Nonlinear Dynamics and Systems Theory, 19 (2) (2019) 227–242

Nonlinear Elliptic Equations with Some Measure Data in Musielak-Orlicz Spaces

A. Aberqi [1*], J. Bennouna [2] and M. Elmassoudi [2]

[1] *University sidi Mohamed ben abdellah- ENSA-Fez. Morocco*
[2] *University sidi Mohamed ben abdellah, Faculty of Sciences Dhar El Mahraz, Atlas Fez, Morocco.*

Received: February 2, 2019; Revised: April 20, 2019

Abstract: Our aim in this paper is to establish an existence result in the framework of Musielak-Orlicz spaces for the following nonlinear Dirichlet problem

$$A(u) + K(x, u, \nabla u) = \mu, \tag{1}$$

where $A(u) = -\mathrm{div}(a(x, u, \nabla u))$ is a Leray-Lions type operator defined on $D(A) \subset W_0^1 L_\varphi(\Omega)$ into its dual and the function K is a lower order term which satisfy some growth condition, and does not satisfy the sign condition. The source data μ is a bounded nonnegative Radon measure on Ω.

Keywords: *Musielak-Orlicz spaces; nonlinear elliptic problems; measure data; weak solution.*

Mathematics Subject Classification (2010): 35J60, 35J66.

1 Introduction

Classical Sobolev spaces do not allow one to solve all problems of the EDP, hence the need to find other spaces, larger and suitable for the recent problems such as the spaces $L^{p(x)}(\Omega)$ or, more generally, the Musielak spaces. These spaces are not always reflexive and separable, adding further difficulties for studying the existence of solutions. Thus all our work will be in these spaces. We consider the following nonlinear Dirichlet problem:

$$A(u) + K(x, u, \nabla u) = f \tag{2}$$

* Corresponding author: mailto:aberqi_ahmed@yahoo.fr,

on a Lipschitz bounded domain in $\mathbb{R}^N$. $A(u) = -\text{div}(a(x,u,\nabla u))$ is a Leary-Lions operator defined on $D(A) \subset W_0^1L_\varphi(\Omega) \to W^{-1}L_\psi(\Omega)$, where φ and ψ are two complementary Musielak-Orlicz functions and K is a nonlinear lower term which satisfies the growth condition without the sign condition. In the framework of Sobolev spaces with variable exponents (the φ-function is $\varphi(x,t) = |t|^{p(x)}$), a series of papers on nonlinear elliptic and parabolic equations without sign condition in the nonlinearity studied (see [8], [4]).

On Orlicz spaces, many papers were devoted to the existence of solutions of (2). In fact, Gossez J.P. [17] solved the problem in the variational case, Elmahi A. et al. [14] proved the existence results for the unilateral problem of (2), where K satisfies the growth condition and the right-hand side belongs to $L^1(\Omega)$. Recently, Dong G. et al. in [13] have taken the source term as a bounded nonnegative Radon measure on Ω.

On Musielak-Orlicz spaces, Ait Khellou M. et al. in [7] have shown the existence of solutions for (2) in the case where K satisfies the sign condition and $f \in L^1(\Omega)$.
The study of nonlinear partial differential equations is motivated by numerous phenomena of physics, namely, the electrorheological fluids, the flow thought the porous media (see the monograph of A. Antsenov [9]).

As an example of operator for which the present result can be applied, we give

$$-\text{div}\Big(\frac{m(x,|\nabla u|).\nabla u}{|\nabla u|}\Big) + u\phi(x,|\nabla u|) = f,$$

where $m(x,s)$ is the derivative of $\phi(x,s)$ with respect to s.

The aim of this paper is the study of the problem (2) in the setting of Musielak-Orlicz spaces overcoming two difficulties. Firstly we do not assume the sign condition on the nonlinearity K, after we prove that there exists at least one solution for approximate equations. Secondly, we show that solutions belong to the Musielak-Sobolev spaces $W_0^1L_\phi(\Omega)$ where ϕ is in a special class of the Musielak-Orlicz functions of the $\mathcal{A}_\varphi$ (see Definition 3.1).

This paper is organized as follows. Section 2 contains some preliminaries in the Musielak-Sobolev spaces. In Section 3, we give some lemmas and we show that the solution of the problem (2) belongs to the space $W_0^1L_\phi(\Omega)$. Section 4 is devoted to specifying the assumptions on $A(u)$, K and μ. In Section 5, we give and we prove principal Theorem 5.1.

2 Musielak-Orlicz Spaces – Notations and Properties

2.1 Musielak-Orlicz function

Let Ω be an open subset of $\mathbb{R}^N$ $(N \geq 2)$ and let φ be a real-valued function defined in $\Omega \times \mathbb{R}^+$. The function φ is called a Musielak-Orlicz function if

- $\varphi(x,\cdot)$ is an N-function for all $x \in \Omega$ (i.e. convex, non-decreasing, continuous, $\varphi(x,0) = 0$, $\varphi(x,0) > 0$ for $t > 0$, $\lim_{t\to 0}\sup_{x\in\Omega}\frac{\varphi(x,t)}{t} = 0$ and $\lim_{t\to\infty}\inf_{x\in\Omega}\frac{\varphi(x,t)}{t} = \infty$).
- $\varphi(\cdot,t)$ is a measurable function for all $t \geq 0$.

