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A Self-Diffusion Mathematical Model to Describe the Toxin Effect on the Zooplankton-Phytoplankton Dynamics

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Abstract: The main goal of this work is the mathematical formulation, the analysis and the numerical simulation of a prey-predator model by taking into account the toxin produced by the phytoplankton species. The mathematical study of the model leads us to have an idea on the existence of solution, the existence of equilibria and the stability of the stationary equilibria. These results are obtained through the principle of comparison. Finally, the numerical simulations in two-dimensional allowed us to establish the formation of spatial patterns and a threshold of release of the toxin, above which we talk about the phytoplankton blooms.

Keywords: *toxin effect; populations dynamics; predator-prey model; reaction-diffusion; pattern formation.*

Mathematics Subject Classification (2010): 65L12, 65M20, 65N40.

1 Introduction

Ecology and harmful toxic release in marine environment are major fields of study in their own right, but there are some common features of these systems. It is interesting and important from biological viewpoints to study ecological systems under the influence of the toxic substance release factors. However, this goal remains difficult to attain due to the complexity of natural systems, especially in the aquatic environment where many processes of all types interact with living organisms. The fundamental basis of all aquatic food chains is plankton, and phytoplankton in particular occupies the first trophic level and the fluctuations in its abundance determine the production of a whole marine

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biological output. The dynamics of rapid (massive) increase or almost equally decrease of phytoplankton populations is a common feature in marine plankton ecology and is known as bloom. This phenomenon can occur in a matter of days and can disappear just as rapidly.

Several authors have argued that there has been a global increase in harmful phytoplankton blooms in recent decades, see [7, 14, 24]. The rapid massive growth of phytoplankton is, general1y, caused by high nutrient levels and favorable conditions (water temperature, salinity levels, etc.). Herbivore grazing takes an important role in the bloom dynamics, see [7,24,26]. Toxic substances produced by phytoplankton species reduce the growth of zooplankton by decreasing grazing pressure and this is one of the important common phenomena in plankton ecology, see [10, 17, 22].

Within the broad perspective drawn above, the present paper explores and compares the coupled dynamics of phytoplankton and zooplankton in a number of mathematical models. The system of phytoplankton-zooplankton has attracted considerable attention from various fields of research, see [10, 21, 25]. It is an important issue in mathematical ecology. The literature abounds in models focusing on various aspects of the problem. Recently, the attention has been focused on the role of the space in explaining heterogeneity and the distribution of the species and the influence of the spatial structure on their abundance, [10, 17, 29]. However, the very question of the interactions between phytoplankton and zooplankton depending on space is far from being fully elucidated.

As part of our work, we will highlight a threshold value of the toxin released by phytoplankton below which the effect of the toxin influences less the dynamics of the zooplankton-fish system. The proposed model consists of two interactive component: zooplankton and toxin-producing phytoplankton that reduces the growth of zooplankton population. The model studied here is of the reaction-diffusion type describing the dynamics of the phytoplankton-zooplankton system in the sense of the works of F. Courchamp [8, 28].

The paper is organized as follows. As far as Section 3 is concerned, we will establish mathematical results such as the existence of a solution, stability of equilibria, persistence, relating to the constructed model in Section 2. Section 4 will be devoted to numerical experiments to illustrate the mathematical results. Finally, Section 5 is devoted to the conclusion and perspectives.

2 Mathematical Model

In this section, we propose a model to describe the dynamics of the phytoplankton-zooplankton system in the presence of toxin. We begin our modeling by a general model describing the dynamics of the prey-predator system, based on the equations with ordinary derivatives. And then we transform this model into a model of reaction-diffusion type while remaining in the logic of the work of F. Courchamp [8] and Bendahmane et al [5]. The aim is to take into account the effect of the toxin on the zooplankton-phytoplankton dynamics.

2.1 Original model formulation

Let P be the density of the prey population and Z be the density of the predator population. According to [5, 8, 18, 28], the general model at any time $T > 0$ is written as follows:

$$\begin{cases} \dfrac{dP}{dT} = \phi_1(P) - g_2(P,Z)Z, \\ \dfrac{dZ}{dT} = g_3(P,Z)Z - g_4(P,Z)Z, \end{cases} \tag{1}$$

where

- ϕ_1, g_2, g_3, g_4 are positive functions and C^∞,
- $\phi_1(P)$ is the growth function of the prey population,
- $g_2(P,Z)$ is the amount of prey consumed by a predator per time unit,
- $g_3(P,Z)$ represents the rate of conversion of the prey into predator,
- $g_4(P,Z)$ is the predator mortality rate due to harmful prey consumption.

We continue our modeling by fixing the expressions of the functions intervening in the model (1), see [5, 8]. The dynamics of the system can be represented by the following figure:

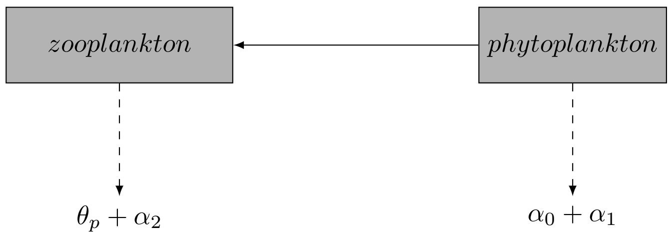

Figure 1: The compartmental model of the zooplankton-phytoplankton system.

