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Real Time Analysis of Signed Marked Random Measures with Applications to Finance and Insurance

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Abstract: This paper deals with stationary and independent increments processes in real time initiated in [14] embellishing it to a two-dimensional signed random measure with position dependent marking. The real-valued component of the associated marked point process is non-monotone presenting an analytical challenge. We manage to investigate various characteristics of that component, including the nth drop or a sharp surge that find applications to finance (like option trading) and risk theory. The need for time sensitive feature of our study (i.e., an analytical association with real time parameter t) allows stochastic control implementation in sharp contrast with time insensitive analysis in the present literature. We proceed with the classical approach of fluctuation analysis of a particle running through a random grid of a convex set that the particle is trying to escape. We find the distribution of the first passage time and its location in space.

Keywords: *random walk; independent and stationary increments processes; fluctuations of stochastic processes; marked point processes; first passage time; signed marked random measures; time sensitive analysis.*

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1 Introduction

In many scientific, financial, and game theoretic processes, timing is of at most importance and a main strategic issue. Several studies have been done on the first passage time in fluctuation theory and their applications to queuing, stochastic games, seismology, and finance (cf. [1,2,8-10,11,12,13,15,16,19,22-24,27,30]). Fluctuation theory pertains to the behavior of an underlying process around a critical threshold and more generally, when a process escapes from a fixed manifold. The time when that passage takes place is referred

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to as the first passage time. Another critical value of that situation is the new location of the process upon its escape. Besides the original topics mentioned above, fluctuation theory has become a stand-alone subject in numerous articles appeared through the decades of intense research, cf. [3-7,17,20,21,29,31].

In our most recent paper [15], we worked with time sensitive functionals of the same entities but under real time observation of a monotone process. We dealt with non-negative random measures and increment processes. In this paper we study a class of signed marked random measures $(\mathcal{A}, \mathit{\Pi}, \mathcal{T}) = \sum_{n=0}^{\infty} (X_n, \pi_n)\, \varepsilon_{t_n}$ with position dependent marking, on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. Marks X_n's are non-negative, while marks π_n's are real-valued, with the support counting random measure $\sum_{n=0}^{\infty} \varepsilon_{t_n}$. This is a significant upgrade from [15], because not only is yet another component added, but it is non-monotone. Studies of non-monotone components are very few in the literature on fluctuations. Most prominent of them was by Lajos Takács [30]. However, the results in [30] were not tractable.

As in the theory of fluctuations, we focus on the behavior of $(\mathcal{A}, \mathit{\Pi}, \mathcal{T})$ around a fixed threshold $M > 0$ with respect to its first component $\mathcal{A}$, referred to as an active component. With

$$A_n = X_0 + X_1 + \ldots + X_n \tag{1.1}$$

we have $\{A_n\}$ monotone non-decreasing, whereas

$$P_n = \pi_0 + \pi_1 + \ldots + \pi_n \tag{1.2}$$

is non-monotone, as π_k's are real-valued marks. Our interest is in an extreme behavior of the marginal process $(\mathit{\Pi}, \mathcal{T}) = \sum_{n=0}^{\infty} \pi_n \varepsilon_{t_n}$ that is difficult to analyze due to the non-monotone nature of its marks. For that reason we introduce active mark X_n being nonnegative and integer-valued that is to oversee π_n. For instance, we might be curious when the process $(\mathit{\Pi}, \mathcal{T})$ changes its monotonicity or when it experiences its first extreme drop or a surge. For example, we set $X_0 = X_1 = \ldots = X_{n-1} = 0, X_n = 1$, if $\pi_0 > a, \pi_1 > a, \ldots, \pi_{n-1} > a$, and $\pi_n \le a$. In the general case, the increments X_i's need not be constant, but they can be random variables with particular marginal distributions. For a fixed positive integer M, we define the exit index as

$$\nu := inf\,\{n = 0, 1, \ldots : A_n \ge M\}\,. \tag{1.3}$$

Then, t_ν is called the first passage time of process $(\mathcal{A}, \mathit{\Pi}, \mathcal{T})$. It is the first epoch when the crossing of M occurs. Obviously, t_ν is a stopping time relative to filtration $\mathcal{F}_t$. The respective excess values of A_ν and P_ν representing active and passive components, $\mathcal{A}$ and $\mathit{\Pi}$, respectively, are also of interest. We further assume that **A1** the increments $\{X_n, \pi_n, \Delta_n = t_n - t_{n-1}\}$ for $n = 0, 1, 2, \ldots,$ $t_{-1} = 0$, of the process $(\mathcal{A}, \mathit{\Pi}, \mathcal{T})$ are independent (position dependent marking), that is, X_n and π_n are dependent only on Δ_n. **A2** for $n = 1, 2, \ldots,$ $\{X_n, \pi_n, \Delta_n\}$ are identically distributed.

Associated with $(\mathcal{A}, \mathit{\Pi}, \mathcal{T})$ is the "time sensitive counting" process

$$(N_t, \mathit{\Pi}_t) = (\mathcal{A}, \mathit{\Pi})\,[0, t] = \sum_{n=0}^{\infty} (X_n, \pi_n)\, \varepsilon_{t_n}\,[0, t]\,, t \ge 0. \tag{1.4}$$

We will be interested in the value of $(N_t, \mathit{\Pi}_t)$ of some t enclosed between $t_{\nu-1}$ and t_ν providing us with the information about $(\mathcal{A}, \mathit{\Pi}, \mathcal{T})$ between two key reference points as well as $(N_t, \mathit{\Pi}_t)$ for $t \in [0, \tau_\nu)$ (that we will discuss later on, in Section 5).

So we target the joint Laplace- and Fourier-Stieltjes transform of the above r.v.'s:

$$\begin{aligned}\Phi_{\nu}(t) &= Ez^{N_t}e^{-i\eta\Pi_t}e^{-i\varphi P_{\nu-1}}u^{A_{\nu-1}}e^{-i\phi P_{\nu}}v^{A_{\nu}}e^{-\vartheta_0 t_{\nu-1}-\vartheta t_{\nu}}\mathbf{1}_{[t_{\nu-1},t_{\nu})}(t), \\ &\|z\|\le 1, \|u\|\le 1, \|v\|\le 1, Re\vartheta_0\ge 0, Re\vartheta\ge 0, \eta\in\mathbb{R}, \varphi\in\mathbb{R}, \phi\in\mathbb{R}.\end{aligned} \tag{1.5}$$

Note that because we manage to observe the process in real time, i.e., upon $t_0, t_1, t_2, \ldots$ (meaning that there are no changes between those epochs), it raises a question about a need in the continuous time interpolation. Indeed, in some past work (cf. Dshalalow and White [17]) when a process was observed over arbitrary time epochs (i.e., unrelated to $t_0, t_1, t_2, \ldots$), its continuous revival made perfect sense. In our case, however, it is more about associating the point process $t_0, t_1, t_2, \ldots$, especially the reference points $t_{\nu-1}, t_{\nu}$, with time t, than anything else. Its very obvious benefit is to know about the process over time related intervals like $[0, t]$ which was impossible with time insensitive versions. From a practical stand point, observing the process over arbitrary time epochs is more realistic than in real time. However, whenever it is possible to render, its second benefit lies in far more tractable results compared to delayed observations that additionally require the named point process to be Poisson or alike. Furthermore, we also obtain explicit characteristics of the continuous time parameter process in interval $[0, t_{\nu})$ giving us a broad spectrum of information about process N_t. The associated functional will read

$$Ez^{N_t}e^{-i\eta\Pi_t}e^{-i\phi P_{\nu}}v^{A_{\nu}}e^{-\vartheta t_{\nu}}\mathbf{1}_{[0,t_{\nu})}(t), \|z\|\le 1, \|v\|\le 1, Re\vartheta\ge 0, \eta\in\mathbb{R}, \phi\in\mathbb{R}. \tag{1.6}$$

Back to the random measure $(\mathcal{A}, \Pi, \mathcal{T})$, we recall that the passive component Π is real-valued making this random measure signed. Studies related to signed random measures have previously been done in various topological and stochastic analysis contexts. In [19] Hellmund extended the idea of completely random measures to completely random signed measure and gave a characterization of this class of signed random measures. He demonstrated that the classes of Lévy random measures (utilized in Lévy adaptive regression kernel models) and Lévy bases (utilized in spatio-temporal modeling) are natural extensions of completely random signed measures and that independence is a fundamental concept in defining Lévy random measures and Lévy bases. Other concepts related to signed random measures are in the work by Smorodina and Faddeev [29] who studied symmetric stable signed measures and showed that they are limit measures of sums of independent random variables.

Various applications of fluctuation theory that we explore can also be found in stochastic signals such as time continuous readings for automated seizure detection and quantification using EEGs, heart attack activity monitoring through detection by EKGs, real time blood pressure monitoring, and the stock market. In this paper, we illustrate the applicability of our study by expounding on the case of stock prices. We are able to predict the time of the first drop of a stock (or first increase if we short it) at t_{ν} and thus, the highest price at $t_{\nu-1}$ at which we can sell it at that point in time.

