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Mathematical Analysis of a Differential Equation Modeling Charged Elements Aggregating in a Relativistic Zero-Magnetic Field

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Abstract: We analyze, in spaces of distributions with finite higher moments, discrete mass and momentum dependent equations describing the movement of charged particles (electrons, ions) aggregating and moving in a relativistic zero-magnetic field. The model is a combination of two processes (kinetic and aggregation), each of which is proven to be separately conservative. Under specific hypothesis, notably on the relativistic work and aggregation rate, we prove existence results for the full model using the perturbation theory and the subordination principle. This result may have a great impact, especially in the full control of the total number of charged particles described by the model.

Keywords: *fractional differential model; magnetic field; perturbation; kinetic processes; subordination principle; aggregation; well-posedness.*

Mathematics Subject Classification (2010): 26A33, 12H20, 34D10, 46S20.

1 Introduction

It is well known [1] that magnetic fields can be produced by charged particles moving in the space. The particles such as electrons or ions, produce complicated but well known magnetic fields that depend on their charge, and their momentum. There are numerous applications and implications of the effects caused by the movements of charged particles in (zero) magnetic fields. The most common example, in consequence of the recent discoveries in the technology of ultrahigh intensity lasers and high current relativistic

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charged bunch sources, is the use of laser pulses together with charged bunches for excitation of strong waves (for example, plasma containing charged particles). The excited waves can be used, for example, for acceleration of charged particles and focusing of bunches [2,3]. Another example in optics is the production of pulses of light of extremely short duration using the mode-locking technique [3]. In biophysics it was proved [4] that the 250-fold screening of the geomagnetic field, which is a "zero" magnetic field with an induction, affects early embryogenesis and the capacity of some animals (a mouse, for instance) to reproduce.

On the other side, various types of pure aggregation equations have been comprehensively analyzed in numerous works (see, e.g., [5–12]). Conservative and nonconservative regimes for pure fragmentation equations have been thoroughly investigated, sometime leading to dishonesty in the process, that is, a process in which models are based on the principle of conservation of mass (individuals, or particles) but which generate solutions that are not conservative.

It is possible to combine the two processes described above into one unique model (the full model). However the analysis and the well posedness of this model are still hardly explored in the domain of mathematical and abstract analysis. Kinetic-type models with diffusion, growth or decay were globally investigated in [13–16], where the authors showed that the transport part does not affect the breach of the conservation laws.

At a macroscopic level, the discrete mass of charged particles (molar or relative molar mass) can be considered during the modeling. Thus, we obtain the following generalized model derived from the combination of Vlasov-Maxwell equations [17] and aggradation equation [18]:

$$\begin{aligned} D_t^\alpha g(t,x,p,n) &= -\frac{\gamma p}{n}\frac{\partial g(t,x,p,n)}{\partial x} + qE\frac{\partial g(t,x,p,n)}{\partial p} - a(x,p,n)g(t,x,p,n) \\ &\quad + \sum_{m=n+1}^{\infty} a(x,p,m)b(x,p,n,m)g(t,x,p,m), \\ g(0,x,p,n) &= \overset{o}{g}(x,p,n), \quad t\in\mathbb{R}, \quad n=1,2,3\ldots, \end{aligned} \tag{1}$$

where D_t^α is defined as

$$D_t^\alpha g(t,x,p,n) = \frac{\partial^\alpha}{\partial t^\alpha} g(t,x,p,n) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-r)^{-\alpha}\frac{\partial}{\partial r} g(r,x,p,n)dr, \tag{2}$$

