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Coexistence of Different Types of Chaos Synchronization Between Non-Identical and Different Dimensional Dynamical Systems

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Abstract: In this paper, based on Lyapunov stability theory, the coexistence of full state hybrid projective synchronization (FSHPS), $\Phi - \Theta$ synchronization, generalized synchronization (GS) and Q-S synchronization between different dimensional chaotic systems is studied. An application example and numerical simulations are presented to validate the main results of this paper.

Keywords: *chaos; full state hybrid projective synchronization; $\Phi - \Theta$ synchronization; generalized synchronization; Q-S synchronization.*

Mathematics Subject Classification (2010): 37B25, 37B55, 93C55, 93D05.

1 Introduction

Over the last few decades, a great deal of attention has been paid to the subject of chaotic dynamical systems and their synchronization control. Synchronization is an adaptive process that works to force the variables of a chaotic slave system to follow those of a corresponding master system [1]. This considerable interest has resulted in many synchronization types and schemes, see [2–5]. Among the most effective types of synchronization for chaotic and hyperchaotic systems are the full state hybrid projective synchronization (FSHPS) [6], Φ-Θ synchronization [7,8], generalized synchronization (GS) [9] and Q-S synchronization [10]. As a natural consequence of defining a variety

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of synchronization types, it became apparent that multiple types could coexist simultaneously, e.g. [11–13], a property that is of particular importance in the fields of secure communications and chaotic encryption schemes.

In this paper, we are concerned with the coexistence of the four types of synchronization mentioned above, i.e. FSHPS, Φ–Θ, GS and Q-S, in four dimensions between a three–dimensional chaotic master system and a four–dimensional hyperchaotic slave system. For this, we employ nonlinear control methods and make use of the well known direct Lyapunov method for establishing the global asymptotic convergence of synchronization errors towards zero. The resulting conditions are simple and their verification is trivial. Also, in order to put the reader's mind at ease and confirm the results of our study, we consider a numerical example, whereby the coexistence of FSHPS, Φ–Θ, GS and Q-S is illustrated for some typical chaotic and hyperchaotic systems. In Section 2 of this paper, the problem formulation and main result are given. Section 3 presents the numerical application of the proposed coexistence result with the aim of demonstrating the effectiveness of the approach developed herein. Section 4 summarizes the work reported in this paper.

2 Problem Formulation and Main Result

We consider the following master and slave systems

$$\dot{x}_i(t) = f_i(X(t)), \quad i = 1, 2, 3, \tag{1}$$

$$\dot{y}_i(t) = \textstyle\sum_{j=1}^{4} b_{ij} y_j(t) + g_i(Y(t)) + u_i, \quad i = 1, 2, 3, 4, \tag{2}$$

where $X(t) = (x_i)_{1\leq i\leq 3}$ and $Y(t) = (y_i)_{1\leq i\leq 4}$ are the states of the master and the slave systems, respectively, $f_i : \mathbb{R}^3 \to \mathbb{R}$, $i = 1, 2, 3$, $(b_{ij})_{4\times 4} \in \mathbb{R}^{4\times 4}$, $g_i : \mathbb{R}^4 \to \mathbb{R}$, $i = 1, 2, 3, 4$, are nonlinear functions, and $U = (u_1, u_2, u_3, u_4)^T$ is a vector–valued controller. The problem of coexistence of FSHPS, $\Theta - \Phi$ synchronization, GS and Q-S synchronization between master system (1) and slave system (2) is to find controllers u_i, $i = 1, 2, 3, 4$, such that the errors

$$\begin{aligned} e_1(t) &= y_1(t) - \Lambda \times X(t), \\ e_2(t) &= \Theta \times Y(t) - \Phi \times X(t), \\ e_3(t) &= y_3(t) - \phi\left(X(t)\right), \\ e_4(t) &= Q\left(Y(t)\right) - S\left(X(t)\right) \end{aligned} \tag{3}$$

satisfy

$$\lim_{t\to\infty} e_i(t) = 0, \quad i = 1, 2, 3, 4,$$

where $\Lambda = (\Lambda_i)_{1\leq i\leq 3}$, $\Theta = (\Theta_i)_{1\leq i\leq 4}$, $\Phi = (\Phi_i)_{1\leq i\leq 4}$ are constant matrices and $\phi : \mathbb{R}^3 \to \mathbb{R}$, $Q : \mathbb{R}^4 \to \mathbb{R}$, $S : \mathbb{R}^3 \to \mathbb{R}$ are differentiable functions. Here e_1 stands for the FSHPS error, e_2 stands for the $\Theta - \Phi$ synchronization error, e_3 denotes the GS error, and e_4 is the Q-S synchronization error.