Our model also applies to the classical risk problem originally posed by Filip Lundberg (see [27]). Assume that an insurance company starts at zero with the initial capital u and let the premium be a linear function with a constant premium rate c, so that the premium income of the company at time t is $u + ct$. Assume that the aggregate claims form a marked point process $\mathcal{Y} = \sum_{k=0}^{\infty} Y_k \varepsilon_{t_k}$, with t_k being the time of the kth claim and Y_k - the amount of claim. Now Lundberg postulated that $\mathcal{Y}$ was a marked Poisson process with position independent marking. We relax either condition by assuming that neither is $\mathcal{Y}$ Poisson, nor is it with position independent marking. If $\Delta_k = t_k - t_{k-1}$,we have $c\Delta_k$ premiums' increase from t_{k-1} to t_k. The mark $\pi_k = c\Delta_k - Y_k$ is the change of company's asset from t_{k-1} to t_k. Now,

$$\Pi = \sum_{k=0}^{\infty} \pi_k \varepsilon_{t_k}$$

is a purely signed marked random measure and

$$P_t = \Pi\,[0,t]$$

is the process describing the asset changes of the insurance company on interval $[0,t]$. Notice that P_t does not give us the true value of the company's asset at time t, because P_t is a piecewise constant interpolation of the true asset value process

$$R_t = u + ct - \sum_{k=0}^{\infty} Y_k \varepsilon_{t_k}\,[0,t]$$

known as the risk process. They coincide upon times $t_0, t_1, t_2, \ldots$ which is exactly what we need. Our process $(\mathcal{A}, \Pi, \mathcal{T})$ is defined through the active component

$$X_k = \begin{cases} 0, & \pi_k > 0, \\ 1, & \pi_k \leq 0. \end{cases}$$

So we are interested in the moment when P_k becomes negative or zero for the first time (which would trigger $X_k = 1$). Thus, $\pi_0, \pi_1, \ldots, \pi_{\nu-1}$ are positive, while π_ν is negative or zero. $\{t_{\nu_n}\}$ is the embedded sequence of consecutive drops of P_t. Then obviously, the risk process R_t will become negative or zero only upon one of the epochs $\{t_{\nu_n}\}$, known as the ruin time of R_t.

Let $\mathcal{F}_t$ be the natural filtration with respect to the risk process R_t. Then, $\{t_{\nu_n}\}$ is a sequence of stopping times relative $\mathcal{F}_t$ that are also locally strong Markov points, that is either R_t and P_t have a locally strong Markov property at each point t_{ν_n}. Therefore, R_t and P_t conditionally regenerate upon these epochs. We can slightly modify P_t to make it semi-regenerative with respect to $\{t_{\nu_n}\}$.

While a further discussion on the risk process and its study as a semi-regenerative process is beyond the scope of this paper, the time of the first or the second or the nth drop of the risk process is of interest for statistics purposes and it is often raised by insurance companies.

We continue this paper in Section 2 through a further formalism of our model and introduce basics of discrete operational calculus earlier developed by Dshalalow [6,7] and Dshalalow and Iwezulu [13]. In Section 3, we use the method of stochastic decomposition previously developed in Dshalalow and Nandyose [15] and Dshalalow and White [17,18], only now embellished for non-monotone components. We establish a key formula for the functional $\Phi_\nu(t)$ of (1.5) that we claim is analytically tractable. This claim is justified throughout Section 4 in a number of examples and special cases. We conclude our paper in Section 5 with time sensitive analysis where time t runs interval $[0, \tau_\nu)$ and find the joint transform of N_t, P_t, N_ν,and the first passage time t_ν in a fully closed form.

2 Formalism and Notation

We now return to the functional Φ_ν. Note that we do not know the distribution of the random vector $(A_\nu - A_{\nu-1}, t_\nu - t_{\nu-1})$ nor is the latter independent of $(A_{\nu-1}, t_{\nu-1})$. The remedy for this predicament is the use of stochastic expansion that will include several steps. In the first step, we introduce the auxiliary sequence $\{\nu(p)\}$ of exit indices relative to the sequence $\{0, 1, \ldots\}$ of thresholds to be crossed by A_n, of which $\nu = \nu(M-1)$ was introduced in (1.3). Namely, let

$$\nu(p) = inf\{n = 0, 1, \ldots : A_n > p\}, p = 0, 1, \ldots \tag{2.1}$$

With p fixed, we have the sequence of functionals

$$\Phi_{\nu(p)}(t) = Ez^{N_t} e^{-i\eta\Pi_t} e^{-i\varphi P_{\nu(p)-1}} u^{A_{\nu(p)-1}} e^{-i\phi P_{\nu(p)}} v^{A_{\nu(p)}} e^{-\vartheta_0 t_{\nu(p)-1} - \vartheta t_{\nu(p)}} \mathbf{1}_{[t_{\nu(p)-1}, t_{\nu(p)})}(t). \tag{2.2}$$

In our second step, we apply to $\Phi_{\nu(p)}$ of (2.2) the transformation D_p defined as

$$D_p\{f(p)\}(x) := \sum_{p=0}^{\infty} x^p f(p)(1-x), ||x|| < 1, \tag{2.3}$$

where f is a real-valued function with the domain $\mathbb{N}_0 = \{0, 1, \ldots\}$. The inverse of D_p is the so-called $\mathcal{D}$-operator previously introduced in Dshalalow [6,7]:

$$\mathcal{D}_x^k \varphi(x, y) = \begin{cases} \lim_{x\to 0} \frac{1}{k!} \frac{\partial^k}{\partial x^k} \left[\frac{1}{1-x}\varphi(x, y)\right], & k \geq 0 \\ 0, & k < 0. \end{cases} \tag{2.4}$$

From $\Phi_{\nu(p)}(t) = \sum_{n=0}^{\infty} \Phi_{\nu(p)}(t) \mathbf{1}_{\{v(p)=n\}}$, we have

$$\Phi(t, x) := D_p\left[\Phi_{\nu(p)}(t)\right](x) = \sum_{n=0}^{\infty} \Phi_{\nu(p)}(t) D_p \mathbf{1}_{\{\nu(p)=n\}}(x)$$

$$= \sum_{n=0}^{\infty} \Phi_{\nu(p)=n}(t) D_p \mathbf{1}_{\{\nu(p)=n\}}(x),$$

with

$$\Phi_{\nu(p)=n}(t) = Ez^{N_t} u^{A_{n-1}} e^{-i\eta\Pi_t} e^{-i\varphi P_{n-1}} v^{A_n} e^{-i\phi P_n} e^{-\vartheta_0 t_{n-1} - \vartheta t_n} \mathbf{1}_{\{t_{n-1} \leq t < t_n\}} = F_n(t). \tag{2.5}$$

From $\mathbf{1}_{\{v(p)=n\}} = \mathbf{1}_{\{A_{n-1} \leq p\}} \mathbf{1}_{\{A_n > p\}}$,

$$D_p \mathbf{1}_{\{v(p)=n\}}(x) = (1-x) \sum_{p=0}^{\infty} x^p \mathbf{1}_{\{A_{n-1} \leq p\}} \mathbf{1}_{\{A_n > p\}}$$

$$= (1-x) \sum_{p=A_{n-1}}^{A_n - 1} x^p$$

$$= (1-x)\left(\sum_{p=0}^{A_n-1} x^p - \sum_{p=0}^{A_{n-1}-1} x^p\right) = (1-x)\left(\frac{1-x^{A_n}}{1-x} - \frac{1-x^{A_{n-1}}}{1-x}\right) = x^{A_{n-1}} - x^{A_n}$$

that yields

$$\Phi(t,x) = \sum_{n=0}^{\infty} F_n(t)\left(x^{A_{n-1}} - x^{A_n}\right)$$
$$= \sum_{n=0}^{\infty} \left[F_n(ux,v,z,\vartheta_0,\vartheta,t) - F_n(u,vx,z,\vartheta_0,\vartheta,t)\right], where A_{-1} = 0. \tag{2.6}$$

Finally, applying the Laplace transform to $\Phi(t,x)$ of (2.6) we have

$$\Phi^*(\theta,x) = \int_{t=0}^{\infty} e^{-\theta t}\Phi(t,x)\,dt = \sum_{n=0}^{\infty}\left[F_n^*(ux,v,z,\vartheta_0,\vartheta,t) - F_n^*(u,vx,z,\vartheta_0,\vartheta,t)\right]. \tag{2.7}$$

Now functionals F_n and their transforms F_n^* are subject to our scrutiny in Section 3.

