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Lie Group Classification of a Generalized Coupled Lane-Emden-Klein-Gordon-Fock System with Central Symmetry

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Abstract: In this paper, we perform a complete symmetry analysis of a generalized Lane-Emden-Klein-Fock system with central symmetry. Several cases for the non-equivalent forms of the arbitrary elements are obtained. Moreover, a symmetry reduction for some cases is performed.

Keywords: *Lie group classification; equivalent transformation; Lie point symmetries; similarity reduction.*

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1 Introduction

In the recent paper [1], the author investigated both the Lie and Noether symmetries of a Lane-Emden-Klein-Fock system with central symmetry of the form

$$
\begin{aligned}
u_{tt} - u_{rr} - \frac{n}{r}u_r + \frac{\gamma v^q}{r^n} &= 0, \\
v_{tt} - v_{rr} - \frac{n}{r}v_r + \frac{\alpha u^p}{r^n} &= 0,
\end{aligned} \tag{1}
$$

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where p, n, γ, α, q are non-zero constants. If the constants $n = 2, \gamma = \alpha = 1$, system (1) reduces to

$$\begin{aligned} u_{tt} - u_{rr} - \frac{2}{r}u_r + \frac{v^q}{r^2} &= 0, \\ v_{tt} - v_{rr} - \frac{2}{r}v_r + \frac{u^p}{r^2} &= 0. \end{aligned} \tag{2}$$

Systems of this type occur in various physical phenomena, see, for example, [1–4] and references therein. Actually, system (1) can also be viewed as a natural extension of the well-known two-component generalization of the nonlinear wave equation, namely

$$u_{tt} - u_{rr} - \frac{m}{r}u_r - u^p = 0, \tag{3}$$

with the real-valued function $u = u(t, r)$, and p representing the interaction power while (t, r) denote time and radial coordinates, respectively, in $m \neq 0$ dimensions [4].

This system has been extensively studied in [2] for its Lie and Noether symmetries and the associated conservation laws for various values of the parameters p and q. More recently, hyperbolic versions of these types of system have also been investigated in [3]. Motivated by the recent results in [1–4], we study a generalized coupled Lane-Emden-Klein-Fock system with central symmetry of the form

$$\begin{aligned} u_{tt} - u_{rr} - \frac{n}{r}u_r + \frac{\Phi(v)}{r^n} &= 0, \\ v_{tt} - v_{rr} - \frac{n}{r}v_r + \frac{\Psi(u)}{r^n} &= 0, \end{aligned} \tag{4}$$

where $\Phi(v)$ and $\Psi(u)$ are arbitrary functions of v and u respectively.

The plan of this paper is as follows. In Section 2, we derive the equivalent generators of system (4). The Lie group classification of system (4) is performed in Section 3. In Section 4, we compute a symmetry reduction for some cases. Concluding remarks are given in Section 5.

2 Equivalence and Composition Transformations

In this section we employ the formulas derived in [5,6]. Applying the classical approach of group classification [7], we conclude that the generalized coupled Lane-Emden-Klein-Fock system (4) admits the following seven equivalence generators spanned by

$$\begin{aligned} &X_1 = \frac{\partial}{\partial t},\ X_2 = \frac{\partial}{\partial u},\ X_3 = \frac{\partial}{\partial v},\ X_4 = u\frac{\partial}{\partial u} + \Phi\frac{\partial}{\partial \Phi},\ X_5 = v\frac{\partial}{\partial v} + \Psi\frac{\partial}{\partial \Psi}, \\ &X_6 = t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r} + (n-2)\Phi\frac{\partial}{\partial \Phi} + (n-2)\Psi\frac{\partial}{\partial \Psi},\ X_7 = \frac{\partial}{\partial r} + \frac{n}{r}\Phi\frac{\partial}{\partial \Phi} + \frac{n}{r}\Psi\frac{\partial}{\partial \Psi} \end{aligned}$$

