Nonlinear Dynamics and Systems Theory, **19** (1-SI) (2019) 178–185

A Phase Change Problem including Space-Dependent Latent Heat and Periodic Heat Flux

Ajay Kumar, Abhishek Kumar Singh and Rajeev *

Department of Mathematical Sciences, IIT(BHU), Varanasi-221005, India

Received: July 8, 2018; Revised: February 5, 2019

Abstract: In this paper, a mathematical model related to a problem of phase-change process with periodic surface heat flux and space-dependent latent heat is considered. We have used the homotopy analysis approach to acquire the solution to the problem. To show the correctness of the calculated result, the comparisons have been discussed with the existing exact solution in a particular case. The effect of various parameters on the movement of the interface is also investigated.

Keywords: *homotopy analysis method; variable latent heat; periodic boundary condition; phase change problem.*

Mathematics Subject Classification (2010): 80A22, 35R37, 35R35, 80A20.

1 Introduction

In recent years, the phase change problem (the Stefan problem) involving diffusion process and variable latent heat is of great interest from mathematical and physical points of views. The research related to the diffusion process and its occurence can be found in many works [1–3]. Physically, a variable latent heat term arises in the Stefan problem governing the processes of movement of a shoreline in a sedimentary ocean basin due to changes in various parameters [4]. Some solutions of the Stefan problems including space-dependent latent heat have been reported in [5–7]. Zhou et al. [8] presented a phase change model (the Stefan problem) that contains a variable latent heat term and they discussed the similarity solution to the problem. After that Zhou and Xia [9] used the Kummer functions to present the similarity solution to a Stefan problem containing a more general variable latent heat term. Mathematically, the Stefan problem with periodic

* Corresponding author: mailto:rajeev.apm@iitbhu.ac.in

boundary is always interesting due to the difficulty associated with its solution. From the literature, it is found that the exact solution to the phase change problem with periodic heat-flux is not known even in its simplest form and a sophisticated scheme is required to solve these problems [10]. Therefore, various numerical [11–13] and approximate analytical techniques [7, 14] have been used by the researchers to solve the phase change problem containing the boundary conditions of periodic nature.

In this study, we consider a Stefan problem containing space-dependent latent heat and a periodic boundary condition. The solution of the problem is obtained by a well-known approximate technique, the homotopy analysis technique, introduced by Liao [12]. From the literature [16–22], it can be seen that this scheme is used by many researchers to solve various problems occurring in science and industries. In this paper, Wolfram Mathematica 8.0.1 has been used for all the computations with the aid of [23]. For the validity of proposed solution, the comparisons have been made with the analytical solution in a particular case. Dependence of movement of interface on some parameters is also analysed.

2 Mathematical Formulation

This section presents a phase change problem involving melting/freezing process in the half plane, i.e. $x > 0$. Motivated by the work of Zhou et al. [8] and Zhou and Xia [9], we have assumed that the latent heat is space-dependent. Moreover, a periodic surface heat flux is supposed in the problem. The mathematical model describing the process is given below:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}, \quad 0 < x < s(t),\ t > 0, \tag{1}$$

$$T(s(t), t) = 0, \quad t > 0, \tag{2}$$

$$k\frac{\partial T(0,t)}{\partial x} = -q(1 + \epsilon \sin \omega t), \quad t > 0, \tag{3}$$

$$k\frac{\partial T(s(t),t)}{\partial x} = -\gamma s \frac{ds}{dt}, \quad t > 0, \tag{4}$$

$$s(0) = 0, \tag{5}$$

where $T(x,t)$ is the temperature profile, x represents the space variable, t is the time, α denotes the thermal diffusivity, $s(t)$ denotes the tracking of moving phase front, k is the thermal conductivity, ω is the oscillation frequency, ϵ is the amplitude, $q(1 + \epsilon \sin \omega t)$ is the periodic heat flux and γs is the latent heat term per unit volume which depends on space.