According to Figure 1, at any time $T > 0$, the dynamics of the phytoplankton (prey)-zooplankton (predator) system is governed by the following ODE system:

$$\begin{cases} \dfrac{dP}{dT} = r_p P - \alpha_0 P^2 - \dfrac{\alpha_1 Z}{P+\gamma_1}P, \quad P(0) = P_0 \geq 0, \\ \dfrac{dZ}{dT} = r_z Z - \theta_p Z - \dfrac{\alpha_2 Z^2}{P+\gamma_2}, \quad Z(0) = Z_0 \geq 0, \end{cases} \tag{2}$$

where

- r_p denotes the phytoplankton growth rates,
- α_0 denotes the mortality rate due to competition between the individuals of the phytoplankton population,
- α_1 is the maximum value that the reduction rate per individual phytoplankton can reach,
- α_2 is the maximum value that the reduction rate per individual zooplankton can reach,

- r_z denotes the zooplankton growth rates,
- θ_p is the rate of toxic phytoplankton consumed by zooplankton,
- γ_1 is the protection of prey P from the environment,
- γ_2 is the protection of predator Z from the environment.

2.2 A spatially structured model

Here, our aim is to introduce the notion of spatial structuring in the model. By considering the relationship between the climate and the diffusion of species and the fact of the existence of diffusion in population, system (1) is developed into a spatial system with diffusion. We expect to explore the effect of climate change on the plankton population by studying the spatial dynamics of the diffusion system. We will introduce the concept of spatial structure in the model, that is to say that the population densities depend now on the time and space. Diffusion models are a simple and reasonable choice for modeling dispersion of populations on a spatial domain, see [3, 12, 18, 25]. Indeed, let $\delta_0(x)$ and $\delta_1(x)$ be respectively the diffusion terms of P and Z. Based on the results established in [5, 8, 28], the reaction-diffusion model associated with the model (1) can be modeled for $x \in \Omega$ as follows:

$$\begin{cases} \partial_T P - div(\delta_0(x)\nabla P) = \phi_1(P) - g_2(P, Z)P, \\ \partial_T Z - div(\delta_1(x)\nabla Z) = g_3(P, Z)Z - g_4(P, Z)Z, \end{cases} \tag{3}$$

where $\Omega \subset \mathbb{R}^n (n \geq 1)$ is the spatial domain in which species occur. We consider the zero-flux boundary condition

$$\delta_i(x)\nabla Q(x, t).\nu(x) = 0, \qquad i = 0, 1, \qquad x \in \partial\Omega, \qquad T > 0$$

for $Q = P, Z$, where ν is the unit normal vector to $\partial\Omega$ on Ω, and the nonnegative initial and bounded conditions

$$Q(x, 0) = Q_0(x) > 0, \qquad Q = P, Z \qquad x \in \Omega.$$

We make the following assumptions:

(H_1) : all demographic parameters of the system (2) are positive constants,

(H_2) : the diffusion coefficients of the system (3) are independent of the spatial variable.

By considering $\delta_0(x) = \delta_0$, $\delta_1(x) = \delta_1$, and taking into account the hypotheses (H_1) and (H_2), the model (3) obtained previously is written as:

$$\begin{cases} \dfrac{\partial P}{\partial T} = \left(r_p - \alpha_0 P - \dfrac{\alpha_1 Z}{P + \gamma_1} \right) P + \delta_0 \Delta P, \\ \dfrac{\partial Z}{\partial T} = \left(r_z - \theta_p - \dfrac{\alpha_2 Z}{P + \gamma_2} \right) Z + \delta_1 \Delta Z, \\ Q(x, 0) = Q_0(x) > 0, \quad Q = P, Z, \quad x \in \Omega. \end{cases} \tag{4}$$

3 Mathematical Results

In this section, we aim to establish the mathematical results of the system (4). The mathematical results are based on the works [5, 6, 25].

3.1 Reduction of model parameters

To simplify the writing, we will make changes of variables in the following way:

$$r = r_z - \theta_p, \qquad t = r_p T, \qquad U_1(t) = \frac{\alpha_0}{r_p} P(T), \qquad U_2(t) = \frac{\alpha_0 \alpha_2}{r_p r} Z(T),$$
$$a = \frac{r\alpha_1}{r_p \alpha_2}, \qquad b = \frac{r}{r_p}, \qquad d_1 = \frac{\alpha_0 \gamma_1}{r_p}, \qquad d_2 = \frac{\alpha_0 \gamma_2}{r_p},$$
$$x = X\left(\frac{r_p}{\delta_0}\right)^{\frac{1}{2}}, \qquad y = Y\left(\frac{r_p}{\delta_0}\right)^{\frac{1}{2}}, \qquad d_z = \frac{\delta_1}{\delta_0}.$$

Thus, the systems (2) and (4) can be written respectively as follows:

$$\begin{cases} \dfrac{dE_1}{dt} = \left(1 - E_1 - \dfrac{aE_2}{E_1 + d_1}\right) E_1 = f(E_1, E_2), \\ \dfrac{dE_2}{dt} = b\left(1 - \dfrac{E_2}{E_1 + d_2}\right) E_2 = g(E_1, E_2), \end{cases} \tag{5}$$

and

$$\begin{cases} \dfrac{\partial U_1}{\partial t} = \left(1 - U_1 - \dfrac{aU_2}{U_1 + d_1}\right) U_1 + \Delta U_1 = f(U_1, U_2) + \Delta U_1 \\ \dfrac{\partial U_2}{\partial t} = b\left(1 - \dfrac{U_2}{U_1 + d_2}\right) U_2 + d_z \triangle U_2 = g(U_1, U_2) + d_z \Delta U_2. \end{cases} \tag{6}$$

3.2 Existence and boundedness of solution

Before stating the boundedness of the solution, we give the local existence of the solution. The following theorem ensures the existence and uniqueness of the local solution of the system (4).

Theorem 3.1 *[1, 2] The system* (4) *has a unique local solution* $(U_1(.,t), U_2(.,t))$ *under the condition* $0 \leq t < T_{max}$, *where* T_{max} *depends on nonnegative initial data* $U_{01}(x)$ *and* $U_{02}(x)$.