and the associated equivalence group is

$$
\begin{array}{lcl}
X_1 & : & \bar{t}=a_1+t,\ \bar{r}=r, \bar{u}=u,\ \bar{v}=v,\ \bar{\Phi}=\Phi,\ \bar{\Psi}=\Psi, \\
X_2 & : & \bar{t}=t,\ \bar{r}=r, \bar{u}=u+a_2,\ \bar{v}=v,\ \bar{\Phi}=\Phi,\ \bar{\Psi}=\Psi, \\
X_3 & : & \bar{t}=t,\ \bar{r}=r,\ \bar{u}=u,\ \bar{v}=v+a_3,\ \bar{\Phi}=\Phi,\ \bar{\Psi}=\Psi, \\
X_4 & : & \bar{t}=t,\ \bar{r}=r,\ \bar{u}=ue^{a_4},\ \bar{v}=v,\ \bar{\Phi}=\Phi e^{a_4}, \bar{\Psi}=\Psi, \\
X_5 & : & \bar{t}=t,\ \bar{r}=r,\ \bar{u}=u,\ \ \bar{v}=ve^{a_5},\ \bar{\Phi}=\Phi, \bar{\Psi}=\Psi e^{a_5}, \\
X_6 & : & \bar{t}=te^{a_6},\ \bar{r}=re^{a_6},\ \bar{u}=u,\ \bar{v}=v,\ \bar{\Phi}=\Phi e^{(n-2)a_6},\ \bar{\Psi}=\Psi e^{(n-2)a_6}, \\
X_7 & : & \bar{t}=t,\ \bar{r}=r+a_7,\ \bar{u}=u,\ \bar{v}=v,\ \bar{\Phi}=(r+a_7)^n\dfrac{\Phi}{r^n},\ \bar{\Psi}=(r+a_7)^n\dfrac{\Psi}{r^n}.
\end{array}
$$

Thus the corresponding composition of the above transformations is

$$
\begin{aligned}
\bar{t} &= e^{a_6}(t+a_1), \\
\bar{r} &= e^{a_6}(r+a_7), \\
\bar{u} &= e^{a_4}(u+a_2), \\
\bar{v} &= e^{a_5}(v+a_3), \\
\bar{\Phi} &= e^{a_4+(n-2)a_6}\big[(r+a_7)^n r^{-n}\Phi\big], \\
\bar{\Psi} &= e^{a_5+(n-2)a_6}\big[(r+a_7)^n r^{-n}\Psi\big].
\end{aligned} \tag{5}
$$

3 Group Classification of System (4)

A generalized coupled Lane-Emden-Klein-Fock system with central symmetry (4) is invariant under the group with the generator

$$
X = \xi^1(t,r,u,v)\frac{\partial}{\partial t}+\xi^2(t,r,u,v)\frac{\partial}{\partial x}+\eta^1(t,r,u,v)\frac{\partial}{\partial u}+\eta^2(t,r,u,v)\frac{\partial}{\partial v} \tag{6}
$$

if and only if

$$
X^{[2]}\left(u_{tt}-u_{rr}-\frac{n}{r}u_r+\frac{\Phi(v)}{r^n}=0\right)\bigg|_{(4)}=0,\ X^{[2]}\left(v_{tt}-v_{rr}-\frac{n}{r}v_r+\frac{\Psi(u)}{r^n}=0\right)\bigg|_{(4)}=0 \tag{7}
$$

with $X^{[2]}$ being the second extension of the generator (6) [4,6–9]. Expanding system (7) and solving the resulting determined system of partial differential equations for arbitrary $\Phi(v)$ and $\Psi(u)$ yield the one-dimensional principal Lie algebra spanned by

$$
X_1=\frac{\partial}{\partial t} \tag{8}
$$

and the classifying relations are

$$
\begin{cases}
(\delta u+\theta)\Psi'(u)+\beta\Psi(u)+\alpha=0, \\[2ex]
(\lambda v+\gamma)\Phi'(v)+\psi\Phi(v)+\omega=0,
\end{cases} \tag{9}
$$

where $\alpha,\beta,\gamma,\delta,\theta,\lambda$ and ω are constants. System (9) is invariant under the equivalence transformations (5) if

$$
\begin{aligned}
&\bar{\delta}=\delta,\quad \bar{\beta}=\beta,\quad \bar{\lambda}=\lambda,\quad \bar{\theta}=\delta a_2+\theta e^{-a_4},\quad \bar{\psi}=\psi,\quad \bar{\gamma}=\lambda a_3+\gamma e^{-a_5}, \\
&\bar{\omega}=e^{(n-2)a_6-a_4}\left(\frac{r^n}{(r+a_7)^n}\right),\quad \bar{\alpha}=e^{(n-2)a_6-a_5}\left(\frac{r^n}{(r+a_7)^n}\right).
\end{aligned}
$$

A complete analysis of equation (9) yields the following cases for the non-equivalent forms of the arbitrary element $\Phi(v)$, $\Psi(u)$ and n:

Case 1: $\Phi(v)$ and $\Psi(u)$ are arbitrary, but not of the form as cases 2-8 given below. In this case, we obtain only the principal Lie algebra (8).