3 Solution of the Problem

According to the homotopy analysis method (HAM) [17, 18], we assume

$$N[\phi(x,t;p)] = \frac{\partial}{\partial t}\phi(x,t;p) - \alpha \frac{\partial^2}{\partial x^2}\phi(x,t;p), \tag{6}$$

and

$$L[\phi(x,t;p)] = \frac{\partial^2}{\partial x^2}\phi(x,t;p) \tag{7}$$

as the non-linear and linear operators, respectively. For equation (1), we first construct the following homotopy:

$$(1-p)L\left[\phi(x,t;p)-T_0(x,t)\right]=p\mu H(x,t)N\left[\phi(x,t;p)\right], \tag{8}$$

where $p\in[0,1]$ denotes the embedding parameter, $T_0(x,t)$ represents the initial guess, $\mu\neq 0$ is the auxiliary parameter, $H(x,t)\neq 0$ is the auxiliary function.

If we substitute $p=0$ and $p=1$ in equation (8), then we simply obtain $\phi(x,t;0)=T_0(x,t)$ and $\phi(x,t;1)=T(x,t)$, respectively. This indicates that when p tends to 1 from 0, the initial estimate $T_0(x,t)$ shifts towards $T(x,t)$ which satisfies the proposed problem.

For equation (1), we can get the $m-th$ order deformation equation [17, 18] as given below:

$$L\left[T_m(x,t)-\chi_m T_{m-1}(x,t)\right]=\mu H(x,t)R_m(\vec{T}_{m-1}), \tag{9}$$

where

$$R_m(\vec{T}_{m-1})=\frac{\partial T_{m-1}(x,t)}{\partial t}-\alpha\frac{\partial^2 T_{m-1}(x,t)}{\partial x^2}$$

and

$$\chi_m=\begin{cases}0, & m<2,\\ 1, & m\geq 2.\end{cases}$$

According to Rajeev et al. [3], we consider the following initial approximation of $T(x,t)$:

$$T_0(x,t)=\frac{q}{k}\left((1+\epsilon\sin\omega t)(s_0-x)\right), \tag{10}$$

where $s_0=\left(\frac{2q}{\gamma}\left(t-\frac{\epsilon}{\omega}\cos\omega t+\frac{\epsilon}{\omega}\right)\right)^{\frac{1}{2}}$.

Using equation (10) in equation (9), we obtain

$$\begin{aligned}T_1(x,t)=&\mu\left(\frac{q^2}{k\gamma}(1+\epsilon\sin\omega t)^2 s_0^{-1}\right)\frac{x^2}{2}+\mu\left(\frac{q}{k}\omega\epsilon\cos\omega t s_0\right)\frac{x^2}{2}\\&-\mu\left(\frac{q}{k}\omega\epsilon\cos\omega t\right)\frac{x^3}{6},\end{aligned} \tag{11}$$

$$\begin{aligned}T_2(x,t)=&T_1(x,t)-\frac{\alpha\mu^2q^2(1+\epsilon\sin\omega t)^2s_0^{-1}}{k\gamma}\frac{x^2}{2}-\frac{\alpha\mu^2q\omega\epsilon\cos\omega t s_0}{k}\frac{x^2}{2}\\&+\frac{\alpha\mu^2q\omega\epsilon\cos\omega t}{k}\frac{x^3}{6}+\frac{\mu^2q^2}{k\gamma}\left\{-\frac{q}{\gamma}(1+\epsilon\sin\omega t)^3s_0^{-3}\right.\\&\left.+2(1+\epsilon\sin\omega t)\omega\epsilon s_0^{-1}\cos\omega t\right\}\frac{x^4}{24}+\frac{\mu^2q}{k}\left\{\frac{\omega q}{\gamma}\epsilon\cos\omega t(1+\epsilon\sin\omega t)s_0^{-1}\right.\\&\left.-(\omega^2\epsilon\sin\omega t)s_0\right\}\frac{x^4}{24}+\frac{\mu^2q\omega^2\epsilon\sin\omega t}{k}\frac{x^5}{120}\end{aligned} \tag{12}$$

and similarly, other components of $T(x,t)$ can be calculated.