We put $\varphi_x(t) = \varphi(x,t)$ and we associate its non-negative reciprocal function φ_x^{-1} with respect to t, that is, $\varphi_x^{-1}(\varphi(x,t)) = \varphi(x,\varphi_x^{-1}(t)) = t$.
Let φ and γ be two Musielak-Orlicz functions, we say that φ dominates γ and we write

$\gamma \prec \varphi$ near infinity (respectively, globally) if there exist two positive constants c and t_0 such that for a.e. $x \in \Omega$ $\gamma(x,t) \leq \varphi(x,ct)$ for all $t \geq t_0$ (respectively, for all $t \geq 0$). We say that φ and γ are equivalents, and we write $\varphi \sim \gamma$ if φ dominates γ and γ dominates φ. Finally, we say that γ grows essentially less rapidly than φ at 0 (respectively, near infinity), and we write $\gamma \prec\prec \varphi$, for every positive constant c, we have $\lim_{t\to 0}\sup_{x\in\Omega}\frac{\gamma(x,ct)}{\varphi(x,t)} = 0$ (respectively, $\lim_{t\to\infty}\sup_{x\in\Omega}\frac{\gamma(x,ct)}{\varphi(x,t)}) = 0$).

Remark 2.1 [12] If $\gamma \prec\prec \varphi$ near infinity, then $\forall\epsilon > 0$ there exists $k(\epsilon) > 0$ such that for almost all $x \in \Omega$ we have

$$\gamma(x,t) \leq k(\epsilon)\varphi(x,\epsilon t) \quad \forall t \geq 0.$$

2.2 Musielak-Orlicz space

Let φ be a Musielak-Orlicz function and a measurable function $u : \Omega \to \mathbb{R}$, we define the functional

$$\varrho_{\varphi,\Omega}(u) = \int_\Omega \varphi(x, |u(x)|)dx.$$

The set $K_\varphi(\Omega) = \{u : \Omega \to \mathbb{R} \quad \text{measurable :} \quad \varrho_{\varphi,\Omega}(u) < \infty\}$ is called the Musielak-Orlicz class. The Musielak-Orlicz space $L_\varphi(\Omega)$ is the vector space generated by $K_\varphi(\Omega)$, that is, $L_\varphi(\Omega)$ is the smallest linear space containing the set $K_\varphi(\Omega)$. Equivalently,

$$L_\varphi(\Omega) = \{u : \Omega \to \mathbb{R} \quad \text{measurable :} \quad \varrho_{\varphi,\Omega}(\frac{u}{\lambda}) < \infty, \quad \text{for some} \quad \lambda > 0\}.$$

On the other hand, we put $\psi(x,s) = \sup_{t\geq 0}(st - \varphi(x,s))$.

ψ is called the Musielak-Orlicz function complementary to φ (or conjugate of φ) in the sense of Young with respect to s. We say that a sequence of function $u_n \in L_\varphi(\Omega)$ is modular convergent to $u \in L_\varphi(\Omega)$ if there exists a constant $\lambda > 0$ such that $\lim_{n\to\infty}\varrho_{\varphi,\Omega}(\frac{u_n-u}{\lambda}) = 0$. This implies convergence for $\sigma(\prod L_\varphi, \prod L_\psi)$ (see [11]).
In the space $L_\varphi(\Omega)$, we define the following two norms:

$$\|u\|_\varphi = \inf\big\{\lambda > 0 : \int_\Omega \varphi(x, \frac{|u(x)|}{\lambda})dx \leq 1\big\},$$

which is called the Luxemburg norm, and the so-called Orlicz norm

$$|||u|||_{\varphi,\Omega} = \sup_{\|v\|_\psi\leq 1}\int_\Omega |u(x)v(x)|dx,$$

where ψ is the Musielak-Orlicz function complementary to φ. These two norms are equivalent [11]. $K_\varphi(\Omega)$ is a convex subset of $L_\varphi(\Omega)$. We define $E_\varphi(\Omega)$ as the subset of $L_\varphi(\Omega)$ of all measurable functions $u : \Omega \mapsto \mathbb{R}$ such that $\displaystyle\int_\Omega \varphi(x, \frac{|u(x)|}{\lambda})dx < \infty$ for all $\lambda > 0$. It is a separable space and $(E_\varphi(\Omega))^* = L_\varphi(\Omega)$. We have $E_\varphi(\Omega) = K_\varphi(\Omega)$ if and only if φ satisfies the Δ_2-condition for the large values of t or for all values of t, according to whether Ω has finite measure or not. We define

$$W^1L_\varphi(\Omega) = \{u \in L_\varphi(\Omega) : D^\alpha u \in L_\varphi(\Omega), \quad \forall\alpha \leq 1\},$$
$$W^1E_\varphi(\Omega) = \{u \in E_\varphi(\Omega) : D^\alpha u \in E_\varphi(\Omega), \quad \forall\alpha \leq 1\},$$