Case 1.1: $n = 2$.

The principal Lie algebra is extended by one symmetry, viz,

$$X_2 = t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r}.$$

Case 2: $\Phi(v) = av^p$ and $\Psi(u) = bu^q$, where a, b, p and q are non-zero constants. This case reduces to the system studied in [1].

Case 3: $\Phi(v) = av^{-1}$ and $\Psi(u)$ is arbitrary, with a and n being non-zero constants. This case extends the principal Lie algebra by one symmetry, namely,

$$X_2 = v(n-2)\frac{\partial}{\partial v} - t\frac{\partial}{\partial t} - r\frac{\partial}{\partial r}. \tag{10}$$

Case 4: $\Phi(v)$ is arbitrary and $\Psi(u) = bu^{-1}$, where b and n are non-zero constants. Again the algebra is two-dimensional and is spanned by (8) and

$$X_2 = u(n-2)\frac{\partial}{\partial u} - t\frac{\partial}{\partial t} - r\frac{\partial}{\partial r}.$$

Case 5: $\Phi(v) = av$ and $\Psi(u) = bu$, where a, b and n are constants. Here the algebra extends by four, with the additional operators,

$$X_2 = \frac{\partial}{\partial u},\ X_3 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}, \quad X_4 = av\frac{\partial}{\partial u} + bu\frac{\partial}{\partial v},$$
$$X_5 = aH\frac{\partial}{\partial u} + \left[nr^{n-1}H_r + r^n H_{rr} - r^n H_{tt}\right]\frac{\partial}{\partial v},$$

where $H(t, r)$ is any solution of partial differential equation

$$\begin{aligned}&br^3(c_1 + aH) + \left[4r^{2n}n^2 - 2r^{2n}n^3 - 2r^{2n}n\right]H_r + \left[3r^{2n+1}n - 5r^{2n+1}n^2\right]H_{rr}\\&-4r^{2n+2}nH_{rrr} - r^{2n+3}H_{rrrr} + \left[2r^{2n+1}n^2 - r^{2n+1}n\right]H_{tt} + 4r^{2n+2}nH_{ttr}\\&-r^{2n+3}H_{tttt} + 2r^{2n+3}H_{ttrr} = 0\end{aligned}$$

and c_1 is an arbitrary constant.

Case 5.1: $n = 2$.

The Lie algebra extends by six additional generators,

$$X_2 = \frac{\partial}{\partial u},\ X_3 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}, \quad X_4 = av\frac{\partial}{\partial u} + bu\frac{\partial}{\partial v},$$
$$X_5 = t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r}, \quad X_6 = 2tu\frac{\partial}{\partial u} + 2tv\frac{\partial}{\partial v} - (t^2 + r^2)\frac{\partial}{\partial t} - 2tr\frac{\partial}{\partial r},$$
$$X_7 = aH\frac{\partial}{\partial u} + \left[2rH_r + r^2H_{rr} - r^2H_{tt}\right]\frac{\partial}{\partial v},$$

where $H(t, r)$ satisfies the partial differential equation

$$\begin{aligned}&b(c_2 + aH) - 4rH_r - 14r^2H_{rr} - 8r^3H_{rrr} - r^4H_{rrrr} + 6r^2H_{tt} + 8r^3H_{ttr}\\&-r^4H_{tttt} + 2r^4H_{ttrr} = 0\end{aligned}$$

and c_2 is an arbitrary constant.

Case 6: $\Phi(v) = de^{-\lambda v}$ and $\Psi(u) = ke^{-au}$, where a, d, λ, k, n are constants.