Now, the solution $T(x,t)$ to the problem can be given by

$$T(x,t)=T_0(x,t)+T_1(x,t)+T_2(x,t)+... \tag{13}$$

Now, by choosing the following linear and non-linear operators, we have

$$L\left[\psi(t;p)\right] = \frac{d\psi(t;p)}{dt}, \tag{14}$$

and

$$N\left[\psi(t;p)\right] = k\frac{\partial T(\psi(t;p),t)}{\partial x} + \gamma\psi(t;p)\frac{d\psi(t;p)}{dt}. \tag{15}$$

We construct the following homotopy for the equation (4):

$$(1-p)\left[\psi(t;p) - s_0(t)\right] = p\hbar N\left[\psi(t;p)\right]. \tag{16}$$

From equation (16), we can easily find

$$\psi(t;0) = s_0, \tag{17}$$

and

$$\psi(t;1) = s(t). \tag{18}$$

According to [17,18], the m-th order deformation equation in the context of equation (4) is

$$L\left[s_m(t) - \chi_m s_{m-1}(t)\right] = \hbar N\left[s_{m-1}(t)\right]. \tag{19}$$

By considering the expression of s_0 (the initial approximation for the moving interface) and equations (13), (19) and (17), the various components of $s(t)$, i.e. $s_1(t)$, $s_2(t)$,..., can be calculated. Hence, the approximate solution for $s(t)$ is given by

$$s(t) = s_0(t) + s_1(t) + ... \tag{20}$$

4 Comparisons and Discussions

To show the accuracy of the obtained solution, we discuss the comparisons of our results for the temperature profile $T(x,t)$ and the location of moving phase front $s(t)$ with the exact solution at $\epsilon = 0$ in Tables 1 and 2, respectively. In case of $\epsilon = 0$, the equations (1)-(5) become a shoreline problem with a fixed line flux and a constant ocean level [4]. In this paper, the comparisons of our calculated results have been made with the exact solution established by Voller et al. [4]. Table 1 represents relative errors of temperature distribution between the obtained results and the exact result (given in [4]) at $\alpha = 1$, $\epsilon = 0$, $k = 1$ and $t = 5.5$. The absolute errors and relative errors of moving phase front are depicted in Table 2 at $\alpha = 1$, $\epsilon = 0$ and $k = 1$. From both tables, it is clear that the obtained computational results agree well with the result of exact solution.

q	x	$T_N(x,t)$	$T_E(x,t)$	Absolute Error	Relative Error
0.5	0.1	0.212321	0.211090	1.20 e-03	5.80 e-03
	0.2	0.162679	0.160212	2.40 e-03	1.50 e-02
	0.3	0.113274	0.109579	3.60 e-03	3.30 e-02
	0.4	0.064106	0.059189	4.90 e-03	8.30 e-02
	0.5	0.015176	0.009037	6.10 e-03	6.70 e-02
1.0	0.1	0.641957	0.637125	4.80 e-03	7.50 e-03
	0.2	0.542968	0.533223	9.70 e-03	1.80 e-02
	0.3	0.444652	0.430042	1.40 e-02	3.30 e-02
	0.4	0.347007	0.327569	1.90 e-02	5.90 e-02
	0.5	0.250031	0.225792	2.40 e-02	1.00 e-01
1.5	0.1	1.213060	1.202430	1.00 e-02	8.80 e-03
	0.2	1.064920	1.043280	2.10 e-02	2.00 e-02
	0.3	0.918012	0.885505	3.20 e-02	3.60 e-02
	0.4	0.772339	0.729075	4.30 e-02	5.90 e-02
	0.5	0.627896	0.573966	5.30 e-02	9.30 e-02

Table 1: Comparison between the exact value $T_E(x,t)$ and the numerical value $T_N(x,t)$ of temperature distribution at $\gamma = 20$.

q	t	$s_N(t)$	$s_E(t)$	Absolute Error	Relative Error
0.5	1	0.199681	0.198055	1.60 e-03	8.20 e-03
	2	0.282205	0.280092	2.10 e-03	7.50 e-03
	3	0.345453	0.343041	2.40 e-03	7.00 e-03
	4	0.398724	0.396109	2.60 e-03	6.60 e-03
	5	0.445619	0.442864	2.70 e-03	6.20 e-03
1.0	1	0.281571	0.277484	4.00 e-03	1.40 e-02
	2	0.397457	0.392422	5.00 e-03	1.20 e-02
	3	0.486084	0.480616	5.40 e-03	1.10 e-02
	4	0.560600	0.554968	5.60 e-03	1.00 e-02
	5	0.626098	0.620473	5.60 e-03	0.90 e-02
2.0	1	0.394948	0.385578	9.30 e-03	2.40 e-02
	2	0.555582	0.545290	10.20 e-03	1.80 e-02
	3	0.677665	0.667841	9.80 e-03	1.40 e-02
	4	0.779793	0.771156	8.60 e-03	1.10 e-02
	5	0.869169	0.862179	6.90 e-03	0.80 e-02

Table 2: Comparison between the exact value $s_E(t)$ and the numerical value $s_N(t)$ of moving interface at $\gamma = 25$.