The following theorem ensures the global solution existence for the system (6).

Theorem 3.2 *For any regular positive functions* $U_{01}(x) \leq 1$ *and* $U_{02}(x)$, *the system* (6) *admits a global solution* $(U_1(.,t), U_2(.,t))$ *for any* $t > 0$.

Proof: Indeed:

- on the one hand, we have $U_1(x,t) \geq 0$ and $U_2(x,t) \geq 0$ because 0 is the sub-solution of each equation of the system (6).
- on the other hand, U_1 satisfies the following problem:

$$\begin{cases} \dfrac{\partial U_1(x,t)}{\partial t} \leq U_1(1 - U_1) + \Delta U_1, \\ \dfrac{\partial U_1}{\partial \nu} = 0 \ , \ t > 0, \\ U_1(x,0) = U_{01}(x) \leq \ U_{01} \equiv \ max_{\overline{\Omega}} \ U_{01}(x), x \in \Omega. \end{cases} \tag{7}$$

According to the principle of comparison, we have $U_1(x,t) \le U(t) \le 1$, where $U(t) = \dfrac{U_0}{U_0 + (1-U_0)e^{-t}}$ is the solution of the problem

$$\frac{\partial U}{\partial t} = U(1-U), \quad U(0) = U_0 \le 1. \tag{8}$$

In the same order U_2 satisfies

$$\frac{\partial U_2}{\partial t} = b\left(1 - \frac{U_2}{U_1 + d_2}\right) U_2 + d_z \Delta U_2,$$

and we obtain the following inequality:

$$\frac{\partial U_2}{\partial t} \le \frac{dE_2}{dt},$$

where E_2 is a solution of the second equation of the system (5) with the initial condition $E_2(0) = max_{\overline{\Omega}}\ U_{02}(x)$.

$$\frac{\partial U_2}{\partial t} \le \frac{dE_2}{dt} + \frac{dE_1}{dt}.$$

Let us denote by $\sigma = E_2 + E_1$, we deduce that

$$\frac{\partial U_2}{\partial t} \le \frac{d\sigma}{dt} \le \frac{5}{4} + \frac{(1+b)^2(1+d_2)}{4b} - \sigma.$$

Using the Gronwall lemma, see [10, 18, 29], we deduce that $U_2 \le \dfrac{5}{4} + \dfrac{(1+b)^2(1+d_2)}{4b}$. Thus, the solutions U_1 and U_2 are bounded.

The following theorem ensures the boundedness of the system (6).

Theorem 3.3 *The domain $\mathbb{R}^+ \times \mathbb{R}^+$ is positively invariant for the system (6). Furthermore, any solution of the system (6) whose initial condition is in $\mathbb{R}^+ \times \mathbb{R}^+$ converges to the set defined by $S_1 = [0,1] \times \left[0, \dfrac{5}{4} + \dfrac{(1+b)^2(1+d_2)}{4b}\right]$.*

Proof: For the initial condition $(U_{01}(x), U_{02}(x))$ of the system (6), we have

$$0 \le U_1 \le E_1, \quad E_1(0) = max_{\overline{\Omega}}\ U_{01}(x),$$

$$0 \le U_2 \le E_2, \quad E_2(0) = max_{\overline{\Omega}}\ U_{02}(x).$$

On the other hand, according to [3, 13], we have

$$\overline{\lim}_{t \longrightarrow +\infty} E_1(t) \le 1, \quad \overline{\lim}_{t \longrightarrow +\infty} (E_1(t) + E_2(t)) \le \frac{5}{4} + \frac{(1+b)^2(1+d_2)}{4b}. \qquad \square$$

3.3 Analysis of stationary solutions

Now, we study the existence of positive equilibrium states of the system (6). Let us consider the following system:

$$\begin{cases} \left(1 - U_1 - \dfrac{aU_2}{U_1 + d_1}\right) U_1 + \Delta U_1 = 0, & x \in \Omega, \\ b\left(1 - \dfrac{U_2}{U_1 + d_2}\right) U_2 + d_z \Delta U_2 = 0, & x \in \Omega, \\ \dfrac{\partial U_1}{\partial \nu} = \dfrac{\partial U_2}{\partial \nu} = 0. \end{cases} \tag{9}$$

Then, (U_1, U_2) is a positive equilibrium state of the system (6) if it satisfies the system (9).

Remark 3.1 Let $V_1(x) = (\overline{U}_1(x), \overline{U}_2(x))$ and $V_2(x) = (\underline{U}_1(x), \underline{U}_2(x))$. According to [4], $V_1(x)$ is an over-solution and $V_2(x)$ is a sub-solution of the system (9) if we have

$$\frac{\overline{U}_1}{\partial \nu} \geq 0 \geq \frac{\underline{U}_1}{\partial \nu} \quad on \ \ \partial\, \Omega, \qquad \frac{\overline{U}_2}{\partial \nu} \geq 0 \geq \frac{\underline{U}_2}{\partial \nu} \quad on \ \ \partial\, \Omega,$$

and

$$-\Delta \overline{U}_1 - \overline{U}_1 \left(1 - \overline{U}_1 - \frac{a\underline{U}_2}{\overline{U}_1 + d_1}\right) \geq 0 \geq -\Delta \underline{U}_1 - \underline{U}_1 \left(1 - \underline{U}_1 - \frac{a\overline{U}_2}{\underline{U}_1 + d_2}\right), \tag{10}$$

$$-\Delta \overline{U}_2 - b\overline{U}_2 \left(1 - \frac{U_2}{\overline{U}_1 + d_2}\right) \geq 0 \geq -\Delta \underline{U}_2 - b\underline{U}_2 \left(1 - \frac{U_2}{\underline{U}_1 + d_2}\right). \tag{11}$$

Let us consider the following conditions

$$a < 1, \quad d_1 - a + 1 - A > 0, \tag{12}$$

where A is an over-solution of the second equation of the system (9).