3 Analysis of F_n

With $n = 1, 2, \ldots$, we work on

$$F_n(t) = E z^{N_t} u^{A_{n-1}} e^{-i\eta\Pi_t} e^{-i\varphi P_{n-1}} v^{A_n} e^{-i\phi P_n} e^{-\vartheta_0 t_{n-1} - \vartheta t_n} \mathbf{1}_{\{t_{n-1} \le t < t_n\}}, \tag{3.1}$$

(defined in (2.5)). (3.1) can be brought to the expression

$$F_n(t) = E[(zuv)^{A_{n-1}} e^{-i(\eta+\varphi+\phi)P_{n-1}} e^{-(\vartheta_0+\vartheta)t_{n-1}} v^{A_n - A_{n-1}}$$
$$\times e^{-i\phi(P_n - P_{n-1})} e^{-\vartheta(t_n - t_{n-1})} \mathbf{1}_{\{t_{n-1} \le t < t_n\}}]$$
$$= E(zuv)^{A_{n-1}} e^{-i(\eta+\varphi+\phi)P_{n-1}} e^{-(\vartheta_0+\vartheta)t_{n-1}} v^{X_n} e^{-i\phi\pi_n} e^{-\vartheta\Delta_n} \mathbf{1}_{\{t_{n-1} \le t < t_n\}}, n = 1, 2\ldots \tag{3.2}$$

The Laplace transform of F_n with the expectation unfolded reads

$$F_n^*(\theta) = \int_{t=0}^{\infty} e^{-\theta t} F_n(t)\,dt$$
$$= \sum_{k=0}^{\infty}(zuv)^k \sum_{j=0}^{\infty} v^j \int_{p=-\infty}^{\infty} e^{-i(\eta+\varphi+\phi)p} \int_{s\ge 0} e^{-(\vartheta_0+\vartheta)s} e^{-\theta s}$$
$$\times \int_{q=-\infty}^{\infty} e^{-i\phi q} \int_{\delta\ge 0} e^{-\vartheta\delta} \int_{t-s=0}^{\delta} e^{-\theta(t-s)} dt$$
$$\times P_{A_{n-1}\otimes P_{n-1}\otimes t_{n-1}\otimes X_n\otimes\pi_n\otimes\Delta_n}(k,j,dp,ds,dq,d\delta)$$
$$= \frac{1}{\theta}\sum_{k=0}^{\infty}(zuv)^k \sum_{j=0}^{\infty} v^j \int_{p=-\infty}^{\infty} e^{-i(\eta+\varphi+\phi)p} \int_{s\ge 0} e^{-(\vartheta_0+\vartheta+\theta)s} \int_{q=-\infty}^{\infty} e^{-i\phi q} \int_{\delta\ge 0} e^{-\vartheta\delta}$$
$$\times P_{A_{n-1}\otimes P_{n-1}\otimes t_{n-1}\otimes X_n\otimes\pi_n\otimes\Delta_n}(k,j,dp,ds,dq,d\delta)$$
$$-\frac{1}{\theta}\sum_{k=0}^{\infty}(zuv)^k \sum_{j=0}^{\infty} v^j \int_{p=-\infty}^{\infty} e^{-i(\eta+\varphi+\phi)p} \int_{s\ge 0} e^{-(\vartheta_0+\vartheta+\theta)s} \int_{q=-\infty}^{\infty} e^{-i\phi q} \int_{\delta\ge 0} e^{-(\vartheta+\theta)\delta}$$
$$\times P_{A_{n-1}\otimes P_{n-1}\otimes t_{n-1}\otimes X_n\otimes\pi_n\otimes\Delta_n}(k,j,dp,ds,dq,d\delta)$$

due to independence of $A_{n-1} \otimes P_{n-1} \otimes t_{n-1}$ and $X_n \otimes \pi_n \otimes \Delta_n$

$$= \frac{1}{\theta} E[(zuv)^{A_{n-1}} e^{-i(\eta+\varphi+\phi)P_{n-1}} e^{-(\vartheta_0+\vartheta+\theta)t_{n-1}}]$$

$$\times\left[Ev^{X_n}e^{-i\phi\pi_n}e^{-\vartheta\Delta_n}-Ev^{X_n}e^{-i\phi\pi_n}e^{-(\vartheta+\theta)\Delta_n}\right]$$
$$=\frac{1}{\theta}\mathit{\Gamma}_{n-1}\left(zuv,\eta+\varphi+\phi,\vartheta_0+\vartheta+\theta\right)\left[\gamma\left(v,\phi,\vartheta\right)-\gamma\left(v,\phi,\vartheta+\theta\right)\right], \qquad (3.3)$$

where

$$\mathit{\Gamma}_{n-1}(zuv,\eta+\varphi+\phi,\vartheta_0+\vartheta+\theta)$$
$$=\gamma_0\left(zuv,\eta+\varphi+\phi,\vartheta_0+\vartheta+\theta\right)\gamma^{n-1}\left(zuv,\eta+\varphi+\phi,\vartheta_0+\vartheta+\theta\right)\text{ for } n\geq 1 \qquad (3.4)$$

and

$$\gamma_0\left(u,\varphi,\vartheta\right)=Eu^{X_0}e^{-i\varphi\pi_0}e^{-\vartheta t_0},\ \gamma\left(u,\varphi,\vartheta\right)=Eu^{X_k}e^{-i\varphi\pi_k}e^{-\vartheta\Delta_k},k=1,2,\ldots. \qquad (3.5)$$

Summing up F_n for all $n=1,2,\ldots$, with (3.3-3.4) in mind, we formally arrive at the expression

$$\sum_{n=1}^{\infty}F_n^*\left(\theta\right)=\frac{1}{\theta}\gamma_0\left(zuv,\eta+\varphi+\phi,\vartheta_0+\vartheta+\theta\right)$$
$$\times\left[\gamma\left(v,\phi,\vartheta\right)-\gamma\left(v,\phi,\vartheta+\theta\right)\right]\frac{1}{1-\gamma\left(zuv,\eta+\varphi+\phi,\vartheta_0+\vartheta+\theta\right)}. \qquad (3.6)$$

To warrant the convergence of the geometric series $\sum_{n=1}^{\infty}F_n^*\left(\theta\right)$, in the proposition below, we show that the norm $\left\|\gamma\left(zuv,\eta+\varphi+\phi,\vartheta_0+\vartheta+\theta\right)\right\|<1$.

Proposition 3.1 *The series*

$$\sum_{n=1}^{\infty}F^*\left(\theta\right)=\sum_{n=1}^{\infty}\int_{t=0}^{\infty}e^{-\theta t}E[(zuv)^{A_{n-1}}e^{-i(\eta+\varphi+\phi)P_{n-1}}$$
$$\times e^{-(\vartheta_0+\vartheta)t_{n-1}}v^{X_n}e^{-i\phi\pi_n}e^{-\vartheta\Delta_n}\mathbf{1}_{\{t_{n-1}\leq t<t_n\}}]dt$$

converges to

$$\sum_{n=1}^{\infty}F_n^*\left(\theta\right)=\frac{1}{\theta}\gamma_0\left(zuv,\eta+\varphi+\phi,\vartheta_0+\vartheta+\theta\right)$$
$$\times\left[\gamma\left(v,\phi,\vartheta\right)-\gamma\left(v,\phi,\vartheta+\theta\right)\right]\frac{1}{1-\gamma\left(zuv,\eta+\varphi+\phi,\vartheta_0+\vartheta+\theta\right)},$$

with

$$\|\gamma(zuv,\eta+\varphi+\phi,\vartheta_0+\vartheta+\theta)\|<1,$$

provided one of the following conditions is met:

$$Re\vartheta_0>0, or Re\vartheta>0, or Re\,\theta>0 or\ \|u\|<1, or\ \|v\|<1, or\ \|z\|<1.$$

Proof. The first part of the proposition is due to the above steps that formally ended in formula (3.6). Inequality (3.7) holds due to the following arguments:

$$\left\|\gamma(uvz,\eta+\varphi+\phi,\vartheta_0+\vartheta+\theta)\right\|\leq E\left\|(uvz)^{X_1}e^{-i(\eta+\varphi+\phi)\pi_n}e^{-(\vartheta_0+\vartheta+\theta)\Delta_1}\right\|$$
$$=\sum_{k=0}^{\infty}\left\|uvz\right\|^k\int_{t=0}^{\infty}e^{-\mathrm{Re}(\vartheta_0+\vartheta+\theta)t}P_{X_1\otimes\Delta_1}\left(k,dt\right)$$
$$=\int_{t=0}^{\infty}e^{-\mathrm{Re}(\vartheta_0+\vartheta+\theta)t}P_{X_1\otimes\Delta_1}\left(0,dt\right)+\sum_{k=1}^{\infty}\left\|uvz\right\|^k\int_{t=0}^{\infty}e^{-\mathrm{Re}(\vartheta_0+\vartheta+\theta)t}P_{X_1\otimes\Delta_1}\left(k,dt\right)$$

$$= \int_{t=0}^{1} e^{-\mathrm{Re}(\vartheta_0+\vartheta+\theta)t} P_{X_1\otimes\Delta_1}(0,dt) + \int_{t=1}^{\infty} e^{-\mathrm{Re}(\vartheta_0+\vartheta+\theta)t} P_{X_1\otimes\Delta_1}(0,dt)$$

$$+ \sum_{k=1}^{\infty} \|uvz\|^k \int_{t=0}^{1} e^{-\mathrm{Re}(\vartheta_0+\vartheta+\theta)t} P_{X_1\otimes\Delta_1}(k,dt)$$

$$+ \sum_{k=1}^{\infty} \|uvz\|^k \int_{t=1}^{\infty} e^{-\mathrm{Re}(\vartheta_0+\vartheta+\theta)t} P_{X_1\otimes\Delta_1}(k,dt)$$

$$\leq \int_{t=0}^{1} P_{X_1\otimes\Delta_1}(0,dt) + e^{-\mathrm{Re}(\vartheta_0+\vartheta+\theta)} \int_{t=1}^{\infty} P_{X_1\otimes\Delta_1}(0,dt)$$

$$+\|uvz\| \sum_{k=1}^{\infty} \int_{t=0}^{1} P_{X_1\otimes\Delta_1}(k,dt) + \|uvz\| \sum_{k=1}^{\infty} e^{-\mathrm{Re}(\vartheta_0+\vartheta+\theta)} \int_{t=1}^{\infty} P_{X_1\otimes\Delta_1}(k,dt),$$

since $\|uvz\| \geq \|uvz\|^k$ for $\|uvz\| \leq 1$ and $k > 1$. Let

$$a := \int_{t=0}^{1} P_{X_i\otimes\Delta_i}(0,dt),\ b := \int_{t=1}^{\infty} P_{X_i\otimes\Delta_i}(0,dt)$$

$$c := \sum_{k=1}^{\infty} \int_{t=0}^{1} P_{X_i\otimes\Delta_i}(k,dt),\ d := \sum_{k=1}^{\infty} \int_{t=1}^{\infty} P_{X_i\otimes\Delta_i}(k,dt).$$