Figures 1 and 2 show the evolution of movement of phase front at the fixed value of thermal diffusivity ($\alpha = 1.0$), the oscillation amplitude ($\epsilon = 0.5$) and the oscillation frequency ($\omega = \frac{\pi}{2}$). In Figure 1 and Figure 2, the effect of periodic heat flux on the movement of phase front is depicted for different values of γ and q, respectively. From Figure 1, it can be seen that the phase front propagates periodically and the movement of phase front becomes slow when we enhance the parameter γ. However, Figure 2 depicts that the periodic propagation of moving boundary $s(t)$ becomes fast as the value of q rises. It is also observed that when we raise the value of q, it makes melting/freezing process fast.

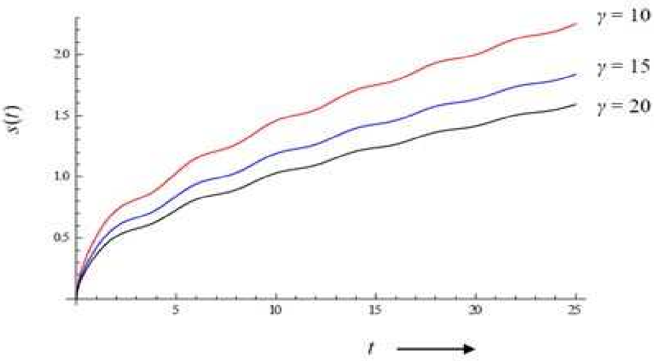

Figure 1: Plot of $s(t)$ vs. t at $\alpha = 1.0$, $q = 1.0$, $\epsilon = 0.5$, $\omega = \pi/2$.

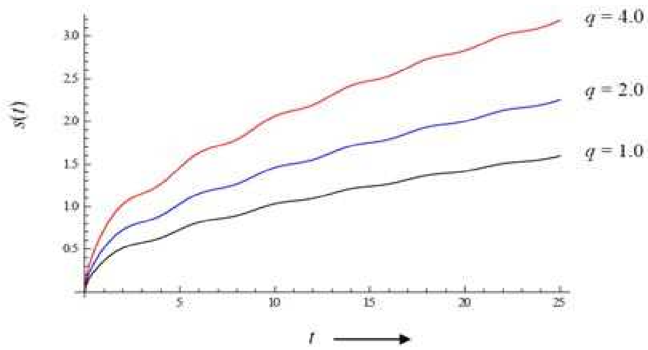

Figure 2: Plot of $s(t)$ vs. t at $\alpha = 1.0$, $\gamma = 20$, $\epsilon = 0.5$, $\omega = \pi/2$.

5 Conclusion

In this work, we study a complicated phase-change problem with a periodic heat flux and variable latent heat term. To the best of our knowledge, the exact solution to the proposed problem is not available in literature yet. Therefore, a homotopy analysis technique has been used to get an approximate analytical solution to the problem, and we have seen that our computed results are sufficiently close to the analytical solution when the surface heat flux is a constant, i.e. the oscillation amplitude is zero. In this paper, we have seen that the movement of interface/phase front is profoundly affected due to the change in various parameters like the oscillation amplitude, oscillation frequency, γ and q. It is also seen that the homotopy analysis technique is a straightforward method.

Moreover, this technique is sufficiently accurate and efficient to solve different types of phase-change problems arising in the various industries.

References

[1] H. Ouedraogo, W. Ouedraogo and B.Sangare. A self-diffusion mathematical model to describe the toxin effect on the zooplankton-phytoplankton dynamics. *Nonlinear Dynamics and Systems Theory* **18** (4) (2008) 392–408.