with $0<\alpha\le 1$ and represents the fractional derivative of the function g in the sense of Caputo [19], where Γ is the gamma-function $\Gamma(\zeta)=\int_0^\infty t^{\zeta-1}e^{-t}dt$. Moreover, the distribution function $g_n \equiv g(t,x,p,n)$ describes the density of groups of size n, that is, the number of particles (electrons or ions) having approximately the momentum p near the position x at time t. Here the independent variables (x,p,n) take values in a set $\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{N}$ and γ is a Lorentz factor. We assume that the mass n of a cluster in motion is dependent on γ and the rest mass n_0, $n=\gamma n_0$. This implies that the relativistic momentum relation takes the same form as for the classical momentum, $p=\gamma p_0$. $a_n=a(x,p,n)\ge 0$ is the average aggregation rate, that is, the average number at which clusters of size n undergo splitting, $b_{n,m}=b(x,p,n,m)\ge 0$ is the average number of n-groups produced upon the splitting of m-groups. Equation (3) is really complex: the first member on its right-hand side represents the kinetic process due to the effect of charged particles in the relativistic zero-magnetic field E, while the second term represents the fission of groups of size n (the loss due to the fragmentation) and the third term is the fission to form groups of size n (the gain due to the fragmentation). The analysis of such a model required us to proceed step by step as we will see in the following sections. To analyse the generalized model (1) with $0 < \alpha \leq 1$, we need to start with the case $\alpha = 1$. We shall therefore fully study the well-posedness for the case $\alpha = 1$ and then extend the analysis to the general case $0 < \alpha \leq 1$ by exploiting the subordination principle [6, 20–22].

2 Existence Results: The Case $\alpha = 1$

2.1 Well-posedness of the full model

The case $\alpha = 1$ yields from (1) the following model

$$\begin{aligned} \frac{\partial g}{\partial t}(t,x,p,n) &= -\frac{\gamma p}{n}\frac{\partial g(t,x,p,n)}{\partial x} + qE\frac{\partial g(t,x,p,n)}{\partial p} - a(x,p,n)g(t,x,p,n) \\ &\quad + \sum_{m=n+1}^{\infty} a(x,p,m)b(x,p,n,m)g(t,x,p,m), \\ g(0,x,p,n) &= \overset{o}{g}(x,p,n), \quad t \in \mathbb{R}, \quad n = 1,2,3\ldots. \end{aligned} \tag{3}$$

Throughout this work we assume that the following hypotheses are satisfied.

(**H1**): $b_{n,m} = 0$ for all $m \leq n$ (since a group of size $m \leq n$ cannot split to form a group of size n);

(**H2**): $a_1 = 0$ (a cluster of size one cannot split);

(**H3**): $\sum_{m=1}^{n-1} m b_{m,n} = n$, $(n = 2,3,...)$, (the sum of all individuals obtained by fragmentation of an n-group is equal to n);

The total number of particles, no matter the momentum in the space, is given by

$$U(t) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\sum_{n=1}^{\infty} ng(t,x,p,n)dxdp = \sum_{n=1}^{\infty} n\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} g(t,x,p,n)dxdp.$$

This number is normally not changed by interactions among groups, so we expect the following conservation law to be satisfied:

$$\frac{d}{dt}U(t) = 0. \tag{4}$$

Since $g_n = g(t,x,p,n)$ is the density of groups of size n with the momentum p near the position x at time t and the total number of particles is expected to be conserved, it is appropriate to work in the Banach space

$$\mathcal{X}_1 := \{\mathbf{h} = (h_n)_{n=1}^{\infty} : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{N} \ni (x,p,n) \to h_n(x,p), \ \|\mathbf{h}\|_1 := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\sum_{n=1}^{\infty} n|h_n(x,p)|dxdp < \infty\}. \tag{5}$$

We choose to restrict our analysis to a smaller class of functions, the class of distributions with finite higher moments

$$\{\ \mathcal{X}_r := \{\mathbf{h} = (h_n)_{n=1}^{\infty} : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{N} \ni (x,p,n) \to h_n(x,p), \ \|\mathbf{h}\|_r := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\sum_{n=1}^{\infty} n^r|h_n(x,p)|dxdp < \infty\}, \tag{6}$$

$r \geq 1$, which coincides with $\mathcal{X}_1$ for $r = 1$. We assume that for each $t \geq 0$, the function $(x, p, n) \longrightarrow g(x, p, n) = g_n(x, p)$ is such that $\mathbf{g} = (g_n(x, p))_{n=1}^{\infty}$ is from the space $\mathcal{X}_r$ with $r \geq 1$. In $\mathcal{X}_r$ we can rewrite (3) in a more compact form

$$\begin{aligned} &\frac{\partial}{\partial t}\mathbf{g} = \mathbf{Tg} - \mathcal{A}\mathbf{g} + \mathfrak{B}\mathcal{A}\mathbf{g} := \mathbf{Tg} + \mathcal{F}\mathbf{g}, \\ &\mathbf{g}_{|_{t=0}} = \overset{\circ}{\mathbf{g}}. \end{aligned} \tag{7}$$

