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Feedback Control of Chaotic Systems by Using Jacobian Matrix Conditions

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Abstract: In this work, we propose, for stabilizing chaotic systems at fixed points, new conditions based on the Jacobian matrix and its relation with the conditions of Routh-Hurwitz. We apply the results of feedback control method to the second type Rössler system, Liu system and Genesio system.

Keywords: *Routh-Hurwitz theorem; Jacobian matrix; feedback control; chaotic systems.*

Mathematics Subject Classification (2010): 34H10, 37N35, 93C10, 93C15, 93C95.

1 Introduction

Chaos, as a very interesting nonlinear phenomenon, has been intensively studied over the past decades. After the pioneering work of Ott et al [1], and Pecora and Carroll [2], research efforts have been devoted to the chaos control problems in many physical systems [3–5]. The control problem attempts to stabilize a chaotic attractor to either a periodic orbit or an equilibrium point [20, 21]. Many potential applications have come true in securing communication, laser and biological systems, and other areas [6–9, 19]. Different control strategies for stabilizing chaos have been proposed, such as adaptive control, time delay control, and fuzzy control. Generally speaking, there are two main approaches for controlling chaos: feedback control and non-feedback control. The feedback control [10, 17, 18] approach offers many advantages such as robustness and computational complexity over the non-feedback control method. The aim of this paper is to apply the feedback control to chaotic systems, with new conditions for the stability at fixed points based on the Jacobian matrix. We present the numerical simulation studies for control of the Rössler, Liu and modified Genesio systems.

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2 Preliminaries

Suppose that A is an $n \times n$ matrix of real constants, its characteristic polynomial is

$$f(\lambda) = \lambda^n + a\lambda^{n-1} + b\lambda^{n-2} + c\lambda^{n-3} + ..., n = 1, 2, 3, 4.$$

The Routh-Hurwitz theorem [10–13] is as follows.

Theorem 2.1 *All the roots of the caracteristic polynomial have negative real parts precisely when the given conditions are satisfied:*
$\lambda^2 + a\lambda + b : a > 0, b > 0.$
$\lambda^3 + a\lambda^2 + b\lambda + c : a > 0, c > 0, ab - c > 0.$
$\lambda^4 + a\lambda^3 + b\lambda^2 + c\lambda + d : a > 0, ab - c > 0, (ab - c)c - a^2 d > 0, d > 0.$

3 Main Results

3.1 The case of third dimension

We assume A is the Jacobian matrix of the third dimension:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \tag{1}$$

then the relation between the coefficients of characteristic polynomial and the Jacobian matrix is

$$\begin{cases} a = -trace(A), \\ b = A_{11} + A_{22} + A_{33}, \\ c = -\det(A), \end{cases} \tag{2}$$

where $A_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, A_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$ and $A_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}, \end{vmatrix}.$
Then $ab - c = -a_{11}(A_{22}+A_{33}) - a_{22}(A_{11}+A_{33}) - a_{33}(A_{22}+A_{11}) - 2a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$.

Remark 3.1 We note that, if $a_{ii} < 0$, $A_{ii} > 0$, $i = 1, 2, 3$ and $\det(A) < 0$ so that $t = a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \geq 0$, then the coefficients of the characteristic polynomial are positive. On the other hand, we have $t = 0$ for the Rössler, Liu and other systems. So, we can ensure the stability of any chaotic systems with the following theorem.

We consider A is the Jacobian matrix at a fixed point, and $t = a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$.

Theorem 3.1 *If $t \geq 0$, all the roots of the characteristic polynomial of A have negative real parts when the given conditions are satisfied:*
$\det(A) < 0$, $a_{ii} < 0$ *and* $A_{ii} > 0$ *for* $i = 1, 2, 3$.

Proof. We have

$$\begin{cases} a = -trace(A) > 0, \\ b = A_{11} + A_{22} + A_{33} > 0, \\ ab - c > 0, \end{cases}$$

then, by the Routh-Hurwitz theorem all the roots of the characteristic polynomial have negative real parts.

Remark 3.2 We can use the condition $t \geq 0$ as an additional condition with the condition of Routh-Hurwitz to get quickly the convergence to the fixed point.