Then clearly, $a + b + c + d = 1$ and thus,

$$a + e^{-\mathrm{Re}(\vartheta_0+\vartheta+\theta)} b + \|uvz\| c + \|uvz\| e^{-\mathrm{Re}(\vartheta_0+\vartheta+\theta)} d < 1$$

whenever $\|uvz\| < 1$ or $\mathrm{Re}(\vartheta_0 + \vartheta + \theta) > 0$ and we are done with the proof. □

We continue with F_n for $n = 0$. F_0 is the functional of the underlying process on interval $[0, t_0)$. With $N_t = \Pi_t = A_{-1} = P_{-1} = t_{-1} = 0$ we have

$$F_0(t) = Ez^{N_t} u^{A_{n-1}} e^{-i\eta\Pi_t} e^{-i\varphi P_{n-1}} v^{A_n} e^{-i\phi P_n} e^{-\vartheta_0 t_{n-1} - \vartheta t_n} \mathbf{1}_{\{0\leq t<t_0\}}$$

$$= Ev^{A_0} e^{-i\phi P_0} e^{-\vartheta t_0} \mathbf{1}_{[0,t_0)}(t).$$

The following is easy to prove.

Proposition 3.2 *Let* $F_0(t) = Ev^{A_0} e^{-i\phi P_0} e^{-\vartheta t_0} \mathbf{1}_{[0,t_0)}(t)$. *Then*

$$F_0^*(\theta) = \frac{1}{\theta}\left[\gamma_0(v,\phi,\vartheta) - \gamma_0(v,\phi,\vartheta+\theta)\right]. \tag{3.7}$$

With Proposition 3.2, we can augment the series $\sum_{n=1}^{\infty} F_n^*$ of formula (3.6) to include F_0^*:

$$\sum_{n=0}^{\infty} F_n^*(\theta) = \frac{1}{\theta}\left[\gamma_0(v,\phi,\vartheta) - \gamma_0(v,\phi,\vartheta+\theta)\right] + \frac{1}{\theta}\gamma_0(zuv,\eta+\varphi+\phi,\vartheta_0+\vartheta+\theta)$$
$$\times \left[\gamma(v,\phi,\vartheta) - \gamma(v,\phi,\vartheta+\theta)\right] \frac{1}{1-\gamma(zuv,\eta+\varphi+\phi,\vartheta_0+\vartheta+\theta)}. \tag{3.8}$$

From (2.7) and (3.8) we arrive at

$$\Phi^{*}(\theta,x)=\sum_{n=0}^{\infty}\left[F_{n}^{*}(ux,v,z,\vartheta_{0},\vartheta,t)-F_{n}^{*}(u,vx,z,\vartheta_{0},\vartheta,t)\right]$$

$$=\frac{1}{\theta}\left[\gamma_{0}(v,\phi,\vartheta)-\gamma_{0}(v,\phi,\vartheta+\theta)\right]-\frac{1}{\theta}\left[\gamma_{0}(vx,\phi,\vartheta)-\gamma_{0}(vx,\phi,\vartheta+\theta)\right]$$

$$+\frac{1}{\theta}\gamma_{0}(zuvx,\eta+\varphi+\phi,\vartheta_{0}+\vartheta+\theta)\frac{1}{1-\gamma(zuvx,\eta+\varphi+\phi,\vartheta_{0}+\vartheta+\theta)}$$

$$\times\left[\gamma(v,\phi,\vartheta)-\gamma(vx,\phi,\vartheta)+\gamma(vx,\phi,\vartheta+\theta)-\gamma(v,\phi,\vartheta+\theta)\right]. \quad (3.9)$$

The Laplace transform $\Phi_{\nu}^{*}(\theta)=\int_{t=0}^{\infty}e^{-\theta t}\Phi_{\nu}(t)\,dt$ of the functional

$$\Phi_{\nu}(t)=Ez^{N_{t}}e^{-i\eta\Pi_{t}}e^{-i\varphi P_{\nu-1}}u^{A_{\nu-1}}e^{-i\phi P_{\nu}}v^{A_{\nu}}e^{-\vartheta_{0}t_{\nu-1}-\vartheta t_{\nu}}\mathbf{1}_{[t_{\nu-1},t_{\nu})}(t)$$

can be extracted from $\Phi^{*}(\theta,x)$ of (4.9) using the $\mathcal{D}$-operator.

The entire effort in this section can be reduced to the following.

Theorem 3.1 *Let $\Phi_{\nu}(\theta)$ denote the Laplace transform of the functional*

$$\Phi_{\nu}(t)=Ez^{N_{t}}e^{-i\eta\Pi_{t}}e^{-i\varphi P_{\nu-1}}u^{A_{\nu-1}}e^{-i\phi P_{\nu}}v^{A_{\nu}}e^{-\vartheta_{0}t_{\nu-1}-\vartheta t_{\nu}}\mathbf{1}_{[t_{\nu-1},t_{\nu})}(t) \quad (3.10)$$

$$\|z\|\leq 1,\|u\|\leq 1,\|v\|\leq 1,Re\vartheta_{0}\geq 0,Re\vartheta\geq 0,\eta,\varphi,\phi\in\mathbb{R},$$

$$Then, with\,\|u\|<1, or\ \ \|v\|<1, or\ \ \|z\|<1,\ or\ Re\vartheta_{0}>0, or\ Re\vartheta>0,\ or\ Re\,\theta>0, \quad (3.11)$$

$$\Phi_{\nu}^{*}(\theta)$$

$$=\mathcal{D}_{x}^{M-1}\Bigg\{\frac{1}{\theta}\left[\gamma_{0}(v,\phi,\vartheta)-\gamma_{0}(v,\phi,\vartheta+\theta)\right]-\frac{1}{\theta}\left[\gamma_{0}(vx,\phi,\vartheta)-\gamma_{0}(vx,\phi,\vartheta+\theta)\right]$$

$$+\frac{1}{\theta}\gamma_{0}(zuvx,\eta+\varphi+\phi,\vartheta_{0}+\vartheta+\theta)\frac{1}{1-\gamma(zuvx,\eta+\varphi+\phi,\vartheta_{0}+\vartheta+\theta)}$$

$$\times\left[\gamma(v,\phi,\vartheta)-\gamma(vx,\phi,\vartheta)+\gamma(vx,\phi,\vartheta+\theta)-\gamma(v,\phi,\vartheta+\theta)\right]\Bigg\}. \quad (3.12)$$

4 Applications to Option Trading

For an illustration, consider the following special case. Suppose that we observe a constantly fluctuating stock price of some company over the times $t_0=0,t_1,t_2,\ldots$ that starts off at time zero with a price π_0.

Case 1. Observation of process P_i upon the first drop.

1a. Suppose we are interested in the characteristics of the process around the period when the stock price drops for the first time. Because the stock prices cannot be modeled by a monotone process, we have the observed prices upon t's as the passive component, and introduce the active component

$$X_n = \begin{cases} 0, & \pi_n \geq 0, \\ 1, & \pi_n < 0. \end{cases} \tag{4.1}$$

Suppose π_0 is a nonnegative r.v. with some specified distribution and let $X_0 = \tau_0 = 0$.

So, $\gamma_0(z, \phi, \theta) = Ee^{-i\phi\pi_0}$ $(innotation) = \gamma_0(\phi)$.