Here $\mathbf{g}$ is the vector $(g(t, x, p, n))_{n\in\mathbb{N}}$, $\mathcal{A}$ is the diagonal matrix $(a_n)_{n\in\mathbb{N}}$, $\mathfrak{B} = (b_{n,m})_{1\leq n\leq m-1, m\geq 2}$, $\mathbf{T}$ is the transport expression defined as $(g(t, x, p, n))_{n\in\mathbb{N}} \longrightarrow \left(\tilde{\mathcal{T}}_n[g(t, x, p, n)]\right)_{n=1}^{\infty}$ with

$$\tilde{\mathcal{T}}_n[g(t, x, p, n)] := -\frac{\gamma p}{n}\frac{\partial g(t, x, p, n)}{\partial x} + qE\frac{\partial g(t, x, p, n)}{\partial p}. \tag{8}$$

$\overset{\circ}{\mathbf{g}}$ is the initial vector $(\overset{\circ}{g}_n(x, p))_{n\in\mathbb{N}}$ which belongs to $\mathcal{X}_r$ and $\mathcal{F}$ is the fragmentation expression defined by

$$\mathcal{F}\mathbf{g} := \left(-a_n g(t, x, p, n) + \sum_{m=n+1}^{\infty} b_{n,m} a_m g(t, x, p, m)\right)_{n=1}^{\infty}. \tag{9}$$

Proposition 2.1 *The fragmentation model described by* (9) *is formally conservative.*

Proof. We aim to show that (4) is satisfied, that is,

$$\frac{d}{dt}U(t) = \frac{d}{dt}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\sum_{n=1}^{\infty} n g(t, x, p, n)dxdp = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\sum_{n=1}^{\infty} n\frac{\partial}{\partial t}g(t, x, p, n)dxdp = 0.$$

It suffices to show that

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\sum_{m=1}^{\infty} a_m|g_m(x, p)|mdxdp = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\sum_{n=1}^{\infty} n\left(\sum_{m=n+1}^{\infty} b_{n,m}a_m|g_m(x, p)|\right)dxdp.$$

Making use of assumptions (**H1**)–(**H3**), we have

$$\begin{aligned} &\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\sum_{n=1}^{\infty} n\left(\sum_{m=n+1}^{\infty} b_{n,m}a_m|g_m(x, p)|\right)dxdp \\ = &\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\sum_{m=2}^{\infty} a_m|g_m(x, p)|\left(\sum_{n=1}^{\infty} nb_{n,m}\right)dxdp \\ = &\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\sum_{m=2}^{\infty} a_m|g_m(x, p)|\left(\sum_{n=1}^{m-1} nb_{n,m}\right)dxdp \\ = &\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\sum_{m=2}^{\infty} a_m|g_m(x, p)|mdxdp \\ = &\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\sum_{m=1}^{\infty} a_m|g_m(x, p)|mdxdp, \end{aligned} \tag{10}$$

which ends the proof.

In this work, for any subspace $S \subseteq \mathcal{X}_r$, we will denote by S_+ the subset of S defined as $S_+ = \{\mathbf{h} = (h_n)_{n=1}^{\infty} \in S; h_n(x,p) \geq 0, n \in \mathbb{N}, x \in \mathbb{R}^3\}$. Note that any $\mathbf{h} \in (\mathcal{X}_r)_+$ possesses moments

$$M_q(\mathbf{h}) := \sum_{n=1}^{\infty} n^q h_n$$

of all orders $q \in [0,r]$. Imposing $r > 1$ ensures that a significant amount of mass after fragmentation is concentrated in small particles. This has the physical interpretation that surface effects are reduced, i.e. it is unlikely that a large cluster will fragment into large groups, therefore making more clusters with small sizes and concentrated at the origin. In $\mathcal{X}_r$, we define the operators $\mathbf{A}$ and $\mathbf{B}$ by

$$\mathbf{A}\mathbf{h} := (a_n h_n)_{n=1}^{\infty}, \quad D(\mathbf{A}) := \{\mathbf{h} \in \mathcal{X}_r : \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \sum_{n=1}^{\infty} n^r a_n |h_n(x,p)| dxdp < \infty\}; \tag{11}$$

$$\mathbf{B}\mathbf{h} := (B_n h_n)_{n=1}^{\infty} = \left(\sum_{m=n+1}^{\infty} b_{n,m} a_m h_m\right)_{n=1}^{\infty}, \quad D(\mathbf{B}) := D(\mathbf{A}). \tag{12}$$