Theorem 2.1 *FSHPS, $\Phi - \Theta$ synchronization, GS and Q-S synchronization coexist between master system (1) and slave system (2) under the following conditions:*

(i) $M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \Theta_1 & \Theta_2 & \Theta_3 & \Theta_4 \\ 0 & 0 & 1 & 0 \\ \frac{\partial Q}{\partial y_1} & \frac{\partial Q}{\partial y_2} & \frac{\partial Q}{\partial y_3} & \frac{\partial Q}{\partial y_4} \end{pmatrix}$ *is an invertible matrix and M^{-1} is its inverse.*

(ii) $U = M^{-1}\left((B-C)\,e\,(t) - R\right)$, *where* $C \in \mathbb{R}^{4\times 4}$ *is a control matrix and*

$$R = \begin{pmatrix} \sum_{j=1}^{4} b_{1j}y_j + g_1 - \sum_{j=1}^{3}\Lambda_j f_j \\ \sum_{i=1}^{4}\Theta_i\left(\sum_{j=1}^{4} b_{ij}y_j + g_i\right) - \sum_{j=1}^{3}\Phi_j f_j \\ \sum_{j=1}^{4} b_{3j}y_j + g_3 - \sum_{j=1}^{3}\frac{\partial\phi}{\partial x_j} f_j \\ \sum_{i=1}^{4}\frac{\partial Q}{\partial y_i}\left(\sum_{j=1}^{4} b_{ij}y_j + g_i\right) - \sum_{j=1}^{3}\frac{\partial S}{\partial x_j} f_j \end{pmatrix}.$$

(iii) $(B-C) + (B-C)^T$ *is a negative definite matrix, where* $B = (b_{ij})_{4\times 4}$.

Proof. The error system (3) can be differentiated as follows:

$$\begin{aligned} \dot{e}_1(t) &= \sum_{j=1}^{4} b_{1j}y_j + g_1 + u_1 - \sum_{j=1}^{3}\Lambda_j f_j, \\ \dot{e}_2(t) &= \sum_{i=1}^{4}\Theta_i\left(\sum_{j=1}^{4} b_{ij}y_j + g_i\right) + \sum_{j=1}^{4}\Theta_j u_j - \sum_{j=1}^{3}\Phi_j f_j, \\ \dot{e}_3(t) &= \sum_{j=1}^{4} b_{3j}y_j + g_3 + u_3 - \sum_{j=1}^{3}\frac{\partial\phi}{\partial x_j} f_j, \\ \dot{e}_4(t) &= \sum_{i=1}^{4}\frac{\partial Q}{\partial y_i}\left(\sum_{j=1}^{4} b_{ij}y_j + g_i\right) + \sum_{j=1}^{4}\frac{\partial Q}{\partial y_j}u_j - \sum_{j=1}^{3}\frac{\partial S}{\partial x_j} f_j. \end{aligned} \tag{4}$$

The error system (4) can be written in the following compact form

$$\dot{e}\,(t) = M \times U + R. \tag{5}$$

By substituting the control law (ii) into equation (5), the error system can be written as

$$\dot{e}(t) = (B-C)\,e(t). \tag{6}$$

Construct the candidate Lyapunov function in the form: $V\,(e(t)) = e^T(t)e(t)$, we obtain

$$\begin{aligned} \dot{V}\,(e(t)) &= \dot{e}^T(t)e(t) + e^T(t)\dot{e}(t) \\ &= e^T(t)(B-C)^T e(t) + e^T(t)\,(B-C)\,e\,(t) \\ &= e^T(t)\left[(B-C)^T + (B-C)\right]e(t). \end{aligned}$$

From (iii), we get $\dot{V}\,(e(t)) < 0$. Thus, from the Lyapunov stability theory, the zero solution of the error system (6) is globally asymptotically stable and, therefore, systems (1) and (2) are globally synchronized.