Next, with $M = 1$ according to our assumption about the first drop, formula (3.12) further reduces to

$$\begin{aligned} \theta\Phi_\nu^*(\theta) = &\gamma_0(\eta + \varphi + \phi) \frac{1}{1 - \gamma(0, \eta + \varphi + \phi, \vartheta_0 + \vartheta + \theta)} \\ &\times [\gamma(v, \phi, \vartheta) - \gamma(0, \phi, \vartheta) + \gamma(0, \phi, \vartheta + \theta) - \gamma(v, \phi, \vartheta + \theta)]. \end{aligned} \tag{4.2}$$

Because the active component is merely auxiliary, we are less interested in any information about $N_t, A_{\nu-1}, A_\nu$, as well as $P_{\nu-1}, t_{\nu-1}$, so we set $z = u = v = 1$ and $\varphi = \vartheta_0 = 0$ restricting the Laplace transform of Φ_ν to the marginal transform

$$\begin{aligned} &\int_{t=0}^{\infty} e^{-\theta t} E e^{-i\eta\Pi_t} e^{-i\phi P_\nu} e^{-\vartheta t_\nu} \mathbf{1}_{[t_{\nu-1}, t_\nu)}(t)\, dt \\ &= \frac{1}{\theta}\gamma_0(\eta + \phi) \frac{1}{1 - \gamma(0, \eta + \phi, \vartheta + \theta)} \\ &\times [\gamma(1, \phi, \vartheta) - \gamma(0, \phi, \vartheta) + \gamma(0, \phi, \vartheta + \theta) - \gamma(1, \phi, \vartheta + \theta)], \end{aligned} \tag{4.3}$$

where

$$\gamma(z, \phi, \theta) = Ez^{X_1} e^{-i\phi\pi_1} e^{-\Delta_1\theta} \ and \gamma(0, \phi, \theta) = Ez^{X_1} e^{-i\phi\pi_1} e^{-\Delta_1\theta}\Big|_{z=0}.$$

From

$$Ez^{X_1}\Big|_{z=0} = P\{X_1 = 0\} + zP\{X_1 = 1\}\Big|_{z=0} = P\{X_1 = 0\} = E\mathbf{1}_{\{X_1=0\}} = E\mathbf{1}_{\{\pi_1 \geq 0\}}$$

we have

$$\gamma(0, \phi, \theta) = E\mathbf{1}_{\{\pi_1 \geq 0\}} e^{-i\phi\pi_1} e^{-\Delta_1\theta}.$$

Suppose now that Δ's and π's are independent, that is, the observation epochs and stock price changes are independent. This may not always apply, but it would simplify establishing of γ. Then

$$\gamma(0, \phi, \theta) = E\mathbf{1}_{\{\pi_1 \geq 0\}} e^{-i\phi\pi_1} E e^{-\Delta_1\theta} = E\mathbf{1}_{\{\pi_1 \geq 0\}} e^{-i\phi\pi_1} \frac{\gamma}{\gamma + \theta},$$

if the observation epochs occur according to a Poisson point process of intensity γ. Our next assumption is that the marginal distribution of π_1 is Laplace with parameter μ and zero shift. That being said, the PDF of π_1 is

$$f_{\pi_1}(x) = \frac{1}{2}\mu e^{-\mu|x|}, x \in \mathbb{R}. \tag{4.4}$$

Then

$$\gamma(0,\phi,0)=E\mathbf{1}_{\{\pi_1\geq 0\}}e^{-i\phi\pi_1}=\int_{x=0}^{\infty}e^{-i\phi x}\frac{1}{2}\mu e^{-\mu x}dx=\frac{1}{2}\frac{\mu}{\mu+i\phi}.$$

Because $Ee^{-i\phi\pi_1}=Ee^{-i\phi\pi_1}\left(\mathbf{1}_{\{\pi_1\geq 0\}}+\mathbf{1}_{\{\pi_1<0\}}\right)$,we have

$$Ee^{-i\phi\pi_1}=\frac{1}{2}\frac{\mu}{\mu+i\phi}+\int_{x=-\infty}^{0}e^{-i\phi x}\frac{1}{2}\mu e^{\mu x}dx$$

$$=\frac{1}{2}\frac{\mu}{\mu+i\phi}+\frac{1}{2}\frac{\mu}{\mu-i\phi}=\frac{1}{2}\mu\frac{2\mu}{\mu^2+\phi^2}=\frac{\mu^2}{\mu^2+\phi^2}.$$

Thus,

$$\gamma(1,\phi,\theta)=Ee^{-i\phi\pi_1}Ee^{-\Delta_1\theta}=Ee^{-i\phi\pi_1}\frac{\gamma}{\gamma+\theta}=\frac{\mu^2}{\mu^2+\phi^2}\frac{\gamma}{\gamma+\theta}.$$

Next the following two further marginals are of interest.

(i) With $\eta=\phi=0$ in (4.3),the functional

$$\int_{t=0}^{\infty}e^{-\theta t}Ee^{-\vartheta t_\nu}\mathbf{1}_{[t_{\nu-1},t_\nu)}(t)\,dt$$

$$=\frac{1}{\theta}\frac{1}{1-\gamma(0,0,\vartheta+\theta)}\left[\gamma(1,0,\vartheta)-\gamma(0,0,\vartheta)+\gamma(0,0,\vartheta+\theta)-\gamma(1,0,\vartheta+\theta)\right] \quad (4.5)$$

represents the Laplace transform of the first passage time t_ν's marginal functional at the first drop with the time t falling between the pre-first passage time $t_{\nu-1}$ and t_ν. Here

$$\gamma(1,0,\vartheta+\theta)=\frac{\gamma}{\gamma+\vartheta+\theta}$$

$$\gamma(0,\phi,\theta)=E\mathbf{1}_{\{\pi_1\geq 0\}}e^{-i\phi\pi_1}Ee^{-\Delta_1\theta}=\frac{1}{2}\frac{\mu}{\mu+i\phi}\frac{\gamma}{\gamma+\theta}$$

$$\gamma(0,0,\vartheta)=\frac{1}{2}\frac{\gamma}{\gamma+\vartheta}.$$

Therefore,

$$\int_{t=0}^{\infty}e^{-\theta t}Ee^{-\vartheta t_\nu}\mathbf{1}_{[t_{\nu-1},t_\nu)}(t)\,dt$$

$$=\left(1+\frac{\gamma}{\gamma+2(\vartheta+\theta)}\right)\frac{\gamma}{2}\frac{1}{(\gamma+\vartheta)(\gamma+\vartheta+\theta)} \quad (4.6)$$

implying that the inverse of the Laplace transform is

$$Ee^{-\vartheta t_\nu}\mathbf{1}_{[t_{\nu-1},t_\nu)}(t)=\mathcal{L}_\theta^{-1}\left\{\int_{t=0}^{\infty}e^{-\theta t}Ee^{-\vartheta t_\nu}\mathbf{1}_{[t_{\nu-1},t_\nu)}(t)\,dt\right\}=\frac{\gamma}{2(\gamma+\vartheta)}e^{-\frac{t}{2}(\gamma+2\vartheta)}. \quad (4.7)$$

(ii) With $\eta=\vartheta=0$ in (4.3), we have the Laplace transform of the P_ν's marginal functional upon the first passage time t_ν jointly with the time t running between $t_{\nu-1}$ and t_ν.

$$\int_{t=0}^{\infty}e^{-\theta t}Ee^{-i\phi P_\nu}\mathbf{1}_{[t_{\nu-1},t_\nu)}(t)\,dt$$

$$=\frac{1}{\theta}\gamma_0(\phi)\frac{1}{1-\gamma(0,\phi,\theta)}\left[\gamma(1,\phi,0)-\gamma(0,\phi,0)+\gamma(0,\phi,\theta)-\gamma(1,\phi,\theta)\right]. \quad (4.8)$$

Because

$$\gamma_0(\phi) = e^{-i\phi p_0}$$

(assuming the initial price $\pi_0 = p_0$ a.s. where p_0 is a constant) and

$$\gamma(1,\phi,\theta) = \frac{\mu^2}{\mu^2+\phi^2}\frac{\gamma}{\gamma+\theta},$$

$$\gamma(0,\phi,\theta) = E\mathbf{1}_{\{\pi_1 \geq 0\}} e^{-i\phi\pi_1} E e^{-\Delta_1\theta} = \frac{1}{2}\frac{\mu}{\mu+i\phi}\frac{\gamma}{\gamma+\theta},$$

$$1-\gamma(0,\phi,\theta) = \frac{2(\mu+i\phi)(\gamma+\theta)-\mu\gamma}{2(\mu+i\phi)(\gamma+\theta)},$$

and

$$\frac{1}{1-\gamma(0,0,\vartheta+\theta)} = 1+\frac{\mu\gamma}{2(\mu+i\phi)(\gamma+\theta)-\mu\gamma},$$

we have that

$$\begin{aligned}\int_{t=0}^{\infty} & e^{-\theta t} E e^{-i\phi P_\nu}\mathbf{1}_{[t_{\nu-1},t_\nu)}(t)\,dt \\ &= \frac{1}{\theta}e^{-i\phi p_0}\left(1+\frac{\mu\gamma}{2(\mu+i\phi)(\gamma+\theta)-\mu\gamma}\right) \\ &\times\left[\frac{\mu^2}{\mu^2+\phi^2}-\frac{1}{2}\frac{\mu}{\mu+i\phi}+\frac{1}{2}\frac{\mu}{\mu+i\phi}\frac{\gamma}{\gamma+\theta}-\frac{\mu^2}{\mu^2+\phi^2}\frac{\gamma}{\gamma+\theta}\right] \\ &= e^{-i\phi p_0}\left[\frac{\mu^2}{\mu^2+\phi^2}-\frac{1}{2}\frac{\mu}{\mu+i\phi}\right]\left(1+\frac{\mu\gamma}{2(\mu+i\phi)(\gamma+\theta)-\mu\gamma}\right)\frac{1}{\gamma+\theta}. \qquad (4.9)\end{aligned}$$