Throughout, we assume that the coefficients a_n and $b_{n,m}$ satisfy the mass conservation conditions (H1)-(H3). Now let us prove that $\mathbf{B}$ is well defined on $D(\mathbf{A})$. Using the condition (H1)–(H3), we can prove that [5]

$$\sum_{m=1}^{n-1} m^r b_{m,n} \leq n^r \tag{13}$$

for $r \geq 1$, $n \geq 2$. Note that the equality holds for $r = 1$. Using this inequality we have, for every $\mathbf{h} \in D(\mathbf{A})$,

$$\begin{aligned}
& \|\mathbf{B}\mathbf{h}\|_r \\
= \;& \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \sum_{n=1}^{\infty} n^r \left(\sum_{m=n+1}^{\infty} b_{n,m} a_m |h_m(x,p)|\right) dxdp \\
= \;& \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \sum_{m=2}^{\infty} a_m |h_m(x,p)| \left(\sum_{n=1}^{\infty} n^r b_{n,m}\right) dxdp \\
= \;& \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \sum_{m=2}^{\infty} a_m |h_m(x,p)| \left(\sum_{n=1}^{m-1} n^r b_{n,m}\right) dxdp = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \sum_{m=2}^{\infty} a_m |h_m(x,p)| m^r dxdp \\
= \;& \|\mathbf{A}\mathbf{h}\|_r < \infty.
\end{aligned}$$

Then $\|\mathbf{B}\mathbf{h}\|_r \leq \|\mathbf{A}\mathbf{h}\|_r$, for all $\mathbf{h} \in D(\mathbf{A})$, so that we can take $D(\mathbf{B}) := D(\mathbf{A})$ and $(\mathbf{A}+\mathbf{B},\, D(\mathbf{A}))$ is well-defined.

3 Analysis of the Transport Operator in $\Lambda = \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{N}$

Our primary objective in this section is to analyze the solvability of the Cauchy problem for the transport equation

$$\frac{\partial}{\partial t} g(t,x,p,n) = -\frac{\gamma p}{n}\frac{\partial g(t,x,p,n)}{\partial x} + qE\frac{\partial g(t,x,p,n)}{\partial p}, \tag{14}$$

$$g(0,x,p,m) = \overset{o}{g}_n(x,p), \quad t \in \mathbb{R}, \;\; n = 1,2,3\ldots$$

or its compact form

$$\frac{\partial}{\partial t}\mathbf{p} = \mathbf{T}\mathbf{p}, \qquad \mathbf{p}_{|_{t=0}} = \overset{\circ}{\mathbf{p}}. \tag{15}$$

in the space $\mathcal{X}_r$.

3.1 Setting

We note that the operators on the right-hand side of (7) have the property that one of the variables is a parameter and, for each value of this parameter, the operator has a certain desirable property (like being the generator of a semigroup) with respect to the other variable. Thus we need to work with parameter-dependent operators that can be "glued"together in such a way that the resulting operator inherits the properties of the individual components. Let us provide a framework for such a technique called the method of semigroups with a parameter. Let us consider the space $\mathcal{X} := L_g(S, X)$ where $1 \leq p < \infty$, (S, dm) is a measure space and X is a Banach space. Let us suppose that we are given a family of operators $\{(A_s, D(A_s))\}_{s\in S}$ in X and define the operator $(\mathbb{A}, D(\mathbb{A}))$ acting in $\mathcal{X}$ according to the following formulae:

$$\mathcal{D}(\mathbb{A}) := \{h \in \mathcal{X}; h(s) \in D(A_s) \text{ for almost every } s \in S, \;\; \mathbb{A}h \in \mathcal{X}\}, \tag{16}$$

and, for $h \in \mathcal{D}(\mathbb{A})$,

$$(\mathbb{A}h)(s) := A_s h(s), \tag{17}$$

for every $s \in S$. We have the following proposition.