[2] A. Raheem. Existence and uniqueness of a solution of fisher-KKP type reaction diffusion equation. *Nonlinear Dynamics and Systems Theory* **13** (2) (2013) 193–202.

[3] P.G. Chhetri and A.S. Vatsala. Generalized monotone method for Riemann-Liouville fractional reaction diffusion equation with applications. *Nonlinear Dynamics and Systems Theory* **18** (3) (2018) 259–272.

[4] V.R. Voller, J.B. Swenson and C. Paola. An analytical solution for a Stefan problem with variable latent heat. *Int. J. Heat Mass Transfer* **47** (2004) 5387–5390.

[5] Rajeev, K.N. Rai and S. Das. Numerical solution of a moving-boundary problem with variable latent heat. *Int. J. Heat Mass Transfer* **52** (2009) 1913–1917.

[6] Rajeev, M.S. Kushwaha and A. Kumar. An approximate solution to a moving boundary problem with space time fractional derivative in fluvio-deltaic sedimentation process. *Ain Shams Engineering Journal* **4** (2013) 889–895.

[7] Rajeev. Homotopy perturbation method for a Stefan problem with variable latent heat. *Thermal Science* **18** (2014) 391–398.

[8] Y. Zhou, Y. Wang and W. Bu. Exact solution for a Stefan problem with latent heat a power function of position. *Int. J. Heat Mass Transfer* **69** (2014) 451–454.

[9] Y. Zhou and Y. Xia. Exact solution for Stefan problem with general power-type latent heat using Kummer function. *Int. J. Heat Mass Transfer* **84** (2015) 114–118.

[10] Rizwan-Uddin, One dimensional phase change problem with periodic boundary conditions. *Numerical Heat Transfer A* **35** (1999) 361–372.

[11] Rizwan-Uddin, An approximate solution for Stefan problem with time dependent boundary conditions. *Numerical Heat Transfer B* **33** (1998) 269–285.

[12] S. Savovic and J. Caldwell. Finite difference solution of one-dimensional Stefan problem with periodic boundary conditions. *Int. J. Heat Mass Transfer* **46** (2003) 2911–2916.

[13] S.G. Ahmed. A new algorithm for moving boundary problem subject to periodic boundary conditions. *Int. J. Numerical Methods for Heat and Fluid Flow* **16** (2006) 18–27.

[14] Rajeev, K.N. Rai and S. Das. Solution of one dimensional moving -boundary problem with periodic boundary conditions by variational iteration method. *Thermal Science* **13** (2009) 199–204.

[15] S.J. Liao. An approximate solution technique which does not depend upon small parameters: a special example. *Int. J. Non. Lin. Mech.* **32** (1997) 815–822.

[16] S.J. Liao. A new branch of solutions of boundary-layer flows over an impermeable stretched plate. *Int. J. Heat Mass Transfer* **48** (2005) 2529–2539.

[17] S. Liao. Notes on the homotopy analysis method-Some definitions and theorems. Common. *Nonlinear Sci. Numer. Simulat.* **14** (2009) 983–997.

[18] S. Abbasbandy. The application of homotopy analysis method to nonlinear equation arising in heat transfer. *Physics Lett.A.* **360** (2006) 109–113.

[19] V. Gorder and R.A. Vajravelu. On the selection of auxiliary functions, operators, and convergence control parameters in the application of the Homotopy Analysis Method to nonlinear differential equations: A general approach. *Commun. Nonlinear Sci. Numer. Simulat.* **14** (2009) 4078–4089.

[20] M.A. Zahran. On the derivation of fractional diffusion equation with an absorbent term and a linear external force. *Applied Mathematical Modelling* **33** (2009) 3088–3092.

[21] H. Jafary, M. Saeid and M.A. Firozjaei. Homotopy analysis method: a tool for solving a Stefan problem. *Journal of Advanced Research in Scientific Computing* **2** (2010) 61–68.

[22] O.N. Onyejekwe. The solution of a one-phase Stefan problem with a forcing term by homotopy analysis method. *International Journal of Advanced Mathematical Sciences* **2** (2) (2014) (2014) 95–100.

[23] I.P. Stavroulakis and S.A. Tersian. *Partial Differential Equations (second edition). An Introduction with Mathematica and Maple.* World Scientific Publishing Company, 2004.