Thus,

$$\begin{aligned}E e^{-i\phi P_\nu}\mathbf{1}_{[t_{\nu-1},t_\nu)}(t) &= \mathcal{L}_\theta^{-1}\left\{\int_{t=0}^{\infty} e^{-\theta t} E e^{-i\phi P_\nu}\mathbf{1}_{[t_{\nu-1},t_\nu)}(t)\right\} \\ &= \left[\frac{\mu^2}{\mu^2+\phi^2}-\frac{1}{2}\frac{\mu}{\mu+i\phi}\right] e^{-\left(\frac{\gamma t}{2}\left(\frac{\mu+2i\phi}{\mu+i\phi}\right)+i\phi p_0\right)} \qquad (4.10)\end{aligned}$$

and

$$EP_\nu\mathbf{1}_{[t_{\nu-1},t_\nu)}(t) = i\lim_{\phi\to 0}\frac{\partial}{\partial\phi}E e^{-i\phi P_\nu}\mathbf{1}_{[t_{\nu-1},t_\nu)}(t) = \frac{1}{2\mu}\left(\frac{\gamma t}{2}+\mu p_0-1\right)e^{-\frac{\gamma t}{2}} \qquad (4.11)$$

$$\begin{aligned}EP_\nu^2\mathbf{1}_{[t_{\nu-1},t_\nu)}(t) &= -\lim_{\phi\to 0}\frac{\partial^2}{\partial\phi^2}E e^{-i\phi P_\nu}\mathbf{1}_{[t_{\nu-1},t_\nu)}(t) \\ &= \frac{1}{2\mu^2}\left(2+\left(\frac{\gamma t}{2}\right)^2+(\mu p_0)^2+2\mu p_0\frac{\gamma t}{2}-2\mu p_0\right)e^{-\frac{\gamma t}{2}}. \qquad (4.12)\end{aligned}$$

So

$$E\mathbf{1}_{[t_{\nu-1},t_\nu)}(t) = P\{t_{\nu-1}\leq t<t_\nu\} = \mathcal{L}_\theta^{-1}\left\{\int_{t=0}^{\infty}e^{-\theta t}E\mathbf{1}_{[t_{\nu-1},t_\nu)}(t)\,dt\right\} = \frac{e^{-\frac{\gamma}{2}t}}{2}. \qquad (4.13)$$

1b. One could be interested in when the passive component drops lower than R, for some $R<0$. Thus the active component reads now

$$X_n = \begin{cases}0, & \pi_n \geq R,\\ 1, & \pi_n < R.\end{cases} \qquad (4.14)$$

With $M = 1$ assumed and because

$$Ez^{X_1}\big|_{z=0} = P\{X_1 = 0\} + zP\{X_1 = 1\}\big|_{z=0} = P\{X_1 = 0\} = E\mathbf{1}_{\{X_1=0\}} = E\mathbf{1}_{\{\pi_1 \geq R\}},$$

we have

$$\gamma(0,\phi,\theta) = E\mathbf{1}_{\{\pi_1\geq R\}}e^{-i\phi\pi_1}e^{-\Delta_1\theta} = E\mathbf{1}_{\{\pi_1\geq R\}}e^{-i\phi\pi_1}\frac{\gamma}{\gamma+\theta},$$

$$\gamma(0,\phi,0) = E\mathbf{1}_{\{\pi_1\geq R\}}e^{-i\phi\pi_1} = \int_{x=R}^{0} e^{-i\phi x}\frac{1}{2}\mu e^{\mu x}dx + \int_{x=0}^{\infty} e^{-i\phi x}\frac{1}{2}\mu e^{-\mu x}dx$$

$$= \frac{1}{2}\frac{\mu}{\mu - i\phi}\left[1 - e^{(\mu - i\phi)R}\right] + \frac{1}{2}\frac{\mu}{\mu + i\phi}.$$

Since

$$Ee^{-i\phi\pi_1} = Ee^{-i\phi\pi_1}\left(\mathbf{1}_{\{\pi_1\geq R\}} + \mathbf{1}_{\{\pi_1<R\}}\right),$$

we have

$$Ee^{-i\phi\pi_1} = \frac{1}{2}\frac{\mu}{\mu - i\phi}\left[1 - e^{(\mu - i\phi)R}\right] + \frac{1}{2}\frac{\mu}{\mu + i\phi} + \int_{x=-\infty}^{R} e^{-i\phi x}\frac{1}{2}\mu e^{\mu x}dx = \frac{\mu^2}{\mu^2+\phi^2}.$$

Thus,

$$\gamma(1,\phi,\theta) = Ee^{-i\phi\pi_1}Ee^{-\Delta_1\theta} = Ee^{-i\phi\pi_1}\frac{\gamma}{\gamma+\theta} = \frac{\mu^2}{\mu^2+\phi^2}\frac{\gamma}{\gamma+\theta}$$

and with $\eta = \phi = 0 = \vartheta$ in (4.3), the functional

$$\int_{t=0}^{\infty} e^{-\theta t}E\mathbf{1}_{[t_{\nu-1},t_\nu)}(t)\,dt$$

$$= \frac{1}{\theta}\frac{1}{1-\gamma(0,0,\theta)}\left[\gamma(1,0,0) - \gamma(0,0,0) + \gamma(0,0,\theta) - \gamma(1,0,\theta)\right] = e^{\mu R}\frac{1}{2\theta + \gamma e^{\mu R}}$$

and

$$E\mathbf{1}_{[t_{\nu-1},t_\nu)}(t) = P\{t_{\nu-1} \leq t < t_\nu\}$$

$$= \mathcal{L}_\theta^{-1}\left\{\int_{t=0}^{\infty} e^{-\theta t}E\mathbf{1}_{[t_{\nu-1},t_\nu)}(t)\,dt\right\} = \frac{1}{2}e^{-\left(\frac{\gamma t e^{\mu R}}{2} - \mu R\right)} \tag{4.15}$$

which reduces to (4.13) when $R = 0$.

Case 2. Observation of process P_i upon general Mth drop.

2a. For the general threshold level M (when the stock price drops Mth times), since the active process increments X_n are Bernoulli with $p = 0.5$ due to the symmetric Laplace PDF of π_n defined in (4.4) above with zero shift and with

$$E\mathbf{1}_{(t_{\nu-1},t_\nu]}(t) = \Phi_\nu(t)\big|_{z,v,u,\vartheta=1,\eta,\varphi,\phi,\vartheta_0,\vartheta=0},$$

$$\Phi_\nu^*(\theta) = \int_{t=0}^{\infty} e^{-\theta t}E\mathbf{1}_{[t_{\nu-1},t_\nu)}(t)\,dt$$

$$= \mathcal{D}_x^{M-1}\frac{1}{\theta}\gamma_0(0)\frac{1}{1-\gamma(x,0,\theta)} \times \left[\gamma(1,0,0) - \gamma(x,0,0) + \gamma(x,0,\theta) - \gamma(1,0,\theta)\right], \tag{4.16}$$

where

$$\gamma(1,0,\theta) = \frac{\gamma}{\gamma+\theta}, \quad \gamma(x,0,0) = \frac{1+x}{2}, \, \gamma(x,0,\theta) = (\frac{1+x}{2})\frac{\gamma}{\gamma+\theta}.$$

Therefore,

$$\Phi_\nu^*(\theta) = \mathcal{D}_x^{M-1}\frac{1}{\theta}\frac{2(\gamma+\theta)}{\gamma+2\theta-\gamma x}\left[1-\frac{1+x}{2}+\frac{1+x}{2}\frac{\gamma}{\gamma+\theta}-\frac{\gamma}{\gamma+\theta}\right]$$

$$= \mathcal{D}_x^{M-1}\frac{1}{\theta}\frac{2(\gamma+\theta)}{\gamma+2\theta-\gamma x}\left[\frac{1-x}{2}\frac{\theta}{\gamma+\theta}\right]$$

$$= \mathcal{D}_x^{M-1}\left\{\frac{1}{\gamma+2\theta-\gamma x}\right\} - \mathcal{D}_x^{M-2}\left\{\frac{1}{\gamma+2\theta-\gamma x}\right\} = \frac{1}{(\gamma+2\theta)}\left(\frac{\gamma}{\gamma+2\theta}\right)^{M-1} = \frac{\gamma^{M-1}}{(\gamma+2\theta)^M}.$$

So

$$E\mathbf{1}_{(t_{\nu-1},t_\nu]}(t) = P\{t_{\nu-1} \le t < t_\nu\} = \mathcal{L}_\theta^{-1}\{\Phi_\nu(\theta)\} = \frac{1}{2}\frac{\left(\frac{\gamma t}{2}\right)^{M-1}}{(M-1)!}e^{-\frac{\gamma}{2}t}. \quad (4.17)$$