where $\alpha=(\alpha_1,...,\alpha_N)$, $|\alpha|=|\alpha_1|+...+|\alpha_N|$ and $D^\alpha u$ denote the distributional derivatives. The space $W^1L_\varphi(\Omega)$ is called the Musielak-Orlicz-Sobolev space. Let
$\overline{\varrho}_{\varphi,\Omega}(u)=\sum_{|\alpha|\leq 1}\varrho_{\varphi,\Omega}(D^\alpha u)$ and $\|u\|^1_{\varphi,\Omega}=\inf\{\lambda>0:\overline{\varrho}_{\varphi,\Omega}(\frac{u}{\lambda})\leq 1\}$ for $u\in W^1L_\varphi(\Omega)$.

These functionals are convex modular and a norm on $W^1L_\varphi(\Omega)$, respectively. Then the pair $(W^1L_\varphi(\Omega),\|u\|^1_{\varphi,\Omega})$ is a Banach space if φ satisfies the following condition (see [19]):

$$\text{There exists a constant} \quad c>0 \quad \text{such that} \quad \inf_{x\in\Omega}\varphi(x,1)>c.$$

The space $W^1L_\varphi(\Omega)$ is identified as a subspace of the product $\prod_{\alpha\leq 1}L_\varphi(\Omega)=\prod L_\varphi$. We denote by $\mathcal{D}(\Omega)$ the Schwartz space of infinitely smooth functions with compact support in Ω and by $\mathcal{D}(\overline{\Omega})$ the restriction of $\mathcal{D}(\mathbb{R})$ on Ω. The space $W^1_0L_\varphi(\Omega)$ is defined as the $\sigma(\prod L_\varphi,\prod E_\psi)$ closure of $\mathcal{D}(\Omega)$ in $W^1L_\varphi(\Omega)$ and the space $W^1_0E_\varphi(\Omega)$ as the(norm) closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^1L_\varphi(\Omega)$.

For two complementary Musielak-Orlicz functions φ and ψ, we have (see [11])
the Young inequality, $\quad st\leq\varphi(x,s)+\psi(x,t)$ for all $s,t\geq 0$, $x\in\Omega$,
the Hölder inequality, $\big|\int_\Omega u(x)v(x)dx\big|\leq\|u\|_{\varphi,\Omega}\||v|\|_{\psi,\Omega}$; for all $u\in L_\varphi(\Omega)$,$v\in L_\psi(\Omega)$.

We say that a sequence u_n converges to u for the modular convergence in $W^1L_\varphi(\Omega)$ (respectively, in $W^1_0L_\varphi(\Omega)$) if, for some $\lambda>0$,

$$\lim_{n\to\infty}\overline{\varrho}_{\varphi,\Omega}\big(\frac{u_n-u}{\lambda}\big)=0.$$

Let us define the following spaces of distributions:

$$W^{-1}L_\psi(\Omega)=\big\{f\in\mathcal{D}^{'}(\Omega):f=\sum_{\alpha\leq 1}(-1)^\alpha D^\alpha f_\alpha,\ \text{where}\ f_\alpha\in L_\psi(\Omega)\big\},$$

$$W^{-1}E_\psi(\Omega)=\big\{f\in\mathcal{D}^{'}(\Omega):f=\sum_{\alpha\leq 1}(-1)^\alpha D^\alpha f_\alpha,\ \text{where}\ f_\alpha\in E_\psi(\Omega)\big\}.$$

Lemma 2.1 *([5]) (Approximation result)* *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ and let φ and ψ be two complementary Musielak-Orlicz functions which satisfy the following conditions:*

- *there exists a constant $c>0$ such that $\inf_{x\in\Omega}\varphi(x,1)>c$,*
- *there exists a constant $A>0$ such that for all $x,y\in\Omega$ with $|x-y|\leq\frac{1}{2}$, we have*

$$\frac{\varphi(x,t)}{\varphi(y,t)}\leq|t|^{\left(\frac{A}{\log(\frac{1}{|x-y|})}\right)} \quad \textit{for all} \quad t\geq 1,$$

- $\int_K\varphi(x,\lambda)dx<\infty$, *for any constant $\lambda>0$ and for every compact $K\subset\Omega$.*
- *there exists a constant* $\quad C>0 \quad$ *such that* $\quad \psi(y,t)\leq C \quad$ *a.e. in* $\quad \Omega$.

Under these assumptions $\mathcal{D}(\Omega)$ is dense in $L_\varphi(\Omega)$ with respect to the modular topology, $\mathcal{D}(\Omega)$ is dense in $W^1_0L_\varphi(\Omega)$ for the modular convergence and $\mathcal{D}(\overline{\Omega})$ is dense in $W^1_0L_\varphi(\Omega)$ for the modular convergence. Consequently, the action of a distribution S in $W^{-1}L\psi$ on an element u of $W^1_0L_\varphi(\Omega)$ is well defined. It will be denoted by $<S,u>$.