4 Application to Chaotic Systems

4.1 The second type Rössler system

The Rössler system [14] is given by the following equations:

$$\begin{cases} \dot{x} = -(y+z), \\ \dot{y} = x + \alpha y, \\ \dot{z} = \beta x + xz - \gamma z, \end{cases}$$

where $\alpha = 0.38, \beta = 0.3, \gamma = 4.5$. The two equilibrium points of system are given by $E_1 = (0,0,0),\ \ E_2 = (\gamma - \alpha\beta, \beta - \frac{\gamma}{\alpha}, \frac{\gamma}{\alpha} - \beta)$.

4.1.1 Control at the equilibrium point E_1

If the controlled Rössler system is given by the equations

$$\begin{cases} \dot{x} = -(y+z) - u_1, \\ \dot{y} = x + 0.38y - u_2, \\ \dot{z} = 0.3x + (x-4.5)z - u_3, \end{cases} \tag{3}$$

where $u_1 = kx, u_2 = ky, u_3 = kz$, and k is the feedback coefficient; when $k > 0.38$, the system (3) will gradually converge to the equilibrium point $(0,0,0)$.

Proof. The Jacobian matrix of system (3) with regard to the equilibrium point $(0,0,0)$ is

$$A = \begin{pmatrix} -k & -1 & -1 \\ 1 & 0.38-k & 0 \\ 0.3 & 0 & -4.5-k \end{pmatrix},$$

where $a_{11} = -k, a_{22} = 0.38 - k, a_{33} = -4.5 - k$, $A_{11} = k^2 + 4.12k - 1.71$, $A_{22} = k^2 + 4.5k + 0.3$, $A_{33} = k^2 - 0.38k + 1$ and $\det(A) = -1k^3 - 4.12k^2 + 0.41k - 4.386$.We have $t = 0$, therefore

$$\begin{cases} a_{11} < 0, \\ a_{22} < 0, \\ a_{33} < 0 \end{cases} \Leftrightarrow \begin{cases} k > 0, \\ k > 0.38, \\ k > -4.5 \end{cases}$$

and

$$\begin{cases} A_{11} > 0, \\ A_{22} > 0, \\ A_{33} > 0, \\ \det(A) < 0, \end{cases} \Leftrightarrow \begin{cases} k \in \,]-\infty, -4.5[\, \cup\,]0.38, \infty[\,, \\ k \in \,]-6.7685 \times 10^{-2}, \infty[\, \cup\,]-\infty, -4.4323[\,, \\ k \in \,]-\infty, \infty[\,, \\ k \in \,]-4.4354, \infty[\,. \end{cases}$$

Obviously, if $k > 0.38$, then $a_{11} < 0, a_{22} < 0, a_{33} < 0, \det A < 0$ and $A_{11} > 0, A_{22} > 0, A_{33} > 0$. According to Theorem 3.1, the system (3)will gradually converge to the unstable equilibrium point $(0,0,0)$, thus the proof is completed (see Figure 1).

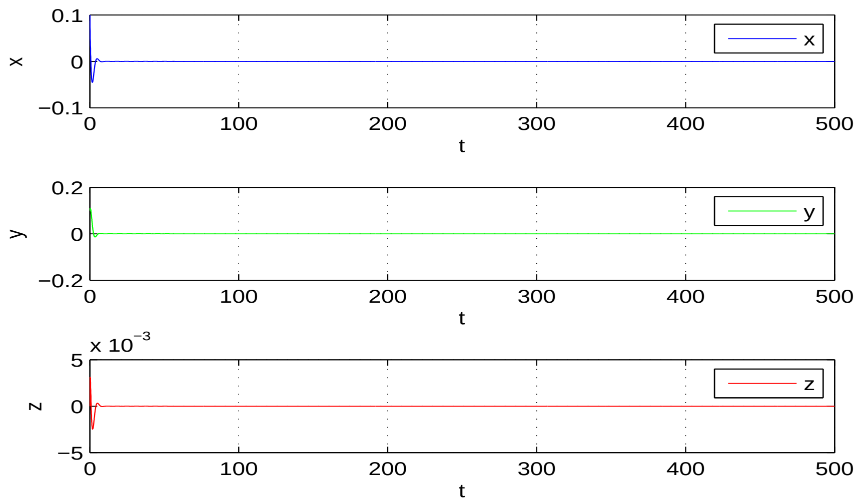

Figure 1: Control of the Rössler system at the equilibrium point E_1.

Remark 4.1 By using the Routh-Hurwitz theorem, we found $k > 0.776\,61$.