Proposition 3.1 *(see [5, 13, 14]). If for almost any $s \in S$ the operator A_s is m-dissipative in X, and the function $s \longrightarrow R(\lambda, A_s)h(s)$ is measurable for any $\lambda > 0$ and $h \in \mathcal{X}$, then the operator $\mathbb{A}$ is an m-dissipative operator in $\mathcal{X}$. If $(G_s(t))_{t\geq 0}$ and $(\mathcal{G}(t))_{t\geq 0}$ are the semigroups generated by A_s and $\mathbb{A}$, respectively, then for almost every $s \in S$, $t \geq 0$, and $h \in \mathcal{X}$ we have*

$$[\mathcal{G}(t)h](s) := G_s(t)h(s). \tag{18}$$

Using the above ideas, we introduce relevant operators in the present applications. In the transport part of (7), the variable n is the parameter and x is the main variable. We set

$$\mathbb{X} := L_1(\mathbb{R}^3 \times \mathbb{R}^3, dxdp) := \{\psi : \|\psi\| = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3} |\psi(x,p)|dxdp < \infty\}$$

and define in $\mathbb{X}$ the operators $(\mathcal{T}_n, D(\mathcal{T}_n))$ as

$$\begin{aligned} \mathcal{T}_n g_n = \tilde{\mathcal{T}}_n g_n, &\quad \text{with } \tilde{\mathcal{T}}_n g_n \;\; \text{represented by (8)} \\ D(\mathcal{T}_n) := \{g_n \in \mathbb{X}, &\;\; \mathcal{T}_n g_n \in \mathbb{X}\}, \quad n \in \mathbb{N}. \end{aligned} \tag{19}$$

Then we introduce the operator $\mathbf{T}$ in $\mathcal{X}_r$ defined by

$$\begin{aligned} &\mathbf{T}\mathbf{g} = (\mathcal{T}_n g_n)_{n\in\mathbb{N}}, \\ &D(\mathbf{T}) = \{\mathbf{g} = (g_n)_{n\in\mathbb{N}} \in \mathcal{X}_r, \; g_n \in D(\mathcal{T}_n) \text{ for almost every } n \in \mathbb{N}, \; \mathbf{T}\mathbf{g} \in \mathcal{X}_r\}. \end{aligned} \tag{20}$$

Making use of Proposition 3.1, we can take $\mathbb{A} = \mathbf{T}$, $\mathcal{X} = \mathcal{X}_r = L_1(\mathbb{N}, \mathbb{X}) = L_1(\Lambda, d\mu dm_r) = L_1(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{N}, d\mu dm_r)$, where $\mathbb{N}$ is equipped with the weighted counting measure dm_r with weight n^r and $d\mu = dxdp = d\mathbf{z}$ is the Lebesgue measure in $\mathbb{R}^6$. In the notation of the proposition, $(\mathbb{N}, dm_r) = (S, dm)$, $\mathbb{X} = X$ and $A_s = \mathcal{T}_n$, therefore $(\mathcal{T}_n, D(\mathcal{T}_n))_{n\in\mathbb{N}}$ is a family of operators in $\mathbb{X}$ and using (17), we have

$$(\mathbf{T}\mathbf{g})_n := \mathcal{T}_n g_n. \tag{21}$$

Here, $\mathcal{T}_n g_n$ is understood in the sense of distribution. Now we can properly study the transport operator $\mathbf{T}$. Let us fix $n \in \mathbb{N}$. We consider the function $\digamma_n : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined by $\digamma_n(x,p) = (-\frac{\gamma p}{n},\ qE(x,p))$. For each $n \in \mathbb{N}$, we assume the following:

(**H4**): $\digamma_n$ is globally Lipschitz continuous;

(**H5**): $\digamma_n \in L^1_{loc}(\mathbb{R}^3 \times \mathbb{R}^3; \mathbb{R}^3 \times \mathbb{R}^3)$; and $\mathrm{div}\digamma_n \in L^1_{loc}(\mathbb{R}^3 \times \mathbb{R}^3)$;

(**H6**): $\overset{o}{g}_n \in L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$. Let us set $\mathbf{z} = (x,p) \in \mathbb{R}^3 \times \mathbb{R}^3$, we rely on the following definition.