Remark 2.2 The second condition in Lemma 2.1 coincides with an alternative log-Hölder continuity condition for the variable exponent p, namely, there exists $A > 0$ such that for x, y close enough and each $t \in \mathbb{R}^N$

$$|p(x) - p(y)| \leq \frac{A}{\log(\frac{1}{|x-y|})}.$$

2.3 Truncation operator

T_k, $k > 0$, denotes the truncation function at level k defined on $\mathbb{R}$ by $T_k(r) = \max(-k, \min(k, r))$. The following abstract lemmas will be applied to the truncation operators.

Lemma 2.2 *([12]) Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. Let φ be an Musielak-Orlicz function and let $u \in W_0^1 L_\varphi(\Omega)$ (respectively, $u \in W^1 E_\varphi(\Omega)$). Then $F(u) \in W^1 L_\varphi(\Omega)$ (respectively, $u \in W_0^1 E_\varphi(\Omega)$). Moreover, if the set of discontinuity points D of F' is finite, then*

$$\frac{\partial}{\partial x_i} F(u) = \begin{cases} F'(x) \frac{\partial u}{\partial x_i}, & a.e. \ in \ \{x \in \Omega; \ u(x) \notin D\}, \\ 0, & a.e. \ in \ \{x \in \Omega; \ u(x) \in D\}. \end{cases}$$

Lemma 2.3 *([12]) Suppose that Ω satisfies the segment property and let $u \in W_0^1 L_\varphi(\Omega)$. Then, there exists a sequence $u_n \in \mathcal{D}(\Omega)$ such that $u_n \to u$ for modular convergence in $W_0^1 L_\varphi(\Omega)$. Furthermore, if $u \in W_0^1 L_\varphi(\Omega) \cap L^\infty(\Omega)$, then $\|u_n\|_\infty \leq (N+1)\|u\|_\infty$.*

Let Ω be an open subset of $\mathbb{R}^N$ and let φ be a Musielak-Orlicz function satisfying the condition

$$\int_0^1 \frac{\varphi_x^{-1}(t)}{t^{\frac{N+1}{N}}} dt = \infty \quad \text{a.e.} \quad x \in \Omega,$$

and the conditions of Lemma 2.1. We may assume, without loss of generality, that

$$\int_0^1 \frac{\varphi_x^{-1}(t)}{t^{\frac{N+1}{N}}} dt < \infty \quad \text{a.e.} \quad x \in \Omega.$$

Define a function $\varphi^* : \Omega \times [0, \infty)$ by $\varphi^*(x, s) = \displaystyle\int_0^s \frac{\varphi_x^{-1}(t)}{t^{\frac{N+1}{N}}} dt$ $x \in \Omega$ and $s \in [0, \infty)$. φ^* is called the Sobolev conjugate function of φ (see [1] for the case of the Orlicz function).

Lemma 2.4 *([15]) Let $u_n, u \in L_\varphi(\Omega)$. If $u_n \to u$ with respect to the modular convergence, then $u_n \rightharpoonup u$ for $\sigma(L_\varphi(\Omega), L_\psi(\Omega))$.*

3 Technical Lemmas

Throughout this paper, we assume also that every Musielak-Orlicz function $\varphi(.,.)$ is decreasing in x in the following sense. For any $x \in \Omega$, let $\Omega_x = \{s \in \Omega / \|x\| \leq \|s\|\}$,

$$\begin{cases} \varphi(s,t) \leq \varphi(x,t) & \text{if} \quad s \in \Omega_x, \\ \varphi(s,t) \geq \varphi(x,t) & \text{if} \quad s \notin \Omega_x \end{cases} \tag{3}$$

for any $t \in \mathbb{R}$.

Lemma 3.1 *([6]) Under the assumptions of Lemma2.1, and by assuming that $\varphi(x,t)$ depends only on $N-1$ coordinate of x, there exists a constant $C_1 > 0$ which depends only on Ω such that*

$$\int_\Omega \varphi(x,|u|)dx \leq \int_\Omega \varphi(x, C_1|\nabla u|)dx. \tag{4}$$

Definition 3.1 Let φ be a Musielak-Orlicz function. We define the following set:

$$\mathcal{A}_\varphi = \left\{ \begin{matrix} \phi : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+ \quad \text{is a Musielak-Orlicz function such that} \\ \phi \prec\prec \varphi \quad \text{and} \quad \displaystyle\int_0^1 \phi(x, \beta H^{-1}(x, \frac{1}{r^{1-\frac{1}{N}}}))dr < \infty \quad \text{a.e. in } \Omega \end{matrix} \right\}$$

for all constant $\beta \geq 1$, where $H(x,r) = \dfrac{\varphi(x,r)}{r}$.

The following lemma generalizes Lemma 2 in [20].

