Intern. Conf. on Mechanics "Seventh Polyakhov's Reading", St. Petersburg, Russia* (2015) 7106712.

[8] Beards, C.F. *Engineering Vibration Analysis with Application to Control Systems*. Edward Arnold, London, 1995.

[9] Gourdon, E., Lamarque, C.H., Pernot, S. Contribution to efficiency of irreversible passive energy pumping with a strong nonlinear attachment. *Nonlinear Dynamics* **50** (4) (2007) 793–808.

[10] Hagedorn, P. On the destabilizing effect of non-linear damping in non-conservative systems with follower forces. *Int. J. Non-Linear Mechanics* **5** (1970) 341–358.

[11] Hale, J.K. and Verduyn Lunel, S.M. *Introduction to Functional Differential Equations*. Springer, New York, 1993.

[12] Hu, H.Y. and Wang, Z.H. *Dynamics of Controlled Mechanical Systems with Delayed Feedback.* Springer, Berlin, 2002.

[13] Kazkurewicz, E. and Bhaya, A. *Matrix Diagonal Stability in Systems and Computation.* Birkhauser, Boston, 1999.

[14] Kosov, A.A. Stability investigation of singular systems by means of the Lyapunov vector functions method. *Vestnik St. Petersburg University, Ser. 10* (4) (2005) 123–129. [Russian]

[15] Lakshmikantham, V., Leela, S. and Martynyuk, A.A. *Stability Analysis of Nonlinear Systems.* Marcel Dekker, New York, 1989.

[16] Lukyanova, T.A. and Martynyuk, A.A. Stability analysis of impulsive Hopfield-type neuron system on time scale. *Nonlinear Dynamics and Systems Theory* **17** (3) (2017) 315–326.

[17] Matrosov, V.M. *The Method of Vector Lyapunov Functions: Analysis of Dynamical Properties of Nonlinear Systems.* Fizmatlit, Moscow, 2001. [Russian]

[18] Melnikov, G.I., Dudarenko, N.A., Malykh, K.S., Ivanova, L.N. and Melnikov, V.G. Mathematical models of nonlinear oscillations of mechanical systems with several degrees of freedom. *Nonlinear Dynamics and Systems Theory* **17** (4) (2017) 369–375.

[19] Ramachandran, P. and Ram, Y.M. Stability boundaries of mechanical controlled system with time delay. *Mechanical Systems and Signal Processing* **27** (2012) 523–533.

[20] Reshmin, S.A. and Chernous'ko, F.L. Synthesis of a control in a non-linear dynamical system based on decomposition. *J. of Applied Mathematics and Mechanics* **62** (1) (1998) 115–122.

[21] Siljak, D.D. *Decentralized Control of Complex Systems.* Academic Press, Boston, 1991.

[22] Tkhai, V.N. and Barabanov, I.N. Extending the property of a system to admit a family of oscillations to coupled systems *Nonlinear Dynamics and Systems Theory* **17** (1) (2017) 95–106.

[23] Zubov, V.I. *Methods of A.M. Lyapunov and Their Applications.* P. Noordhoff Ltd, Groningen, 1964.

[24] Zubov, V.I. *Analytical Dynamics of Gyroscopic Systems.* Sudostroenie, Leningrad, 1970. [Russian]