2b. Next we obtain the result for $E\mathbf{1}_{(t_{\nu-1},t_\nu]}(t)$ for general M and general shift parameter a in our model such that

$$f_{\pi_1}(x) = \frac{1}{2}\mu e^{-\mu|x-a|}, x \in \mathbb{R}.$$

After some algebra we have

$$\gamma(0,\phi,0) = E\mathbf{1}_{\{\pi_1 \ge 0\}}e^{-i\phi\pi_1} = \int_{x=0}^{a} e^{-i\phi x}\frac{1}{2}\mu e^{\mu(x-a)}dx + \int_{x=a}^{\infty} e^{-i\phi x}\frac{1}{2}\mu e^{-\mu(x-a)}dx$$

$$= \frac{1}{2}\mu\frac{2\mu e^{-i\phi a} - e^{-\mu a}(\mu+i\phi)}{\mu^2+\phi^2},$$

$$\gamma(0,\phi,\theta) = \frac{1}{2}\mu\frac{2\mu e^{-i\phi a} - e^{-\mu a}(\mu+i\phi)}{\mu^2+\phi^2}\frac{\gamma}{\gamma+\theta}$$

and

$$\gamma(0,\phi,0) = \frac{1}{2}\mu\frac{2\mu e^{-i\phi a} - e^{-\mu a}(\mu+i\phi)}{\mu^2+\phi^2},$$

$$Ee^{-i\phi\pi_1} = \frac{2\mu^2 e^{-i\phi a}}{\mu^2+\phi^2} - \frac{1}{2}\frac{\mu}{\mu^2+\phi^2}\left[(\mu+i\phi)e^{-\mu a} + (\mu-i\phi)e^{\mu a}\right]$$

$$\gamma(1,\phi,\theta) = \left[\frac{2\mu^2 e^{-i\phi a}}{\mu^2+\phi^2} - \frac{1}{2}\frac{\mu}{\mu^2+\phi^2}\left[(\mu+i\phi)e^{-\mu a} + (\mu-i\phi)e^{\mu a}\right]\right]\frac{\gamma}{\gamma+\theta}$$

$$\gamma(x,\phi,\theta) = (p+qx)\left[\frac{2\mu^2 e^{-i\phi a}}{\mu^2+\phi^2} - \frac{1}{2}\frac{\mu}{\mu^2+\phi^2}\left[(\mu+i\phi)e^{-\mu a} + (\mu-i\phi)e^{\mu a}\right]\right]\frac{\gamma}{\gamma+\theta}.$$

Hence

$$\Phi_\nu^*(\theta) = \int_{t=0}^{\infty} e^{-\theta t}E\mathbf{1}_{(t_{\nu-1},t_\nu]}(t)\,dt$$

$$= \mathcal{D}_x^{M-1}\frac{1}{\theta}\gamma_0(0)\frac{1}{1-\gamma(x,0,\theta)} \times [\gamma(1,0,0) - \gamma(x,0,0) + \gamma(x,0,\theta) - \gamma(1,0,\theta)]$$

$$= \mathcal{D}_x^{M-1}\left\{\frac{1}{\theta}\frac{(\gamma+\theta)}{\gamma+\theta-p\gamma(2-\cosh(\mu a)) - q\gamma(2-\cosh(\mu a))x}\right.$$

$$\times\left[(2-\cosh(\mu a))-(p+qx)\,(2-\cosh\mu a))+(p+qx)\,(2-\cosh(\mu a))\,\frac{\gamma}{\gamma+\theta}\right.$$
$$\left.\left.-\,(2-\cosh(\mu a))\,\frac{\gamma}{\gamma+\theta}\right]\right\}$$
$$=\frac{\gamma^{M-1}(q\,(2-\cosh(\mu a)))^{M}}{(\gamma+\theta-p\gamma\,(2-\cosh(\mu a)))^{M}} \tag{4.18}$$

by the $\mathcal{D}$-operator inversion formulas from [12].

$$E\mathbf{1}_{[t_{\nu-1},t_\nu)}\,(t)=P\,\{t_{\nu-1}\le t<t_\nu\}=\mathcal{L}_\theta^{-1}\,\{\varPhi_\nu^*\,(\theta)\}$$
$$=\gamma^{M-1}(q\,(2-\cosh(\mu a)))^{M}\,\frac{t^{M-1}}{(M-1)!}e^{-(\gamma-p\gamma(2-\cosh(\mu a)))t}$$
$$=q\,(2-\cosh(\mu a))\,\frac{(\gamma q\,(2-\cosh(\mu a))\,t)^{M-1}}{(M-1)!}e^{-(\gamma-p\gamma(2-\cosh(\mu a)))t}. \tag{4.19}$$

Notice that when $a=0$ (in the symmetric case), (4.19) reduces to (4.17) and the value of μ is irrelevant given it is finite.

5 Continuous Time Parameter Process on Interval $[0,t_\nu)$

Now consider the functional of passive process P being observed over the period $[0,t_\nu)$, jointly with the active process A_ν, the first passage time t_ν, and the counting processes N_t and $\varPi_t$. The functional satisfies the formula:

$$\overset{\wedge}{\varPhi}_\nu\,(t)=Ez^{N_t}e^{-i\eta\varPi_t}e^{-i\phi P_\nu}\upsilon^{A_\nu}e^{-\vartheta t_\nu}\mathbf{1}_{[0,t_\nu)}(t)$$
$$=\sum_{k=0}^{\infty}Ez^{N_t}e^{-i\eta\varPi_t}e^{-i\phi P_\nu}\upsilon^{A_\nu}e^{-\vartheta t_\nu}\mathbf{1}_{[0,t_\nu)}(t)\mathbf{1}_{\{\nu=k\}}.$$

Since $\sum_{j=0}^{\nu}E\mathbf{1}_{[t_{\nu-j-1},t_{\nu-j})}(t)=E\mathbf{1}_{[0,t_\nu)}(t)$,

$$\overset{\wedge}{\varPhi}_\nu\,(t)=\sum_{k=0}^{\infty}\sum_{j=0}^{k}E[z^{A_{k-j-1}}\upsilon^{A_{k-j-1}}\upsilon^{\sum_{i=k-j}^{k}X_i}e^{-i\eta\varPi_{k-j-1}-i\phi P_{k-j-1}}$$
$$\times e^{-i\phi\sum_{i=k-j}^{k}\pi_i}e^{-\vartheta t_{k-j-1}}e^{-\vartheta\sum_{i=k-j}^{k}\Delta_i}\mathbf{1}_{[t_{k-j-1},t_{k-j})}(t)],$$

and applying the transformation D_p to $\overset{\wedge}{\varPhi}_\nu\,(t)$ we have:

$$D_p\left[\overset{\wedge}{\varPhi}_\nu\,(t)\right](x)=\sum_{k=0}^{\infty}\sum_{j=0}^{k}F_{jk}\,(t)\,x^{X_{k-j+1}+\cdots+X_{k-1}}$$
$$\times E\,(vx)^{X_{k-j+1}+\cdots+X_{k-1}}\,e^{-i\phi(\pi_{k-j+1}+\cdots+\pi_{k-1})}e^{-\vartheta(\Delta_{k-j+1}+\cdots+\Delta_{k-1})}$$
$$\times E\left(1-x^{X_k}\right)e^{-i\phi\pi_k}e^{-\vartheta\Delta_k}\upsilon^{X_k},$$

where

$$F_{jk}\,(t)=$$
$$E\,(zvx)^{A_{k-j-1}}\,e^{-i(\eta+\phi)P_{k-j-1}}e^{-\vartheta t_{k-j-1}}\mathbf{1}_{[t_{k-j-1},t_{k-j})}(t)\,(vx)^{X_{k-j}}\,e^{-i\phi(\pi_{k-j})}e^{-\vartheta(\Delta_{k-j})},$$

$$\widetilde{F}(t) = Ez^A e^{-i\eta P} e^{-\vartheta T} \mathbf{1}_{[T,T+\Delta)}(t)\, v^X e^{-i\phi\pi} e^{-\vartheta\Delta}$$

under the assumptions that random vectors $A \otimes P \otimes T$ and $X \otimes \pi \otimes \Delta$ are independent. Then

$$\widetilde{F}^*(\theta) = \sum_r z^r \sum_m v^m \int_p e^{-i\eta p} \int_w e^{-i\phi w} \int_{s\geq 0} e^{-\vartheta s} e^{-\theta s}.$$

$$\times \frac{1}{\theta} \int_\delta \left(e^{-\vartheta\delta} - e^{-(\vartheta+\theta)\delta}\right) P_{A\otimes P\otimes T\otimes X\otimes\pi\otimes\Delta}(r, m, dp, ds, dw, d\delta)$$

and because $A \otimes P \otimes T$ and $X \otimes \pi \otimes \Delta$ are independent,

$$= \frac{1}{\theta} E\left[z^A e^{-i\eta P} e^{-(\vartheta+\theta)T}\right] \left[\gamma(v,\phi,\vartheta) - \gamma(v,\phi,\vartheta+\theta)\right].$$

Thus

$$F^*_{jk}(\theta) = \frac{1}{\theta}\gamma_0 {\delta^1}^{j-1}\left[\delta - \delta^1\right]\gamma^{k-j-1}\left[\delta^1 - \delta^{13}\right], \tag{5.1}$$

(i)
$$\sum_{k>0}\sum_{j=1}^{k-1} F^*_{jk}(\theta) = \frac{1}{\theta}\gamma_0 \Psi\delta \sum_{k>0}\gamma^{k-2}\sum_{j=1}^{k-1}\left(\frac{\delta^1}{\gamma}\right)^{j-1} = \frac{1}{\theta}\gamma_0 \frac{\Psi\delta}{(1-\gamma)(1-\delta^1)}, \tag{5.2}$$

with notation $\gamma := \gamma(zvx, \eta+\phi, \vartheta)$ and $\gamma_0 := \gamma_0(zvx, \eta+\phi, \vartheta)$, and further

$$\delta^1 = \gamma(vx,\phi,\vartheta), \delta^1_0 = \gamma_0(vx,\phi,\vartheta), \delta = \gamma(v,\phi,\vartheta),$$

$$\delta^3 = \gamma(v,\phi,\vartheta+\theta), \delta^{13} = \gamma(vx,\phi,\vartheta+\theta), \ \ \delta_0 = \gamma_0(v,\phi,\vartheta), \delta^{13}_0 = \gamma_0(vx,\phi,\vartheta+\theta),$$

$$\Gamma\delta = \delta - \delta^3 - \delta^1 + \delta^{13},\ \Lambda\delta = \frac{\Psi\delta}{1-\delta^1} + \Gamma\delta, \Psi\delta = \left(\delta - \delta^1\right)\left(\delta^1 - \delta^{13}\right).$$