Lemma 3.2 *Let Ω be an open subset of $\mathbb{R}^N$ with finite measure. Let φ be a Musielak-Orlicz function under assumption (3) and the assumptions of Lemma 2.1.*
For any $u \in W_0^1 L_\varphi(\Omega)$ such that $\displaystyle\int_\Omega \varphi(x,|\nabla u|)dx < \infty$, we have for all $x \in \Omega$,

$$-\mu'(t) \geq N C_N^{\frac{1}{N}} \mu(t)^{1-\frac{1}{N}} C\Big(x, \frac{-1}{C_N^{\frac{1}{N}} \mu(t)^{1-\frac{1}{N}}} \frac{d}{dt} \int_{\{|u|>t\}} \varphi(s, |\nabla u|)ds\Big) \tag{5}$$

for a.e. $t > 0$. Here μ is the distribution function of u, and the function $C(.,.)$ is defined by $C(x,t) = \frac{1}{H_x^{-1}(x,t)}$ with $H(x,t) = \frac{\varphi(x,t)}{t}$, C_N is the measure of the unit ball of $\mathbb{R}^N$, and $\mu(t) = meas\{|u| > t\}$.

Proof. By definition of the Musielak-Orlicz function, φ is an increasing convex function in t, then H is an increasing convex function in t, and $C(.,.)$ is a decreasing convex function in t.
Fix $x \in \Omega$. Jensen's inequality for a convex function gives

$$\begin{aligned} C\left(x, \frac{\displaystyle\int_{\{t<|u|\leq t+h\}} \varphi(s,|\nabla u|)\,ds}{\displaystyle\int_{\{t<|u|\leq t+h\}} |\nabla u|\,ds}\right) &= C\left(x, \frac{\displaystyle\int_{\{t<|u|\leq t+h\}} H(s,|\nabla u|)|\nabla u|\,ds}{\displaystyle\int_{\{t<|u|\leq t+h\}} |\nabla u|\,ds}\right) \\ &\leq \frac{\displaystyle\int_{\{t<|u|\leq t+h\}} C\Big(x, H(s,|\nabla u|)\Big)|\nabla u| ds}{\displaystyle\int_{\{t<|u|\leq t+h\}} |\nabla u|\,ds} \\ &\leq \frac{\displaystyle\int_{\{t<|u|\leq t+h\}\cap\Omega_x} C\Big(x, H(s,|\nabla u|)\Big)|\nabla u| ds}{\displaystyle\int_{\{t<|u|\leq t+h\}} |\nabla u|\,ds} \\ &\quad + \frac{\displaystyle\int_{\{t<|u|\leq t+h\}\cap(\Omega\setminus\Omega_x)} C\Big(x, H(s,|\nabla u|)\Big)|\nabla u|\,ds}{\displaystyle\int_{\{t<|u|\leq t+h\}} |\nabla u| ds}. \end{aligned}$$

By (3) and for all $t > 0$ we have
for $s \in \Omega_x$, $H(s, |\nabla u|) \leq H(x, |\nabla u|)$ and $C(x, H(s, |\nabla u|)) \leq C(x, H(x, |\nabla u|)) = \frac{1}{|\nabla u|}$,
for $s \in \Omega \backslash \Omega_x$, $H(x, |\nabla u|) \leq H(s, |\nabla u|)$ and $|\nabla u| \leq H_x^{-1}(H(s, |\nabla u|))$,
then $C(x, H(s, |\nabla u|)) = \frac{1}{H_x^{-1}(H(s,|\nabla u|))} \leq \frac{1}{|\nabla u|}$. Hence

$$C\Bigg(x, \frac{\displaystyle\int_{\{t<|u|\leq t+h\}} \varphi(s, |\nabla u|)\, ds}{\displaystyle\int_{\{t<|u|\leq t+h\}} |\nabla u|\, ds}\Bigg) \leq \frac{-\mu(t+h) + \mu(t)}{\displaystyle\int_{\{t<|u|\leq t+h\}} |\nabla u|\, ds},$$

letting $h \to 0$, we have

$$C\Bigg(x, \frac{(\frac{d}{dt})\displaystyle\int_{\{|u|>t\}} \varphi(s, |\nabla u|)\, ds}{(\frac{d}{dt})\displaystyle\int_{\{|u|>t\}} |\nabla u|\, ds}\Bigg) \leq \frac{\mu^{'}(t)}{(\frac{d}{dt})\displaystyle\int_{\{|u|>t\}} |\nabla u|\, ds} \tag{6}$$

for all $t > 0$.

On the other hand we can follow [16] to prove that

$$-\frac{d}{dt}\int_{\{|u|>t\}} |\nabla u|\, dx \geq NC_N^{\frac{1}{N}}\mu(t)^{1-\frac{1}{N}} \tag{7}$$

for a.e. $t > 0$. Finally, combining (6), (7) and the monotony of $C(.,.)$ we get (5).

Lemma 3.3 *Let φ be a Musielak-Orlicz function under assumption* (3) *and the assumptions of Lemma 2.1 and $\phi \in \mathcal{A}_\varphi$ with $\phi \sim \varphi$, there exists a constant $\beta \geq 1$ such that*

$$\frac{d}{dt}\int_{\{|u|>t\}} \phi(s, |\nabla u|)\, ds \leq -\mu^{'}(t)\phi\Bigg(x, \beta H_x^{-1}\Big(\frac{1}{NC_N^{\frac{1}{N}}\mu(t)^{1-\frac{1}{N}}}\frac{d}{dt}\int_{\{|u|>t\}} \varphi(s, |\nabla u|)\, ds\Big)\Bigg)$$

for each $x \in \Omega$ and for any $u \in W_0^1L_\varphi(\Omega)$ such that $\displaystyle\int_\Omega \varphi(x, |\nabla u|)dx < \infty$.