Theorem 3.4 *[6, 12] If the conditions* (12) *are satisfied, then the system* (9) *admits at least one positive solution* $(U_1(x), U_2(x))$.

Proof: We write the system (9) as follows:

$$\begin{cases} -\Delta U_1 = \left(1 - U_1 + \dfrac{aU_2}{U_1 + d_1}\right) U_1 = f(U_1, U_2), & x \in \Omega, \\ -d_z \Delta U_2 = \left(1 - \dfrac{U_2}{U_1 + d_2}\right) U_2 = g(U_1, U_2) & x \in \Omega, \\ \dfrac{\partial U_1}{\partial \nu} = \dfrac{\partial U_2}{\partial \nu} = 0. \end{cases} \tag{13}$$

If $U_1 \geq 0$, $U_2 \geq 0$, we obtain

$$\frac{\partial f}{\partial U_2} = \frac{-aU_1}{U_1 + d_1} \leq 0, \quad \frac{\partial g}{\partial U_2} = \frac{bU_2^2}{(U_1 + d_2)^2} \geq 0.$$

This means that the function f is quasi-monotone decreasing and the function g is quasi-monotone increasing. The system (9) is then called a quasi-monotonic mixed system.

We will now construct a pair of over-solution and sub-solution of the system (9) that we denote respectively by $V_1(x) = (\overline{U}_1(x), \overline{U}_2(x))$ and $V_2(x) = (\underline{U}_1(x), \underline{U}_2(x))$.

Let $\overline{U}_1(x) = 1$, then for every $\overline{U}_2 \geq 0$, the first inequality of (10) is satisfied. By fixing A such that $A \geq \frac{5}{4} + \frac{(1+b)^2(1+d_2)}{4b}$ and by considering $\overline{U}_2(x) = A$, the inequality of (11) is satisfied. If we consider that $\overline{U}_2(x) = A$, the second inequality of (10) becomes

$$-\Delta\underline{U}_1 - \underline{U}_1\left(1 - \underline{U}_1 - \frac{aA}{\underline{U}_1 + d_2}\right) \leq 0.$$

Let $\underline{U}_1(x)$ be the strictly positive solution of the following system:

$$\begin{cases} \Delta\underline{U}_1 - \underline{U}_1\left(1 - \underline{U}_1 - \dfrac{aA}{\underline{U}_1 + d_2}\right) = 0, \\ \dfrac{\partial \underline{U}_1}{\partial \nu} = 0. \end{cases} \tag{14}$$

We will show that if the conditions $a < 1$ and $d_2 - a + 1 - M > 0$ are satisfied, then the system (14) will admit a positive solution. If $a < 1$ and $d_2 - a + 1 - A > 0$, then one can easily verify that $(1; 1-a)$ is a pair of over-solution and sub-solution of the equation (14). This equation admits a positive solution $\underline{U}_1(x)$ which checks $1 - a \leq \underline{U}_1(x) \leq 1$. Obviously, we have $\overline{U}_1(x) \geq \underline{U}_1(x)$.

We take arbitrarily $\overline{U}_1(x)$, $\underline{U}_1(x)$ and $\overline{U}_2(x)$. If $d_2 - a + 1 - A > 0$, then we can choose $\underline{U}_2(x)$ constant positive and small enough so that the following inequality is satisfied:

$$-\Delta\underline{U}_2 - b\underline{U}_2\left(1 - \frac{\underline{U}_2}{\underline{U}_1 + d_2}\right) \leq 0.$$

Note that this inequality is satisfied as soon as $0 < \underline{U}_2 < 1 - a - d_2$. Thus we build a pair of over-solution and sub-solution $(\overline{U}_1(x), \overline{U}_2(x))$ and $(\underline{U}_1(x), \underline{U}_2(x))$ of the system (9).

Thus, the system (9) admits at least one solution $(U_1(x), U_2(x))$ which satisfies

$$\underline{U}_1(x) \leq U_1(x) \leq \overline{U}_1(x), \quad \underline{U}_2(x) \leq U_2(x) \leq \overline{U}_2(x). \qquad \square$$

3.4 Stability of homogeneous stationary solutions

The following result gives the stationary states and their stability condition for the system (6).

Proposition 3.1 *[3, 13, 29]*

(i) $E_0 = (0, 0)$ *is the trivial state. This equilibrium is unstable.*

(ii) $E_1 = (1, 0)$ *is an equilibrium point. This equilibrium is unstable.*

(iii) $E_2 = (0, d_2)$ *is an equilibrium point. This state is unstable if* $d_1 > ad_2$ *and stable if* $d_1 < ad_2$

For the proof of the local stability of E_i, we consider the eigenvalue problem of the corresponding linearized operator, see [3,4,13,28,29]. In fact, we consider $(U_1(x,t), U_2(x,t))$ the solution of the system (6), then, we have

$$(U_1(x,t), U_2(x,t)) = E_i + W(x,t) = E_i + (w_1(x,t), w_2(x,t)).$$

We will make the following hypothesis:

$$(H_3): \quad a \geq \frac{1}{2} \text{ and } 0 < d_1 < \overline{d}_1 \text{ with } \overline{d}_1 = -(a+1)+\sqrt{\xi},\ \xi = (a+1)^2 + 2a(1+2a) - 1.$$

Theorem 3.5 *[6, 12, 27, 29] The interior equilibrium point $E_3 = (U_1^*, U_2^*)$ of the system (6) is stable if the hypothesis (H_3) is satisfied.*

With regard to the analysis of the global stability of the interior equilibrium state, we will make the following hypothesis:
$(H_4): \quad 1 \leq d_1 \leq d_2$.