Here the principle algebra enlarges by one operator,

$$X_2 = \lambda(n-2)\frac{\partial}{\partial u} + a(n-2)\frac{\partial}{\partial v} - \lambda at\frac{\partial}{\partial t} - \lambda ar\frac{\partial}{\partial r}. \tag{11}$$

Case 7: $\Phi(v) = mv^p$ and $\Psi(u) = ke^{-au}$, where p, a, m, k, n are arbitrary constants.

Again the Lie algebra extends by one generator,

$$X_2 = va(n-2)\frac{\partial}{\partial v} - (p+1)(n-2)\frac{\partial}{\partial u} + pat\frac{\partial}{\partial t} + par\frac{\partial}{\partial r}. \tag{12}$$

Case 8: $\Phi(v) = de^{-\lambda v}$ and $\Psi(u) = ku^q$, where λ, d, k, n are constants.

The principle algebra also enlarges by one generator,

$$X_2 = u\lambda(n-2)\frac{\partial}{\partial u} - (q+1)(n-2)\frac{\partial}{\partial v} + \lambda qt\frac{\partial}{\partial t} + \lambda qr\frac{\partial}{\partial r}.$$

4 Reduction of System (4)

This section aims to perform reduction of system (4) using some symmetries obtained in Section 3. To obtain the symmetry reduction of system (4), we begin with the principle Lie algebra (8) and take $\Phi(v)$ and $\Psi(u)$ arbitrary. Solving the invariant surface condition

$$\frac{dt}{1} = \frac{dr}{0} = \frac{du}{0} = \frac{dv}{0}$$

yields the following group invariant solution $u(t,r) = \phi(r)$, $v(t,r) = \psi(r)$ of system (4) where $\phi(r)$ and $\psi(r)$ satisfy

$$\begin{aligned} \phi'' + \frac{n}{r}\phi' - \frac{\Psi}{r^n} &= 0, \\ \psi'' + \frac{n}{r}\psi' - \frac{\Phi}{r^n} &= 0. \end{aligned} \tag{13}$$

We now choose case 3 with the generator (10). The integration of the invariant surface condition

$$\frac{dt}{-t} = \frac{dr}{-r} = \frac{du}{0} = \frac{dv}{v(n-2)}$$

gives the following invariant solution of system (4); $u(t,r) = \phi(z)$, $v(t,r) = r^{-(n-2)}\psi(z)$ with the similarity variable $z = \frac{t}{r}$. Substituting the values of u and v into system (4) we get

$$\begin{aligned} (z^2-1)\phi'' - (n-2)\phi' - \frac{a}{\psi} &= 0, \\ (z^2-1)\psi'' + (n-2)z\psi' - (n-2)\psi + \phi &= 0, \end{aligned} \tag{14}$$

where $\phi(z)$ and $\psi(z)$ are any solutions of the system of ordinary differential equations (14).

We now choose case 6 and the generator (11). After some straightforward but lengthy computations, we obtain the invariant $z = \frac{t}{r}$ and $u(t,r) = \phi(z) + \frac{n\ln(r)}{a} - \frac{2\ln(r)}{a}$, $v(t,r) = \psi(z) + \frac{n\ln(r)}{\lambda} - \frac{2\ln(r)}{\lambda}$ as the group invariant solution of system (4), with $\phi(z)$ and $\psi(z)$ being any solutions of the system of ordinary differential equations

$$\begin{aligned}(z^2-1)\phi'' - (n-2)z\phi' - de^{-\lambda\phi} - \frac{(n-2)(n-1)}{a} &= 0,\\ (z^2-1)\psi'' - (n-2)z\psi' - ke^{-a\psi} - \frac{(n-2)(n-1)}{\lambda} &= 0.\end{aligned} \tag{15}$$

Another general group invariant solution of system (4) will be derived from case 7 with the operator (12). Considering the invariant surface condition

$$\frac{dt}{apt} = \frac{dr}{apr} = \frac{du}{(2-p)(p+1)} = \frac{dv}{av(n-2)}$$

we conclude that the group invariant solution of system (4) is $u(t,r) = \phi(z) + \frac{n\ln(r)}{a} - \frac{2\ln(r)}{a} + \frac{n\ln(r)}{ap} - \frac{2\ln(r)}{ap}$, $v(t,r) = r^{\frac{-(n-2)}{p}}\psi(z)$ with the invariant $z = \frac{t}{r}$, where $\phi(z)$ and $\psi(z)$ satisfy the system of ordinary differential equations

$$\begin{aligned}(z^2-1)\phi'' - z(n-2)\phi' - m\psi^p - \frac{(p+1)(n-2)(n-1)}{ap} &= 0,\\ (z^2-1)\psi'' - \frac{(p+2)(n-2)}{p}z\psi' + \frac{(n-2)}{p^2}(p(n-1)+(n-2))\psi - ke^{-a\phi} &= 0.\end{aligned} \tag{16}$$