(ii) Consider $j = k = 0$. $A_{-1} = t_{-1} = P_{-1} = 0$ for $t \in [0, t_0)$ and $N_t = A_{-1} = \Pi_t = 0$.

$$F_{00}(t) = E\mathbf{1}_{[0,t_0)}(t) e^{-\vartheta t_0} v^{A_0} e^{-i\phi P_0}\left(1 - x^{A_0}\right).$$

$$F^*_{00}(\theta) = \sum_r v^r \int_p e^{-i\phi p}\int_s e^{-\vartheta s}\int_{t=0}^s e^{-\theta t} dt P_{A_0\otimes P_0\otimes t_0}(r, dp, ds)$$

$$= \sum_r v^r \int_p e^{-i\phi p}\int_s e^{-\vartheta s}\frac{1}{\theta}\left[e^{-\vartheta s} - e^{-(\vartheta+\theta)s}\right] P_{A_0\otimes P_0\otimes t_0}(r, dp, ds) = \frac{1}{\theta}\Gamma\delta_0. \tag{5.3}$$

(iii) Consider $j = 0, k > 0$.

$$F_{0k}(t) = Ez^{N_t} v^{A_k} e^{-i\eta P_{k-1}} e^{-i\phi P_k} e^{-\vartheta t_k}\mathbf{1}_{[t_{k-1},t_k)}(t)\left(x^{A_{k-1}} - x^{A_k}\right)$$

$$= E(zvx)^{A_{k-1}} e^{-i(\eta+\phi)P_{k-1}} e^{-\vartheta t_{k-1}}\mathbf{1}_{[t_{k-1},t_k)}(t)\, v^{X_k}\left(1 - x^{X_k}\right) e^{-i\phi\pi_k} e^{-\vartheta\Delta_k}.$$

$$F^*_{0k}(\theta) = \int_t e^{-\theta t}\sum_r (zvx)^r \int_p e^{-i(\eta+\phi)p}\sum_m v^m \int_q e^{-i\phi q}\int_s e^{-\vartheta s}$$

$$\times \int_\delta e^{-\vartheta\delta}\mathbf{1}_{[s,s+\delta)}(t)\, dt P_{A_{k-1}\otimes P_{k-1}\otimes T_{k-1}\otimes X_k\otimes\pi_k\otimes\Delta_k}(r, dp, ds, m, dq, d\delta)$$

$$= \sum_r (zvx)^r \int_p e^{-i(\eta+\phi)p}\sum_m v^m \int_q e^{-i\phi q}\int_s e^{-\vartheta s} e^{-\theta s}$$

$$\times \int_{t-s=0}^{\delta} e^{-\theta(t-s)} dt P_{A_{k-1}\otimes P_{k-1}\otimes T_{k-1}\otimes X_k\otimes \pi_k\otimes \Delta_k} (r, dp, ds, m, dq, d\delta)$$

$$= \sum_r (zvx)^r \int_p e^{-i(\eta+\phi)p} \sum_m v^m \int_q e^{-i\phi q} \int_\delta \left[e^{-\vartheta\delta} - e^{-(\vartheta+\theta)\delta} \right]$$

$$\times P_{A_{k-1}\otimes P_{k-1}\otimes X_k\otimes \pi_k\otimes \Delta_k} (r, dp, m, dq, d\delta)$$

$$= \frac{1}{\theta} \gamma^{k-1} (zvx, \eta+\phi, \vartheta) \gamma_0 (zvx, \eta+\phi, \vartheta)$$

$$\times \left[\gamma(v,\phi,\vartheta) - \gamma(v,\phi,\vartheta+\theta) - \gamma(vx,\phi,\vartheta) + \gamma(vx,\phi,\vartheta+\theta) \right]$$

and

$$\sum_{k>0} F_{0k}^* (\theta) = \frac{1}{\theta} \gamma_0 (zvx, \eta+\phi, \vartheta) \frac{1}{1-\gamma(zvx, \eta+\phi, \vartheta)}$$

$$\times \left[\gamma(v,\phi,\vartheta) - \gamma(v,\phi,\vartheta+\theta) - \gamma(vx,\phi,\vartheta) + \gamma(vx,\phi,\vartheta+\theta) \right] = \frac{1}{\theta} \frac{\gamma_0}{1-\gamma} \mathit{\Gamma} \delta. \tag{5.4}$$

(iv) Consider $j = k > 0$. $F_{kk}(t) = E\mathbf{1}_{[0,t_0)}(t) e^{-\vartheta t_k} v^{A_k} e^{-i\phi P_k} \left(x^{A_{k-1}} - x^{A_k} \right)$

$$= E\mathbf{1}_{[0,t_0)}(t) e^{-\vartheta t_k} (vx)^{A_0} e^{-i\phi P_0} e^{-\vartheta t_0} (vx)^{X_1+\cdots+X_{k-1}} e^{-i\phi(\pi_1+\cdots+\pi_{k-1})} e^{-\vartheta(\Delta_1+\cdots+\Delta_{k-1})}$$

$$\times \left[v^{X_k} - (vx)^{X_k} \right] e^{-i\phi\pi_k} e^{-\vartheta\Delta_k}$$

$$= E\mathbf{1}_{[0,t_0)}(t) e^{-\vartheta t_k} (vx)^{A_0} e^{-i\phi P_0} e^{-\vartheta t_0} \gamma^{k-1} (vx,\phi,\vartheta) \left[\gamma(v,\phi,\vartheta) - \gamma(vx,\phi,\vartheta) \right].$$

So,

$$\sum_{k>0} F_{kk}^* (\theta) = \frac{1}{\theta} \frac{\mathit{\Psi}\delta_0}{1-\delta^1}. \tag{5.5}$$

Altogether, from (i) through (iv) we have

$$\overset{\wedge}{\mathit{\Phi}}{}_\nu^* (\theta) = \int_{t=0}^{\infty} e^{-\theta t} \mathit{\Phi}_\nu (t)\, dt = \mathcal{D}_x^{M-1} \Big\{ \sum_{k>0} \sum_{j=1}^{k-1} F_{jk}^* (\theta) + F_{00}^* (t) + \sum_{k>0} F_{0k}^* (\theta) + \sum_{k>0} F_{kk}^* (\theta) \Big\}$$

$$= \mathcal{D}_x^{M-1} \Big\{ \frac{1}{\theta} \Big(\Lambda\delta_0 + \frac{\gamma_0}{1-\gamma} \Lambda\delta \Big) \Big\} \tag{5.6}$$

where $\Lambda\alpha = \mathit{\Gamma}\alpha + \frac{\mathit{\Psi}\alpha}{1-\delta^1}$ and $\alpha = \delta$ or δ_0. The Laplace inverse of (5.6) will permit the recovery of $\overset{\wedge}{\mathit{\Phi}}_\nu (t)$.

6 Conclusion

In this paper we study a class of signed marked random measures $(\mathcal{A}, \mathit{\Pi}, \mathcal{T}) = \sum_{n=0}^{\infty} (X_n, \pi_n) \varepsilon_{t_n}$ with position dependent marking, on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. We target the critical behavior of the underlying stochastic process about a fixed threshold in the context of time sensitivity. The latter means that all related characteristics, such as first passage time and the location of the process upon crossing the threshold relate to deterministic time $t \geq 0$. The major benefit of this study is to utilize stochastic control over the process that must traditionally be considered on time interval $[0,t]$, $t \geq 0$. Using and further embellishing fluctuation theory, we find explicitly the functionals

$$\Phi_\nu(t) = Ez^{N_t} e^{-i\eta \Pi_t} e^{-i\varphi P_{\nu-1}} u^{A_{\nu-1}} e^{-i\phi P_\nu} v^{A_\nu} e^{-\vartheta_0 t_{\nu-1} - \vartheta t_\nu} \mathbf{1}_{[t_{\nu-1}, t_\nu)}(t)$$

and

$$\overset{\wedge}{\Phi}_\nu(t) = Ez^{N_t} e^{-i\eta \Pi_t} e^{-i\phi P_\nu} v^{A_\nu} e^{-\vartheta t_\nu} \mathbf{1}_{[0,t_\nu)}(t)$$

with respect to time $t \in [\tau_{\nu-1}, \tau_\nu)$ and $t \in [0, \tau_\nu)$, respectively. These functionals describe the status of underlying processes $N_t = \sum_{n=0}^{\infty} X_n \varepsilon_{t_n}[0,t]$ and $\Pi_t = \sum_{n=0}^{\infty} \pi_n \varepsilon_{t_n}[0,t]$, along with other characteristics like the values of these processes upon the crossing as well as just prior to crossing the threshold.

We discuss various applications to the finance (stock option trading) and risk theory. A number of special cases and examples demonstrate analytic tractability of the results obtained.

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