Theorem 3.6 *Suppose that (H_4) is satisfied, then the equilibrium (U_1^*, U_2^*) of the system (9) is globally asymptotically stable.*

Proof: Let us consider the functions l and L defined by

$$l(U_1, U_2) = \int_{U_1^*}^{U_1} \frac{(\eta - U_1^*)(\eta + d_1)}{a\eta(\eta + d_2)} d\eta + \frac{U_1^* + d_2}{bU_2^*} \int_{U_2^*}^{U_2} \frac{\eta - U_2^*}{\eta} d\eta,$$

$$L(U_1, U_2) = \int_\Omega l(U_1, U_2) dx = \int_\Omega \left(\int_{U_1^*}^{U_1} \frac{(\eta - U_1^*)(\eta + d_1)}{a\eta(\eta + d_2)} d\eta + \frac{U_1^* + d_2}{bU_2^*} \int_{U_2^*}^{U_2} \frac{\eta - U_2^*}{\eta} d\eta \right) dx.$$

Our goal is to show that L is a Lyapunov function, with a negative orbital derivative. For any solution (U_1, U_2) of (6) whose initial condition $(U_{01}(x), U_{02}(x))$ is in the positive quadrant, $L(U_1, U_2)$ is positive. Moreover, $L(U_1, U_2) = 0$ if and only if $(U_1, U_2) = (U_1^*, U_2^*)$. It remains to prove the following inequality $\frac{dL}{dt} < 0$.

Indeed,

$$\begin{aligned}
\frac{dL}{dt} &= \int_\Omega \left(\frac{(U_1 - U_1^*)(U_1 + d_1)}{aU_1(U_1 + d_2)} \right) \left(\Delta U_1 + U_1(1 - U_1) - \frac{aU_1U_2}{U_1 + d_1} \right) dx \\
&+ \int_\Omega \frac{U_1^* + d_2}{bU_2^*} \frac{U_2 - U_2^*}{U_2} \left(d_z \Delta U_2 + b \left(1 - \frac{U_2}{U_1 + d_2} \right) U_2 \right) dx \\
&= \int_\Omega \left(\frac{(U_1 - U_1^*)(U_1 + d_1)}{aU_1(U_1 + d_2)} \right) \left(U_1(1 - U_1) - \frac{aU_1U_2}{U_1 + d_1} \right) dx \qquad (15) \\
&+ \int_\Omega \frac{U_1^* + d_2}{bU_2^*} \frac{U_2 - U_2^*}{U_2} bU_2 \left(1 - \frac{U_2}{U_1 + d_2} \right) dx \\
&+ \int_\Omega \left(\Delta U_1 \frac{(U_1 - U_1^*)(U_1 + d_1)}{aU_1(U_1 + d_2)} + d_z \Delta U_2 \frac{U_1^* + d_2}{bU_2^*} \frac{U_2 - U_2^*}{U_2} \right) dx.
\end{aligned}$$

Let us denote by T_1 the first two right terms of the equality (15) and by T_2 the last term on the right. After a simple calculation followed by a reduction, T_1 becomes:

$$T_1 = -\int_\Omega \left((U_1 + U_1^* + d_1 - 1)\frac{(U_1 - U_1^*)^2}{a(U_1 + d_2)} + \frac{(U_2 - U_2^*)^2}{U_1 + d_2} \right) dx \tag{16}$$

By the green formula, T_2 becomes

$$\begin{aligned} T_2 = &-\int_\Omega \left(\mid \nabla U_1 \mid^2 \frac{d}{dU_1} \left(\frac{(U_1 - U_1^*)(U_1 + d_1)}{aU_1(U_1 + d_2)} \right) \right. \\ &\left. + d_z \frac{U_1^* + d_2}{bU_2^*} \mid \nabla U_2 \mid^2 \frac{d}{dU_2} \left(\frac{U_2 - U_2^*}{U_2} \right) \right) dx \\ = &-\int_\Omega \left(\mid \nabla U_1 \mid^2 \left(\frac{d_2 - d_1 + 1 + U_1^*}{a(U_1 + d_2)^2} + \frac{U_1^* d_1 (2U_1 + d_2)}{a(U_1^2 + d_2 U_1)^2} \right) \right. \\ &\left. + d_z \frac{U_1^* + d_2}{bU_2^*} \mid \nabla U_2 \mid^2 \frac{U_2^*}{U_2^2} \right) dx \end{aligned} \tag{17}$$

According to the expressions of T_1 and T_2, we have $\frac{dL}{dt}(U_1, U_2) < 0$.

Therefore, according to the LaSalle's theorem [7], the equilibrium point (U_1^*, U_2^*) is globally asymptotically stable.

4 Numerical Results

In this section, we perform extensive numerical simulations of the spatial model system (9) in two dimensional space using the forward finite difference method. The set of parameter values used for the numerical simulation is given in Table 1, see [5, 9, 11, 12]. Here, the system is studied on a spatial domain $\Omega = [0, 50] \times [0, 50]$. It is assumed that the zooplankton and phytoplankton populations are spread over the whole domain at the beginning. These results show that for every strictly positive initial condition, under the assumptions $(H1) - (H4)$, the non-homogeneous equilibrium is always globally asymptotically stable.

Param	Description	Values	Refs
r_p	the natural growth-rate of phytoplankton	1.58	[9, 20]
δ_0	diffusivity coefficient of P	5	[9, 20]
δ_1	diffusivity coefficient of Z	600.5	[5]
α_0	mortality rate due to competition between the individuals of P	0.30	[9]
γ_1	the protection of prey P from the environment	0.00661	[5]
r_z	the zooplankton growth rates	0.25	[9]
α_2	maximum value of the reduction rate per individual of Z	0.26	[9]
γ_2	the protection of predator Z from the environment	0.231	[15]
P_0	initial condition of the phytoplankton	150	[16]
F_0	initial condition of the zooplankton	100	[15]

Table 1: Parameters values for the numerical simulation of the system.