Following the aforementioned procedure, one can obtain more group invariant solutions for the generalized coupled Lane-Emden-Klein-Fock system with central symmetry system (4). It is worthy mentioning that all the cases that do not extend the principle Lie algebra have been excluded.

5 Conclusion

In this paper we performed a complete Lie symmetry classification of a generalized coupled Lane-Emden-Klein-Fock system with central symmetry (4). Several cases which resulted in Lie symmetries have been obtained. Moreover, some symmetry reductions for some cases were derived. In future, we would like to extend the results obtained in this manuscript by employing the techniques in [10–15].

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References

[1] B. Muatjetjeja. Coupled Lane-Emden-Klein-Gordon-Fock system with central symmetry: Symmetries and Conservation laws. *J. Differ. Equ.* **263** (2017) 8322–8328.

[2] I. L. Freire and B. Muatjetjeja. Symmetry analysis of a Lane-Emden-Klein-Gordon-Fock system with central symmetry. *Discrete. Contin. Dyn Syst. Ser A.* **11** (2018) 667–673.

[3] B. Muatjetjeja and T. E. Mogorosi. Variational principle and conservation laws of a generalized hyperbolic Lane-Emden system. *J. Comput. Nonlinear Dyn.* **13** (2018) 121002.

[4] T. E. Mogorosi, I. L. Freire, B. Muatjetjeja and C. M. Khalique. Group anylsis of a hyperbolic Lane-Emden system. *Appl. Math. Comput.* **292** (2017) 157–164.

[5] T. E. Mogorosi and B. Muatjetjeja. Group classification of a generalized coupled hyperbolic Lane-Emden system. DOI: 10.1007/s40995-018-0575-z.

[6] M. Molati and C. M. Khalique. Lie group classification for a generalised coupled Lane-Emden type system in two dimensions, *J. Appl. Maths.* **2012** (2012) 10 pages.

[7] L. V. Ovsiannikov. *Group Analysis of Differential Equations.* Academic Press, New York, USA, 1982.

[8] Y. Bozkhov and I. L. Freire. Symmetry anylsis of the bidimensional Lane-Emden system. *J. Math. Anal. Appl.* **328** (2012) 1279–1284.

[9] Y. Bozkhov and A. C. G. Martins. Lie point symmetries of the Lane-Emden systems. *J. Math. Anal. Appl.* **294** (2004) 334–344.

[10] H. Jafari, N. Kadkhoda and C. M. Khalique. Exact solutions of ϕ^4 equation using Lie symmetry approach along with the simplest equation and Exp-function methods, *Abstr. Appl. Anal.* **2012** (2012) Article ID 350287, 7 pages.

[11] H. Jafari, N. Kadkhoda and C. M. Khalique. Application of Lie symmetry analysis and simplest equation method for finding exact solutions of Boussinesq equations, *Math. Probl. Eng.* **2013** (2013) ID 452576 4 pages.

[12] E. R.Attia, V. Benekas, H. A. El-Morshedy and I.P. Stavroulakis. Oscillation of first order linear differential equations with several non-monotone delays. *Open Math.* **16** (2018) 83–94.

[13] G. M. Moremedi and I. P. Stavroulakis. A Survey on the Oscillation of Differential Equations with Several Non-Monotone Arguments. *Appl. Math. Inf. Sci.* **12** (2018) 1047–1053.

[14] M. Azadi and H. Jafari. Lie Symmetry Reductions of a Coupled Kdv System of Fractional Order. *Nonlinear Dyn. Systems Theory* **18** (2018) 22–28.

[15] M. Kharrat. Closed-Form Solution of European Option under Fractional Heston Model. *Nonlinear Dyn. Systems Theory* **18** (2018) 191–195.