Similarly, the system can also be controlled at $E_2(4.386, -11.542, 11.542)$ by the similar control method. The controlled Rössler system is

$$\begin{cases} \dot{x} = -(y+z) - u_1, \\ \dot{y} = x + 0.38y - u_2, \\ \dot{z} = 0.3x + (x-4.5)z - u_3, \end{cases} \tag{4}$$

where $u_1 = k(x - 4.386), , u_2 = k(y + 11.542), u_3 = k(z - 11.542)$.

For demonstrating this conclusion, we do the following transformations: $x_1 = x - \beta$, $y_1 = y + \alpha$,$z_1 = z - \alpha$. When $\alpha = 11.542$, $\beta = 4.386$, then the system (4) has the following form:

$$\begin{cases} \dot{x_1} = -(y_1+z_1) - kx_1, \\ \dot{y_1} = x_1 + 0.38y_1 - ky_1, \\ \dot{z_1} = 11.842x_1 + (x_1 - 0.114)z_1 - kz_1. \end{cases} \tag{5}$$

The Jacobian matrix of the system (5) is

$$A = \begin{pmatrix} -k & -1 & -1 \\ 1 & 0.38-k & 0 \\ 11.842 & 0 & -0.114-k \end{pmatrix},$$

where k is the feedback coefficient; when $k > 0.38$, we found that the system (5) will converge to the equilibrium point $E_2'(0,0,0)$, that is system (4) will gradually converge to the equilibrium point $E_2(4.386, -11.542, 11.542)$.

Proof. We have $\det(A) = -1.0k^3 + 0.266\,k^2 - 12.799k + 4.386\,0$,

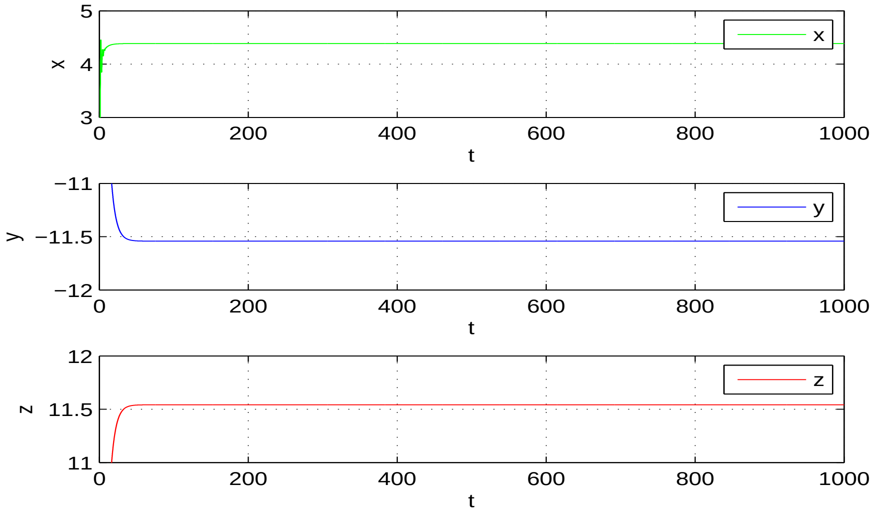

Figure 2: Control of the Rössler system at the equilibrium point E_2.

$a_{11} = -k, a_{22} = 0.38 - k, a_{33} = -0.114 - k,\ A_{11} = k^2 - 0.266\,k - 0.043\,32, A_{22} = k^2 + 0.114\,k + 11.842, A_{33} = 11.842k - 4.500\,0$, and $t = 0$, then

$$\begin{cases} a_{11} < 0, \\ a_{22} < 0, \\ a_{33} < 0, \end{cases} \Leftrightarrow \begin{cases} k > 0, \\ k > 0.38, \\ k > -0.114, \end{cases}$$

and

$$\begin{cases} A_{11} > 0, \\ A_{22} > 0, \\ A_{33} > 0, \\ \det(A) < 0, \end{cases} \Leftrightarrow \begin{cases} k \in]-\infty, -0.114[\cup]0.38, \infty[\,, \\ k \in \mathbb{R}, \\ k \in]0.38, \infty[\,, \\ k \in]0.341\,99, \infty[\,. \end{cases}$$

.