Proof. For $x \in \Omega$, let $C(x,t) = \dfrac{1}{H_x^{-1}(x,t)}$, then $C(x,t) = \dfrac{t}{\varphi \circ H_x^{-1}(x,t)}$.
By (5), we have

$$-\mu^{'}(t) \geq NC_N^{\frac{1}{N}}\mu(t)^{1-\frac{1}{N}} C\Bigg(x, \frac{-1}{NC_N^{\frac{1}{N}}\mu(t)^{1-\frac{1}{N}}}\frac{d}{dt}\int_{\{|u|>t\}} \varphi(s, |\nabla u|)\, ds\Bigg),$$

then

$$\begin{aligned}&-\mu^{'}(t)\varphi \circ H_x^{-1}\Bigg(\frac{-1}{NC_N^{\frac{1}{N}}\mu(t)^{1-\frac{1}{N}}}\frac{d}{dt}\int_{\{|u|>t\}} \varphi(s, |\nabla u|)\, ds\Bigg)\\ &\qquad \geq NC_N^{\frac{1}{N}}\mu(t)^{1-\frac{1}{N}}\Big(-\frac{1}{NC_N^{\frac{1}{N}}\mu(t)^{1-\frac{1}{N}}}\frac{d}{dt}\int_{\{|u|>t\}} \varphi(s, |\nabla u|)\, ds\Big)\end{aligned}$$

$$-\mu^{'}(t)\varphi\circ H_x^{-1}\Big(\frac{-1}{NC_N^{\frac{1}{N}}\mu(t)^{1-\frac{1}{N}}}\frac{d}{dt}\int_{\{|u|>t\}}\varphi(s,|\nabla u|)ds\Big)\geq-\frac{d}{dt}\int_{\{|u|>t\}}\varphi(s,|\nabla u|)\,ds,$$

and also

$$\frac{1}{\mu^{'}(t)}\frac{d}{dt}\int_{\{|u|>t\}}\varphi(s,|\nabla u|)\,ds\leq\varphi\circ H_x^{-1}\left(\frac{-1}{NC_N^{\frac{1}{N}}\mu(t)^{1-\frac{1}{N}}}\frac{d}{dt}\int_{\{|u|>t\}}\varphi(s,|\nabla u|)\,ds\right),$$

using the monotony of the function φ_x^{-1}, we obtain

$$\varphi_x^{-1}\Big(\frac{1}{\mu^{'}(t)}\frac{d}{dt}\int_{\{|u|>t\}}\varphi(s,|\nabla u|)\,ds\Big)\leq H_x^{-1}\Big(\frac{-1}{NC_N^{\frac{1}{N}}\mu(t)^{1-\frac{1}{N}}}\frac{d}{dt}\int_{\{|u|>t\}}\varphi(s,|\nabla u|)\,ds\Big).$$

Let $\phi\in\mathcal{A}_\varphi$ and let $D(x,t)=\varphi(x,\phi_x^{-1}(t))$, then D is convex and by Jensen's inequality we have

$$D\Big(x,\frac{\int_{\{t<|u|<t+h\}}\phi(s,|\nabla u|)\,ds}{-\mu(t+h)+\mu(t)}\Big)\leq\frac{\int_{\{t<|u|<t+h\}}D\Big(x,\phi(s,|\nabla u|)\Big)\,ds}{-\mu(t+h)+\mu(t)}.$$

Since $\phi\sim\varphi$, there exists a constant $\lambda>0$ such that $\varphi(x,t)\leq\lambda\phi(x,t)$. Then for every $\phi\in\mathcal{A}_\varphi$ with $\lambda\leq 1$ and by the monotony of the functions φ_x and ϕ_x^{-1} for any x and s in Ω, we have

$$D\left(x,\phi(s,|\nabla u|)\right)=\varphi\Big(x,\phi_x^{-1}(\phi(s,|\nabla u|))\Big)\leq\phi(s,|\nabla u|),$$

and by Remark 2.1, there exists $\beta>0$ such that $D\Big(x,\phi(s,|\nabla u|)\Big)\leq\beta\varphi(s,|\nabla u|)$, then

$$D\left(x,\frac{\int_{\{t<|u|<t+h\}}\phi(s,|\nabla u|)\,ds}{-\mu(t+h)+\mu(t)}\right)\leq\frac{\beta\int_{\{t<|u|<t+h\}}\varphi\Big(s,|\nabla u|\Big)\,ds}{-\mu(t+h)+\mu(t)},$$

using the definition of D(.,.) and the monotony of φ_x^{-1} we have

$$phi_x^{-1}\left(x,\frac{1}{\mu^{'}(t)}\frac{d}{dt}\int_{\{|u|>t\}}\phi(s,|\nabla u|)\,ds\right)\leq\beta\varphi_x^{-1}\left(x,\frac{1}{\mu^{'}(t)}\frac{d}{dt}\int_{\{|u|>t\}}\varphi(s,|\nabla u|)\,ds\right),$$

$$\leq\beta H_x^{-1}\left(\frac{-1}{C_N^{\frac{1}{N}}\mu(t)^{1-\frac{1}{N}}}\frac{d}{dt}\int_{\{|u|>t\}}\varphi(s,|\nabla u|)\,ds\right),$$

which gives our result.