4.1 Pattern formation

Here, we will illustrate the mathematical predictions, by numerical simulations, concerning the behavior of the dynamics under the hypotheses $(H_1) - (H_4)$. The qualitative results of different pattern formations due to the variation of α_1 are shown. We consider the value of released toxin $\theta_p = 0.06$. These numerical results show that for every strictly positive initial condition, under the assumptions $(H_1)-(H_4)$, the non-homogeneous equilibrium is always globally asymptotically stable. Figure $2-7$ show the spatial structures formation for the two species described in (9). This numerical results confirm the mathematical results for the existence of positive equilibrium and its stability according to the values of α_1. In this case, we will speak of a subsistence phenomenon of the zooplankton population.

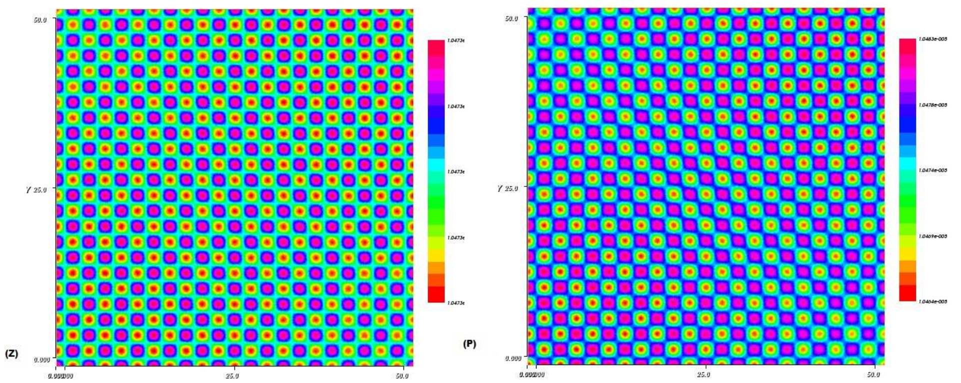

Figure 2: Spatial distribution of the two species, zooplankton and phytoplankton, if we consider in the system $\alpha_1 = 0.125$ and $d_z = 120.1$.

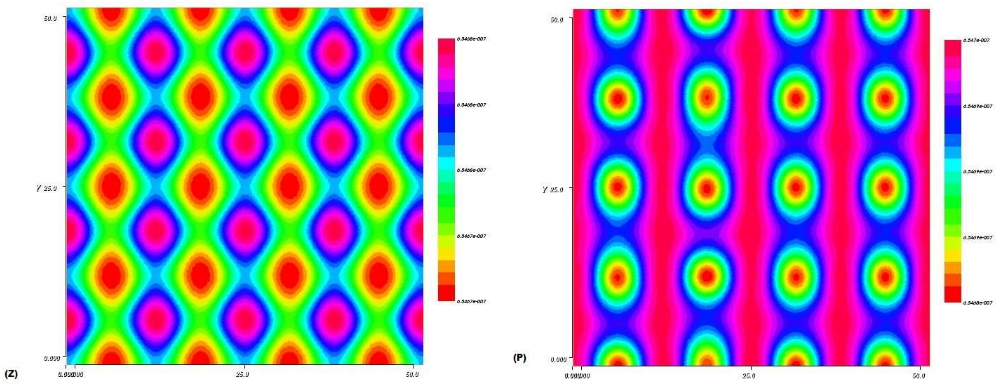

Figure 3: Spatial distribution of the two species, zooplankton and phytoplankton, if we consider in the system $\alpha_1 = 0.195$ and $d_z = 120.1$.

Remark 4.1 From a biological point of view, these results (Figure $2-7$) show that there is coexistence between the two populations despite the release of the toxin into the aquatic environment. This means that despite the harmful effects of the toxin released by phytoplankton, the zooplankton population persists.

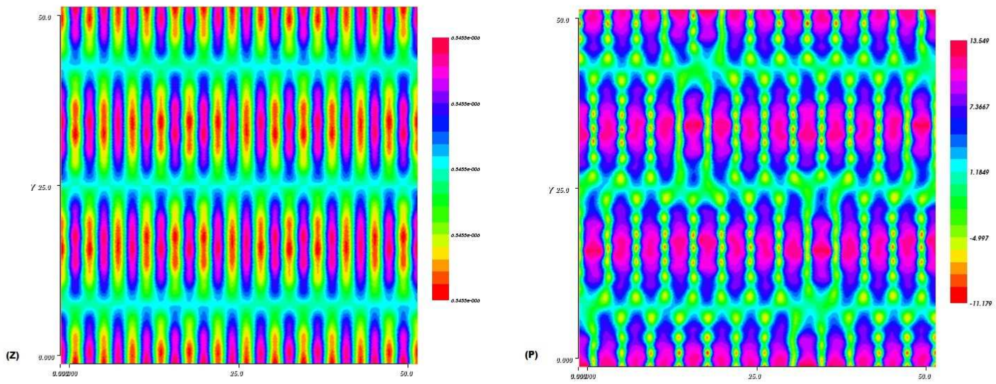

Figure 4: Spatial distribution of the two species, zooplankton and phytoplankton, if we consider in the system $\alpha_1 = 0.198$ and $d_z = 120.1$.