When $k > 0.38$, we have $a_{11} < 0, a_{22} < 0, a_{33} < 0, \det A < 0, A_{11} > 0,\ A_{22} > 0$, and $A_{33} > 0$. According to Theorem 3.1, the system (5) will gradually converge to the unstable equilibrium point E_2. Hence the proof is completed (see Figure 2).

4.2 Control of the Liu system

The Liu system [15] is given by

$$\begin{cases} \dot{x} = \alpha(y - x), \\ \dot{y} = x(\lambda - \gamma z), \\ \dot{z} = \delta x^2 - \beta z, \end{cases}$$

where $\alpha = 10, \lambda = 40, \gamma = 1, \delta = 4, \beta = 2.5$. The fixed points are $E_1 : (0, 0, 0)$, $E_{2,3} : (\pm\sqrt{\frac{\beta\delta}{\gamma\lambda}}, \pm\sqrt{\frac{\beta\delta}{\gamma\lambda}}, \frac{\beta}{\gamma})$.

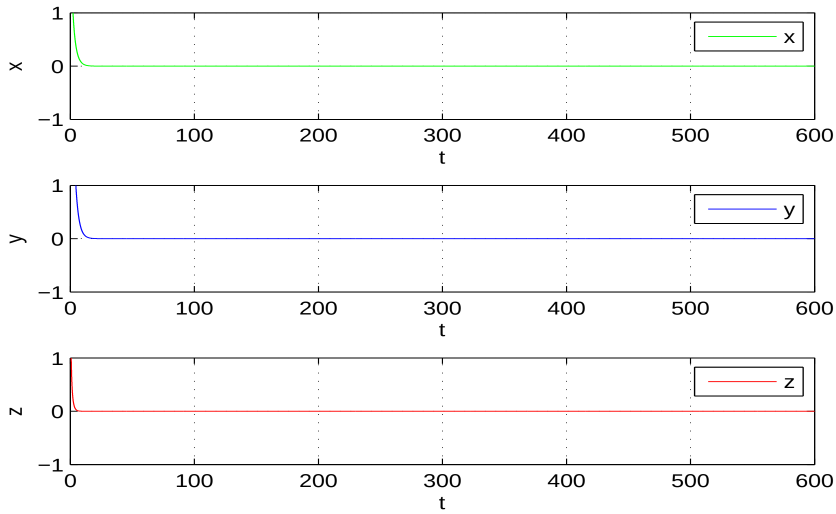

Figure 3: Control of the Liu system at the equilibrium point E_1.

4.2.1 Control at the equilibrium point E_1

The controlled Liu system is

$$\begin{cases} \dot{x} = \alpha(y - x) - u_1, \\ \dot{y} = x(\lambda - \gamma z) - u_2, \\ \dot{z} = \delta x^2 - \beta z - u_3, \end{cases} \tag{6}$$

where $u_1 = kx, u_2 = ky, u_3 = kz$
and k is the feedback coefficient; when we have $k > 15.616$, the system (6) will gradually converge to the equilibrium point $(0, 0, 0)$.

Proof. The Jacobian matrix of the system (6) with regard to the equilibrium point $(0, 0, 0)$ is

$$A = \begin{pmatrix} -10 - k & 10 & 0 \\ 40 & -k & 0 \\ 0 & 0 & -2.5 - k \end{pmatrix},$$

thus $\det(A) = -1k^3 - 12.5k^2 + 375k + 1000 < 0$,
$a_{11} = -10 - k, a_{22} = -k, a_{33} = -2.5 - k$,
$A_{11} = k^2 + 2.5k$,
$A_{22} = k^2 + 12.5k + 25$,
$A_{33} = k^2 + 10k - 400$
with $t = 0$. So,

$$\begin{cases} a_{11} < 0, \\ a_{22} < 0, \\ a_{33} < 0, \end{cases} \Leftrightarrow \begin{cases} k > -10, \\ k > 0, \\ k > -2.5. \end{cases} \quad \text{and}$$