Definition 3.1 A function g_n is called a (weak) L^∞-solution to (14) if $g_n \in L^\infty([0,T] \times \mathbb{R}^3 \times \mathbb{R}^3$ and moreover, for every test function $\Psi \in C_0^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$,

$$\int_{\mathbb{R}^6} \Psi(\mathbf{z}) g_n(t,\mathbf{z}) d\mathbf{z} = \int_{\mathbb{R}^6} \Psi(\mathbf{z}) \overset{o}{g}_n(\mathbf{z}) d\mathbf{z} + \int_0^t d\sigma \int_{\mathbb{R}^6} g_n(\sigma,\mathbf{z}) (\digamma_n(\sigma,\mathbf{z}) \cdot \nabla\Psi(\mathbf{z}) + \Psi(\mathbf{z}) \mathrm{div}\digamma_n(\sigma,\mathbf{z})) d\mathbf{z},$$

$t \in \mathbb{R}$.

Lemma 3.1 *In $\mathbb{X}$ the existence and uniqueness of L^∞-solutions to (14) hold if the above assumptions (**H4**)-(**H6**) are satisfied.*

We prove it by uniquely solving the characteristic ordinary differential equations

$$\begin{aligned} \dot{\gimel}_n(s) &= \digamma_n(\gimel_n(s)), \qquad s \in \mathbb{R}, \\ \gimel_n(t) &= \mathbf{z}, \end{aligned} \tag{22}$$

with $\mathbf{z} \in \mathbb{R}^3 \times \mathbb{R}^3$ and $t \in \mathbb{R}$, which have one and only one solution $\gimel_n(s)$ taking values in $\mathbb{R}^3 \times \mathbb{R}^3$. Thus we find the flow $(\phi^n_{t,s})$, $t,\ s \in \mathbb{R}$ generated by $\digamma_n$ with $\phi^n_{t,s} : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \times \mathbb{R}^3$, that is,

1. $\phi^n_{t,s}(\mathbf{z}) = \gimel_n(s)$, where $\gimel_n(s)\ \ s \in \mathbb{R}$, solves (22),
2. $\phi^n_{t,s}(\mathbf{z}) = \phi^n_{\tau,s}(\phi^n_{t,\tau}(\mathbf{z}))$, $t,\ s,\ \tau \in \mathbb{R}$,
3. The transformations $\phi^n_{t,s} : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \times \mathbb{R}^3$ are Lipschitz-homeomorphism.

Note that the functions $\phi^n_{t,s}$ possess many more desirable properties as listed in [5, 23–25] that are relevant for studying the transport operator in $\mathcal{X}_r$. Then making use of $g_n(t, \phi^n_{0,t}(\mathbf{z})) = \overset{o}{g}_n(\mathbf{z})$, we obtain the unique solution to (14) given by

$$g_n(t,x,p) = \overset{o}{g}_n((\phi^n_{0,t})^{-1}(x,p)).$$

It is obvious that this solution belongs to $D(\mathcal{T}_n)$. Therefore the operator $(\mathcal{T}_n, D(\mathcal{T}_n))$ generates a semigroup given by

$$[G_{\mathcal{T}_n}(t) g_n](x,p) = g_n((\phi^n_{0,t})^{-1}(x,p)), \tag{23}$$

$g_n \in \mathbb{X}$. For existence and uniqueness in the full space $\mathcal{X}_r$, we state the following.

Proposition 3.2 *Under the conditions of Lemma 3.1, there is one and only one L^∞-solution to (15) holding in $\mathcal{X}_r$ and belonging to $D(\boldsymbol{T})$.*

Proof. The proof follows immediately from relation (21) and Lemma 3.1

4 Generalization: Existence Results for $0 < \alpha \leq 1$

Now, as we have fully analized the special case (3), proved its well-posedness and shown its existence results, we can come back to the general model (1):

$$\begin{aligned} D_t^\alpha g(t,x,p,n) &= -\frac{\gamma p}{n}\frac{\partial g(t,x,p,n)}{\partial x} + qE\frac{\partial g(t,x,p,n)}{\partial p} - a(x,p,n)g(t,x,p,n) \\ &+ \sum_{m=n+1}^{\infty} a(x,p,m)b(x,p,n,m)g(t,x,p,m), \\ g(0,x,p,n) &= \overset{o}{g}(x,p,n), \quad t \in \mathbb{R}, \quad n = 1,2,3\ldots. \end{aligned} \tag{24}$$

This model can be written in the same way as the perturbed transport equation (7) above to read as

$$\begin{aligned} D_t^\alpha \mathbf{g} &= \mathbf{Tg} - \mathbf{Ag} + \mathbf{Bg}, \\ \mathbf{g}_{|t=0} &= \overset{o}{\mathbf{g}}. \end{aligned} \tag{25}$$

To process we need the following.