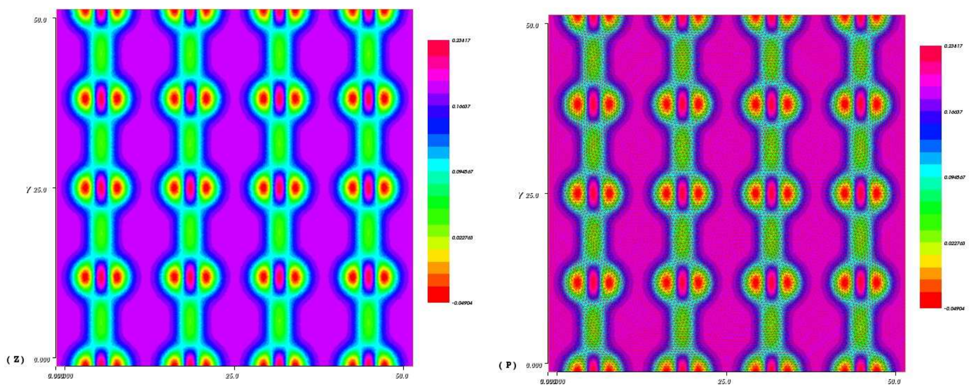

Figure 5: Spatial distribution of the two species, zooplankton and phytoplankton if we consider in the system $\alpha_1 = 0.205$ and $d_z = 120.1$.

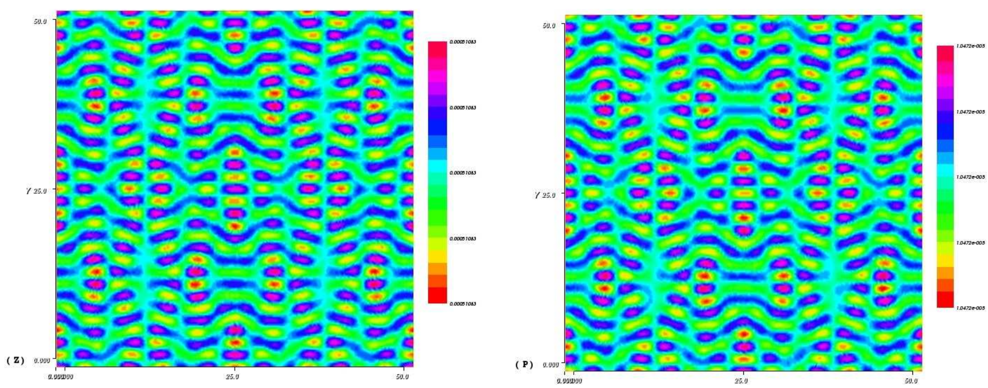

Figure 6: Spatial distribution of the two species, zooplankton and phytoplankton, if we consider in the system $\alpha_1 = 0.23$ and $d_z = 120.1$.

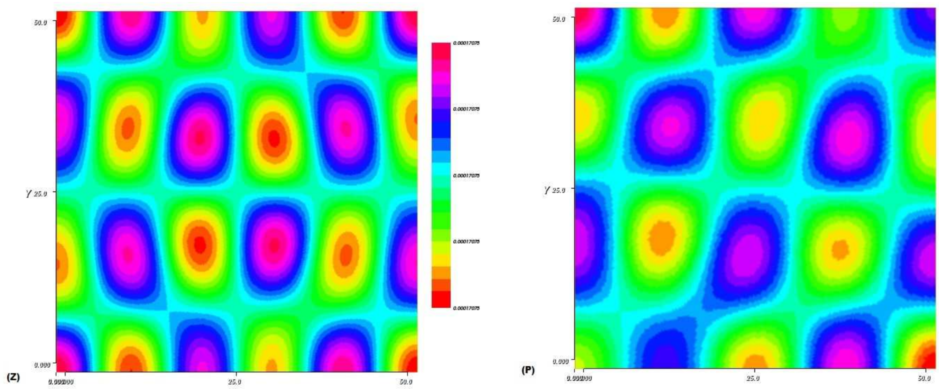

Figure 7: Spatial distribution of the two species, zooplankton and phytoplankton, if we consider in the system $\alpha_1 = 0.26$ and $d_z = 120.1$.

4.2 Analysis of the dynamics behavior with toxin effect

We continue our numerical study in this sub-section to look at the dynamics behavior of the system by considering different values of the toxin parameter. Here, we consider that $\alpha_1 = 0.25$. The numerical simulations show that after a transitional phase, the equilibrium can be established with coexistence of the two populations. Figure $8 - 11$ show the behavior of the two populations. As a biological interpretation we can say that if the toxin is released below this value the impact is not significant on the zooplankton population (Figure $8 - 10$). In fact, the effect does not disrupt the survival of other species. Figure 11 shows the spatial distributions of the two populations. A less dense distribution of the zooplankton population than the previous one was observed. This explains the considerable decrease of these species due to the increase in the number of toxic phytoplankton. There is a strong distribution of the phytoplankton population. Since the distribution is high, this explains the release of the toxin in large quantities by this population. This period corresponds to the phytoplankton bloom.

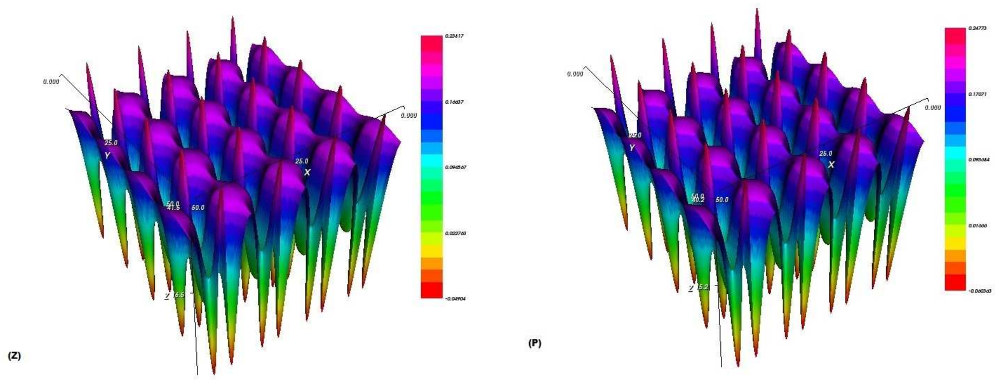

Figure 8: Dynamics behavior of the two species with $\alpha_1 = 0.26$, $d_z = 120.1$ and $\theta_p = 0.2$