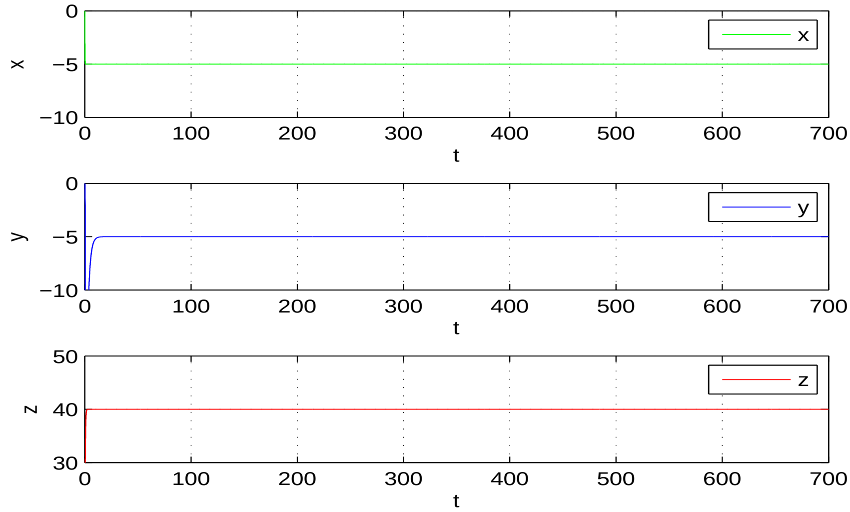

Figure 4: Control of the Liu system at the equilibrium point E_2.

$$\left\{\begin{array}{r} A_{11} > 0, \\ A_{22} > 0, \\ A_{33} > 0, \\ \det(A) < 0, \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} k \in \,]-\infty, -2.5[\cup]0, \infty[\,, \\ k \in \,]-\infty, -10.0[\cup]-2.5, \infty[\,, \\ k \in \,]-\infty, -25.616[\cup], 15.616, \infty[\,, \\ k \in \,]-10, -2.5[\cup]0, \infty[\,. \end{array}\right.$$

It can be easily seen when $k > 15.616$, so $a_{11} < 0, a_{22} < 0, a_{33} < 0, \det A < 0, A_{11} > 0$, and $A_{22} > 0, A_{33} > 0$. According to Theorem 3.1, the system (6) will gradually converge to the unstable equilibrium point $(0, 0, 0)$ (see Figure 3).

4.2.2 Control at the equilibrium point E_2

We consider the controlled Liu system given by

$$\left\{\begin{array}{l} \dot{x} = \alpha(y - x) - u_1, \\ \dot{y} = x(\lambda - \gamma z) - u_2, \\ \dot{z} = \delta x^2 - \beta z - u_3, \end{array}\right. \tag{7}$$

where $u_1 = k(x + 5) + 10(y + 5), u_2 = k(y + 5), u_3 = k(z - 40)$. Here k is the feedback coefficient; when $k > 0$, it can be demonstrated that system (7) will gradually converge to the equilibrium point $(-5, -5, 40)$.

Proof. The Jacobian matrix of the system(7) at $(-5, -5, 40)$ is

$$A = \begin{pmatrix} -10 - k & 0 & 0 \\ 0 & -k & 5 \\ -40 & 0 & -2.5 - k \end{pmatrix},$$

where $\det(A) = -1.0k^3 - 12.5k^2 - 25k$,

$a_{11} = -10 - k, a_{22} = -k, a_{33} = -2.5 - k,$
$A_{11} = k^2 + 2.5k,$
$A_{22} = k^2 + 12.5k + 25,$
$A_{33} = k^2 + 10k$
with $t = 0$, then

$$\begin{cases} a_{11} < 0, \\ a_{22} < 0, \\ a_{33} < 0, \end{cases} \Leftrightarrow \begin{cases} k > -10, \\ k > 0, \\ k > -2.5, \end{cases} \quad \text{and}$$

$$\begin{cases} A_{11} > 0, \\ A_{22} > 0, \\ A_{33} > 0, \\ \det(A) < 0, \end{cases} \Leftrightarrow \begin{cases} k \in]-\infty, -2.5[\cup]0, \infty[, \\ k \in]-\infty, -10.0[\cup]-2.5, \infty[, \\ k \in]-\infty, -10[\cup]0, \infty[, \\ k \in]-10, -2.5[\cup]0, \infty[. \end{cases}$$

When $k > 0$, we have $a_{11} < 0, a_{22} < 0, a_{33} < 0, \det A < 0, A_{11} > 0$, and $A_{22} > 0, A_{33} > 0$. According to Theorem 3.1, the system (7) will gradually converge to the unstable equilibrium point $(-5, -5, 40)$ (see Figure 4).

Remark 4.2 Similarly, the system can also be controlled at $E_3(5, 5, 40)$ by the similar control method if $k > 0$.