4 Essential Assumptions

Let φ and γ be two Musielak-Orlicz functions such that φ and its complementary ψ satisfy the previous conditions and $\gamma\prec\prec\varphi$.

$A:D(A)\subset W_0^1L_\varphi(\Omega)\to W^{-1}L_\psi(\Omega)$ is defined by $A(u)=-\text{div}(a(x,u,\nabla u))$, where $a:\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}^N$ is a Carathéodory function such that for a.e. $x\in\Omega$ and for all $s\in\mathbb{R}$, $\xi,\xi^*\in\mathbb{R}^N$, $\xi\neq\xi^*$.

$$|a(x,s,\xi)|\leq\beta(c(x)+\psi_x^{-1}(\gamma(x,\nu_1|s|))+\psi_x^{-1}(\varphi(x,\nu_2|\xi|))),\beta>0,c(x)\in E_\psi(\Omega),\quad(8)$$

$$(a(x,s,\xi)-a(x,s,\xi^*))(\xi-\xi^*)>0,\quad(9)$$

$$a(x, s, \xi).\xi \geq \alpha\varphi(x, |\xi|), \tag{10}$$

$K : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function such that

$$|K(x, s, \xi)| \leq b(x) + \rho(s)\varphi(x, |\xi|), \tag{11}$$

$\rho : \mathbb{R} \to \mathbb{R}^+$ is a continuous positive function which belongs to $L^1(\mathbb{R})$ and b(x) belongs to $L^1(\Omega)$.

$$\mu \in \mathcal{M}_b(\Omega), \tag{12}$$

assume that there exists $\phi \in \mathcal{A}_\varphi$ such that

$$\phi \circ H^{-1} \text{ is a Musielak-Orlicz function.} \tag{13}$$

5 Main Results

Let Ω be an open bounded subset of $\mathbb{R}^N$ $(N \geq 2)$, and let φ and ψ be two complementary Musielak-Orlicz functions.
Define the set $\mathcal{T}_0^{1,\varphi}(\Omega) = \{u : \Omega \mapsto \mathbb{R} \text{ is measurable and } T_k(u) \in D(A)\}$.

Theorem 5.1 *Assume that (8) − (12) hold true with $\mathcal{A}_\varphi \neq \emptyset$. Then there exists at least one solution of the following problem:*

$$\begin{cases} u \in \mathcal{T}_0^{1,\varphi}(\Omega) \cap W_0^1 L_\phi(\Omega), \quad \forall \phi \in \mathcal{A}_\varphi, \\ < A(u), v > + \displaystyle\int_\Omega K(x, u, \nabla u) v dx =< \mu, v >, \quad \forall v \in \mathcal{D}(\Omega). \end{cases} \tag{14}$$

Example 5.1 We give an example of equations to which the present result can be applied.

1. We give an example of the Musielak-Orlicz-functions φ for which the set $\mathcal{A}_\varphi$ is not empty. Let $a(.), b(.)$ be two functions in $L^\infty(\Omega)$ such that $a(.), b(.)$ are decreasing strict positive and there exist two constants $\lambda_1 > 0, \lambda_2 > 0$ such that $\lambda_1 \leq \frac{a(x)}{b(x)} \leq \lambda_2$. Take now $\varphi(x, t) = a(x)|t|^p$ and $\phi(x, t) = b(x)|t|^p$ such that $p > N$, then $\phi \in \mathcal{A}_\varphi$.

2. Let us take the functions mentioned above and consider the following problem:

$$\begin{cases} \operatorname{div}(a(x)|\nabla u|^{p-2}\nabla u) + b(x) + \rho(u)\varphi(x, |\nabla u|) = \mu, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

   Here $a(x, u, \nabla u) = a(x)|\nabla u|^{p-2}\nabla u$ satifies the hypotheses (8)-(10), $K(x, u, \nabla u) = b(x) + \rho(u)\varphi(x, |\nabla u|)$, where $\rho : \mathbb{R} \to \mathbb{R}^+$ is a continuous positive function which belongs to $L^1(\mathbb{R})$ and $\mu \in \mathcal{M}_b(\Omega)$.

Proof of Theorem 5.1. The proof is divided into four steps.

Step 1: Existence of weak solutions for approximate problems.
We consider the following approximate equation for any $n \in \mathbb{N}$:

$$\int_\Omega [a(x,u,\nabla u)\nabla v + K_n(x,u,\nabla u)v]dx = \int_\Omega \mu_n v\, dx, \quad \forall v \in W_0^1 L_\varphi(\Omega), \tag{15}$$

where $K_n(x,u,\nabla u) = \dfrac{K(x,u,\nabla u)}{1+\frac{1}{n}|K(x,u,\nabla u)|}$, and $(\mu_n)_n \in \mathcal{D}(\Omega)$ is a sequence such that

$$\mu_n \to \mu \quad \text{in the sense of the distributions.} \tag{16}$$

We will prove that, for every n, there exists at least one bounded solution u_n of (15) with $u_n \in W_0^1 E_\varphi(\Omega)$.