3 Numerical Application

In this example, the master system is chosen as the following 3D system

$$\begin{aligned} \dot{x}_1 &= a_1\,(x_2 - x_1)\,, \\ \dot{x}_2 &= x_1x_3, \\ \dot{x}_3 &= 50 - a_2x_1^2 - a_3x_3. \end{aligned} \tag{7}$$

When $a_1 = 2.9$, $a_2 = 0.7$ and $a_3 = 0.6$, system (7) exhibits chaotic attractors [14]. The salve system is described by

$$\begin{aligned} \dot{y}_1 &= \alpha(y_2 - y_1) + \gamma y_4 + u_1, \\ \dot{y}_2 &= -y_1 y_3 - y_2 + \gamma y_4 + u_2, \\ \dot{y}_3 &= y_1 y_2 - y_3 - \beta + u_3, \\ \dot{y}_4 &= -\delta(y_1 + y_2) + u_4. \end{aligned} \tag{8}$$

The uncontrolled system (8) (i.e. with $u_1 = u_2 = u_3 = u_4 = 0$) exhibits strange hyperchaotic attractors for the parameter values $\alpha = 4$, $\beta = 20$, $\gamma = 0.2$ and $\delta = 0.5$ [15]. The linear part B and nonlinear part g of the slave system (8) can be formulated as

$$B = \begin{pmatrix} -4 & 4 & 0 & 0.2 \\ 0 & -1 & 0 & 0.2 \\ 0 & 0 & -1 & 0 \\ -0.5 & -0.5 & 0 & 0 \end{pmatrix} \text{ and } g = \begin{pmatrix} 0 \\ -y_1 y_3 \\ y_1 y_2 - \beta \\ 0 \end{pmatrix}.$$

According to our approach, the error system between systems (7) and (8) is described by

$$\begin{aligned} e_1 &= y_1 - \Lambda \times (x_1, x_2, x_3)^T, \\ e_2 &= \Theta \times (y_1, y_2, y_3, y_4)^T - \Phi \times (x_1, x_2, x_3)^T, \\ e_3 &= y_4 - \phi(x_1, x_2, x_3), \\ e_4 &= Q(y_1, y_2, y_3, y_4) - S(x_1, x_2, x_3), \end{aligned} \tag{9}$$

where $\Lambda = (-1, 0, 2)$, $\Theta = (0, 2, 0, 0)$, $\Phi = (1, 2, 3)$, $\phi(x_1, x_2, x_3) = x_1 x_2 + x_3, Q(y_1, y_2, y_3, y_4) = 1 + 3y_4$ and $S(x_1, x_2, x_3) = x_1 x_2 x_3$. Based on the notations used in Section 2, the matrix M is given by

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}, \tag{10}$$

and thus

$$M^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{3} \end{pmatrix}. \tag{11}$$

Then, the control matrix C can be selected as

$$C = \begin{pmatrix} 0 & 4 & 0 & 0.2 \\ 0 & 2 & 0 & 0.2 \\ 0 & 0 & 1 & 0 \\ -0.5 & -0.5 & 0 & 1 \end{pmatrix}. \tag{12}$$

Using matrices (10), (11) and (12) we can easily construct the control law (ii) described in Theorem 1. We can see that $(B - C) + (B - C)^T$ is a negative–definite matrix and all conditions of Theorem 1 are satisfied. Therefore, systems (7) and (8) are globally synchronized in 4-D. The time evolution of errors between systems (7) and (8) is shown in Figure 1.

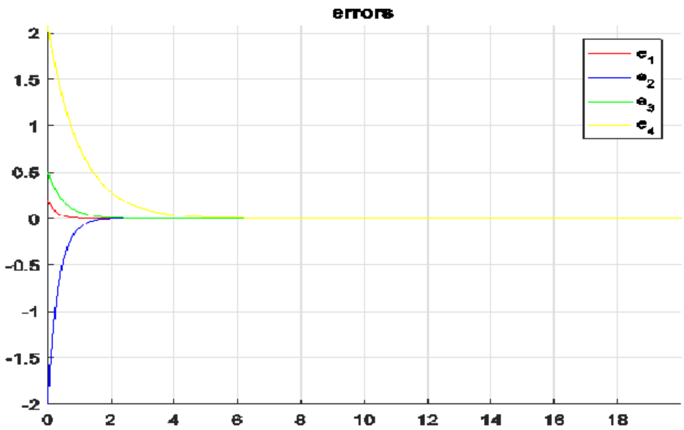

Figure 1: Time evolution of the synchronization errors e_1,e_2, e_3 and e_4 between the master system (7) and the slave system (8).

4 Conclusion

A new synchronization scheme has been used to achieve coexistence of several types of synchronization between an arbitrary 3-dimensional master and a 4-dimensional slave system. By using Lyapunov stability theory, the paper analysed the coexistence of full state hybrid projective synchronization (FSHPS), $\Phi - \Theta$ synchronization, generalized synchronization (GS) and Q-S synchronization based on the control of the linear part of the master system. The numerical example detailed in the previous section confirms the effectiveness of the theoretical analysis.

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