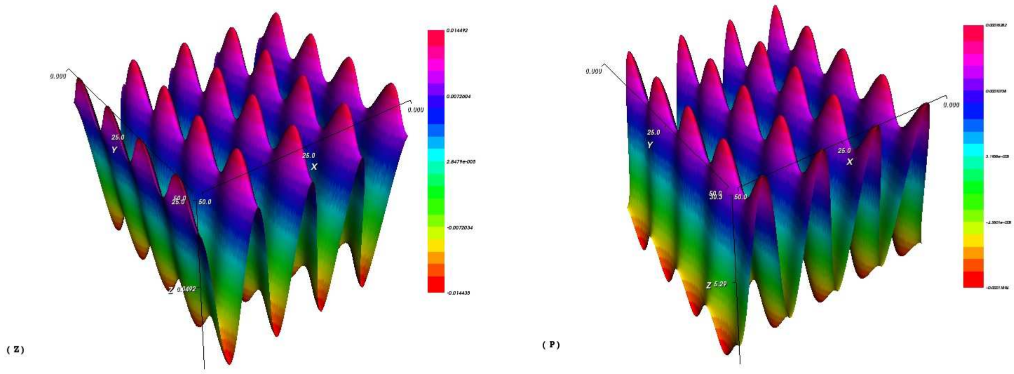

Figure 9: Dynamics behavior of the two species with $\alpha_1 = 0.26, \quad d_z = 120.1$ and $\theta_p = 0.35$.

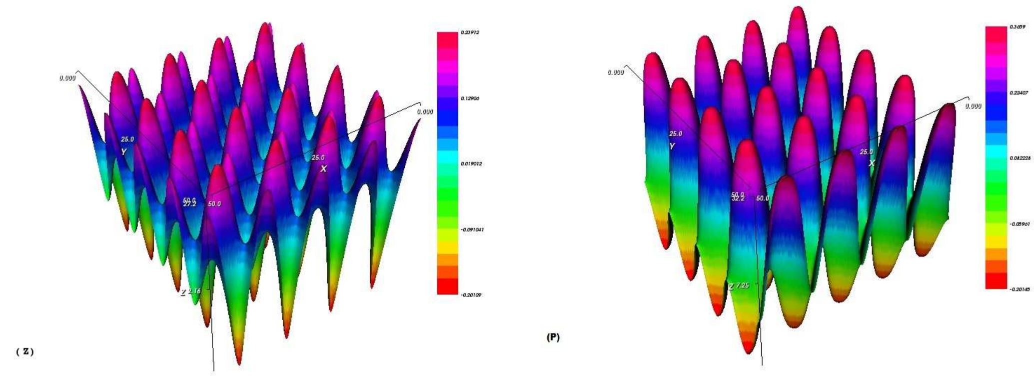

Figure 10: Dynamics behavior of the two species with $\alpha_1 = 0.26, \quad d_z = 120.1$ and $\theta_p = 0.504$.

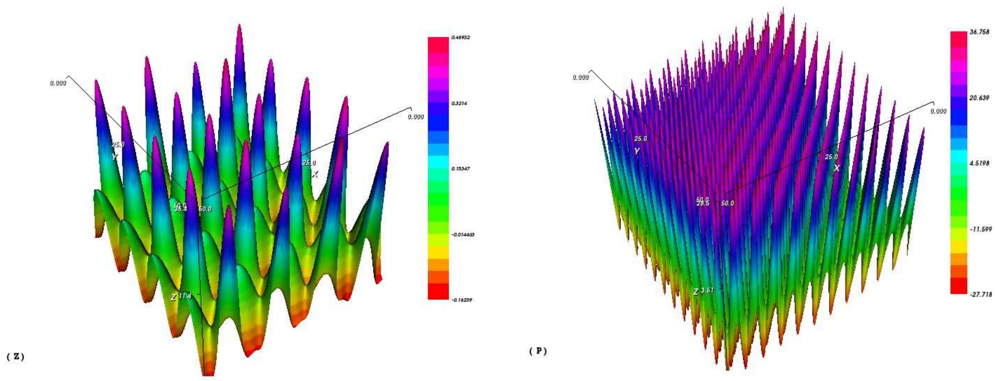

Figure 11: Dynamics behavior of the two species with $\alpha_1 = 0.26, \quad d_z = 120.1$ and $\theta_p = 1.4$.

5 Conclusion

In this paper, our interest is the formulation of a reaction-diffusion model to represent the dynamics of zooplankton and phytoplankton population by taking into account the effect of the toxin. The model construction is derived from an ODE system by considering an isotropic distribution as in [5, 8]. It should be noted that we consider diffusion independently of the spatial variable in the construction of the reaction-diffusion model. The mathematical results allowed us to establish conditions of existence of equilibrium which depend on the demographic parameters. We also gave some results about the stability of the stationary equilibria and we established the conditions on the non existence of the equilibrium with strictly positive components.

We continued our study through numerical experiments in order to confirm our mathematical results. The numerical results have yielded interesting results on the effect of the toxin on the dynamics. This is why we are led to conclude that the release of the toxin under certain conditions, in the aquatic environment contributes to the regulation of the system. The phytoplankton bloom was observed during our simulations and is in perfect agreement with the biological observations.

Despite of important results on this dynamics, in order to further our study, we consider, for our future work, to clearly subdivide the phytoplankton population into toxic phytoplankton and non-toxic phytoplankton to extend our results to these types of cross-diffusion system.

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