Definition 4.1 ([21, 26]) Consider an operator Q applied in the fractional model

$$D_t^\alpha(g(x,t)) = Qg(x,t), \quad 0 < \alpha < 1, \;\; x,\; t > 0, \tag{26}$$

subject to the initial condition

$$g(x,0) = f(x), \qquad x > 0 \tag{27}$$

and defined in a Banach space X_1. A family $(G_Q(t))_{t>0}$ of bounded operators on X_1 is called a solution operator of the fractional Cauchy problem (26)-(27) if

(i) : $G_Q(0) = I_{X_1}$;
(ii) : $G_Q(t)$ is strongly continuous for every $t \geq 0$;
(iii) :$QG_Q(t)f = G_Q(t)Qf$ for all $f \in D(Q)$;
(iv) : $G_Q(t)D(Q) \subset D(Q)$;
(v) : $G_Q(t)f$ is a (classical) solution of the model $(26) - (27)$ for all $f \in D(Q)$, $t \geq 0$.

It is well known [5] that an operator $\widetilde{Q} \in \mathcal{G}(M,\omega)$ means $\widetilde{Q}$ generates a C_0-semigroup $(G_{\widetilde{Q}}(t))_{t>0}$ so that there exists $M > 0$ and ω such that

$$\|G_{\widetilde{Q}}(t)\| \leq Me^{\omega t}. \tag{28}$$

Whence, by analogy, if the fractional Cauchy problem (26)-(27) has a solution operator $(G_Q(t))_{t>0}$ verifying (28), then we say that $Q \in \mathcal{G}^\alpha(M,\omega)$. The solution operator $(G_Q(t))_{t>0}$ is positive if

$$G_Q(t) \geq 0$$

and contractive if

$$\|G_Q(t)\|_{X_1} \leq 1, \tag{29}$$

and we say $Q \in \mathcal{G}^\alpha(1,0)$.

This leads to the following existence result.

Proposition 4.1 *Assume that the conditions of Lemma 3.1 hold, then for (25) there is an extension* $(\mathcal{K}_\alpha, D(\mathcal{K}_\alpha))$ *of* $(\boldsymbol{T} - \boldsymbol{A} + \boldsymbol{B},\ \ D(\boldsymbol{T}) \cap D(\boldsymbol{A}))$ *that generates a positive solution operator on* $\mathcal{X}_r$*, denoted by* $(G_{\mathcal{K}_\alpha}(t))_{t\geq 0}$.

Proof. The proof follows from the subordination principle [6, 20–22], by considering the existence result for (7) with $\alpha = 1$ and extending it to $0 < \alpha \leq 1$.

5 Results and Conclusion

We have analyzed, in the space $\mathcal{X}_r$ of distributions with finite higher moments, the generalized mass dependent discrete model (1), describing the movement of charged particles (electrons, ions) aggregating and moving in a relativistic zero-magnetic field. We showed existence of a solution g to (1) that is positive. Therefore, the evolution of the number of charged particles, given by this solution, is the same as the one predicted by the local law given in (4) which was used to construct the model. This is not always true since the analysis of certain models sometimes leads to the breach of the mass conservation law (called shattering) and that has been attributed to a phase transition creating a dust of "zero-size" particles with nonzero mass [9], which are beyond the model's resolution. Then we can use the full combination model (1) to study and control the dynamics of a number of charged particles moving in a relativistic zero-magnetic field. This work generalizes the preceding ones with the combination of the mass dependent relativistic kinetic and aggregation kernels which were not considered before. This work will therefore help addressing the problem of identifying and characterizing the full generator of our model which is still an unsolved issue.

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