4.3 The modified Genesio system

We have the modified Genesio system [16, 17] as

$$\begin{cases} \dot{x} = y, \\ \dot{y} = \alpha_1 y + z, \\ \dot{z} = \alpha_2 x^2 + \alpha_3 x + \alpha_4 y + \alpha_5 z, \end{cases}$$

where $\alpha_1 = -0.5, \alpha_2 = 3, \alpha_3 = -6, \alpha_4 = -2.85, \alpha_5 = -0.5$, and the fixed points are $E_1 = (0, 0, 0)$, $E_2 = (2, 0, 0)$.

4.3.1 Control at the equilibrium point E_1

The controlled modified Genesio system is given by

$$\begin{cases} \dot{x} = y - u_1, \\ \dot{y} = \alpha_1 y + z - u_2, \\ \dot{z} = \alpha_2 x^2 + \alpha_3 x + \alpha_4 y + \alpha_5 z - u_3, \end{cases} \tag{8}$$

where $u_1 = kx, u_2 = ky - z, u_3 = kz$. Here k is the feedback coefficient; when $k > 0$, we found that the system (8)will gradually converge to the equilibrium point $(0, 0, 0)$.

Proof. The Jacobian matrix of the system (8) with regard to the equilibrium point (0.0.0) is

$$A = \begin{pmatrix} -k & 1 & 0 \\ 0 & -0.5 - k & 0 \\ -6 & -2.85 & -0.5 - k \end{pmatrix},$$

where $\det(A) = -2k^3 - 1.5k^2 - 0.25k,$
$a_{11} = -k, a_{22} = -0.5 - k, a_{33} = -0.5 - k,$

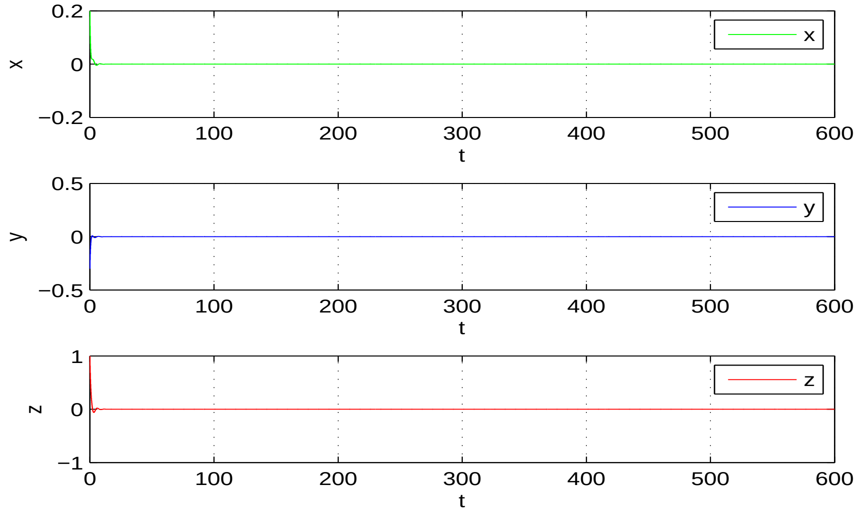

Figure 5: Control of the Modified Genesio System to the original equilibrium point.

$A_{11} = 2k^2 + 1.5k + 0.25$,
$A_{22} = k^2 + 0.5k$,
$A_{33} = k^2 + 0.5k$ and $t = 0$, then

$$\begin{cases} a_{11} < 0, \\ a_{22} < 0, \\ a_{33} < 0, \end{cases} \Leftrightarrow \begin{cases} k > 0, \\ k > -0.5, \\ k > -0.5, \end{cases} \text{and}$$

$$\begin{cases} A_{11} > 0, \\ A_{22} > 0, \\ A_{33} > 0, \\ \det(A) < 0, \end{cases} \Leftrightarrow \begin{cases} k \in]-\infty, -0.5[\cup]-0.25, \infty[\,, \\ k \in]-\infty, -0.5[\cup]0, \infty[\,, \\ k \in]-\infty, -0.5[\cup]0, \infty[\,, \\ k \in]-0.5, -0.25[\cup]0, \infty[\,. \end{cases}$$

Obviously, when $k > 0$,then $a_{ii} < 0$, $A_{ii} > 0$, $i = 1, 2, 3$ and $\det(A) < 0$. According to Theorem 3.1, the system (8) will gradually converge to the unstable equilibrium point $(0, 0, 0)$. Hence the proof is completed (see Figure 5).

4.3.2 Control at the equilibrium point $E_2 : (2, 0, 0)$

The controlled modified Genesio system is given by

$$\begin{cases} \dot{x} = y - u_1, \\ \dot{y} = \alpha_1 y + z - u_2, \\ \dot{z} = \alpha_2 x^2 + \alpha_3 x + \alpha_4 y + \alpha_5 z - u_3, \end{cases} \tag{9}$$

where $u_1 = k(x - 2), u_2 = ky - z, u_3 = kz$, and k is the feedback coefficient, if $k > 0$, the system (9) will gradually converge to the equilibrium point $(2, 0, 0)$.

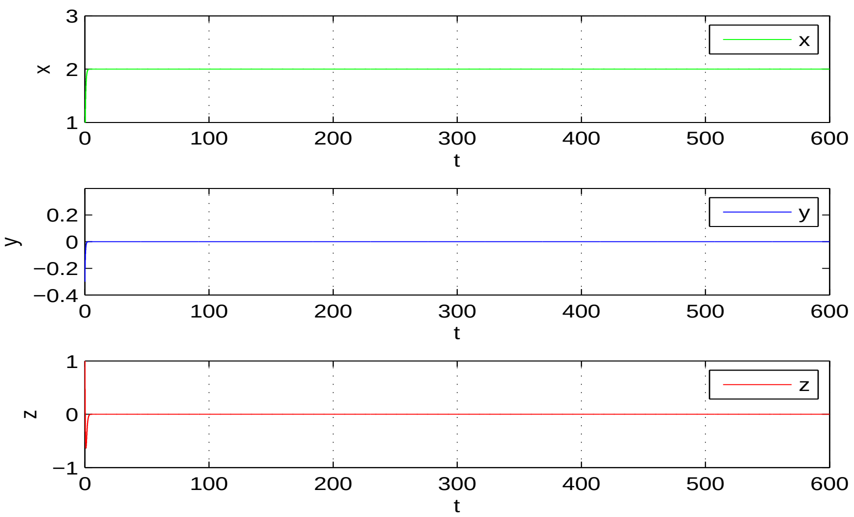

Figure 6: Control of the modified Genesio system at the equilibrium point E_2.

Proof. The Jacobian matrix of the system (9) at (2.0.0) is

$$A = \begin{pmatrix} -k & 1 & 0 \\ 0 & -0.5-k & 0 \\ 6 & -2.85 & -0.5-k \end{pmatrix},$$

where $\det(A) = -1k^3 - 1k^2 - 0.25k$,
$a_{11} = -k, a_{22} = -0.5-k, a_{33} = -0.5-k$,
$A_{11} = k^2 + k + 0.25$,
$A_{22} = k^2 + 0.5k$,
$A_{33} = k^2 + 0.5k$ with $t = 0$. So

$$\begin{cases} a_{11} < 0, \\ a_{22} < 0, \\ a_{33} < 0, \end{cases} \Leftrightarrow \begin{cases} k > 0, \\ k > -0.5, \quad \text{and} \\ k > -0.5, \end{cases}$$

$$\begin{cases} A_{11} > 0, \\ A_{22} > 0, \\ A_{33} > 0, \\ \det(A) < 0, \end{cases} \Leftrightarrow \begin{cases} k \in]-\infty, -0.5[\, \cup \,]-0.5, \infty[\,, \\ k \in]-\infty, -0.5[\, \cup \,]0, \infty[\,, \\ k \in]-\infty, -0.5[\, \cup \,]0, \infty[\,, \\ k \in]0, \infty[\,. \end{cases}$$

Obviously, when $k > 0$, we have, $a_{ii} < 0$, $A_{ii} > 0$, $i = 1, 2, 3$ and $\det(A) < 0$. According to Theorem 3.1, the system (9) will gradually converge to the unstable equilibrium point $(2, 0, 0)$, thus the proof is completed (see Figure 6).

5 Conclusion

This work presents the feedback control at fixed points of the second type Rössler, Liu and modified Genesio chaotic systems. By using new conditions for the stability based on the Jacobian matrix, we simplified and modified the calculations for the Routh-Hurwitz coefficient.

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