Proposition 5.1 *(See [13]) Let φ and ψ be two complementary Musielak-Orlicz functions satisfying the conditions of Lemma 2.1, assume that* (8)-(12) *hold, then, for any $n \in \mathbb{N}^*$, there exists at least one solution $u_n \in W_0^1 E_\varphi(\Omega)$ of (15).*

Step 2: Consider the following approximate problems:

$$\begin{cases} u_n \in \mathcal{T}_0^{1,\varphi}(\Omega) \cap W_0^1 E_\varphi(\Omega) \\ < A(u_n), v > + \displaystyle\int_\Omega K_n(x,u_n,\nabla u_n) v dx =< \mu_n, v >, \quad \forall v \in W_0^1 L_\varphi(\Omega). \end{cases} \tag{17}$$

By proposition, there exists at least one solution u_n of (17).

Lemma 5.1 *Let u_n be a solution of the approximate problem* (15)*, then*

1. *for all $k > 0$, there exists a constant C (which does not depend on n and k) such that*
$$\int_\Omega a(x, T_k(u_n), \nabla T_k(u_n))\nabla T_k(u_n) dx \le C_2 k, \tag{18}$$
and
$$\int_\Omega \varphi(x, |\nabla T_k(u_n)|) dx \le C_3 k. \tag{19}$$

2. *There exists a measurable function u such that*
$$u_n \to u \quad a.e. \ in \quad \Omega. \tag{20}$$

3. $$a(x, T_k(u_n), \nabla T_k(u_n)) \rightharpoonup \varpi_k \ \textit{weakly in} \ \big(L_\psi(\Omega)\big)^N \ \textit{for} \ \sigma(\Pi L_\psi, \Pi E_\varphi). \tag{21}$$

Proof of Lemma 5.1. (1) Let $v_0 \in W_0^1 L_\varphi(\Omega) \cap L^\infty(\Omega)$ with $v_0 \ge 0$.
On the one hand, taking $\exp(G(u_n))v_0$ as a test function in (15), where $G(s) = \displaystyle\int_0^s \frac{1}{\alpha}\rho(r)dr$, we obtain

$$\int_\Omega a(x,u_n,\nabla u_n)\exp(G(u_n))\frac{\rho(u_n)}{\alpha}\nabla u_n v_0\, dx + \int_\Omega a(x,u_n,\nabla u_n)\exp(G(u_n))\nabla v_0 dx$$

$$+\int_\Omega K_n(x,u_n,\nabla u_n)\exp(G(u_n))v_0dx = \int_\Omega \mu_n \exp(G(u_n))v_0dx,$$

by (10) and (11) we simplify by the term $\int_\Omega \rho(u_n)\varphi(x,|\nabla u_n|)v_0\,dx$ and we have

$$\int_\Omega a(x,u_n,\nabla u_n)\exp(G(u_n))\nabla v_0\,dx \le \int_\Omega \mu_n \exp(G(u_n))v_0dx + \int_\Omega b(x)\exp(G(u_n))v_0\,dx. \tag{22}$$

On the other hand, taking $\exp(-G(u_n))v_0$ as a test function in (15), we deduce also

$$\int_\Omega a(x,u_n,\nabla u_n)\exp(-G(u_n))\nabla v_0dx + \int_\Omega b(x)\exp(-G(u_n))v_0\,dx \ge \int_\Omega \mu_n \exp(-G(u_n))v_0dx. \tag{23}$$

By choosing $v_0 = T_k(u_n)^+$ in (22), we otain

$$\int_\Omega a(x,u_n,\nabla u_n)\exp(G(u_n)))\nabla T_k(u_n)^+dx$$

$$\le \int_\Omega \mu_n \exp(G(u_n)))T_k(u_n)^+dx + \int_\Omega b(x)\exp(G(u_n)))T_k(u_n)^+\,dx.$$

Since $\rho \in L^1(\mathbb{R})$, we see that $G(-\infty) \le G(s) \le G(+\infty)$ and $|G(\pm\infty)| \le \frac{1}{\alpha}\|\rho\|_{L^1(\mathbb{R})}$, then we have

$$\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)^+\,dx \le \exp\big(\frac{\|\rho\|_{L^1(\mathbb{R})}}{\alpha}\big)k[\|\mu\|_{\mathcal{M}_b(\Omega)} + \|b\|_{L^1(\Omega)}] = kC_4,$$

and using (10) we get

$$\int_\Omega \varphi(x,|\nabla T_k(u_n)^+|)dx \le kC_5.$$

Choosing again $v_0 = T_k(u_n)^-$ in (23) we get

$$-\int_{\{-k\le u_n\le 0\}} a(x,u_n,\nabla u_n)\exp(-G(u_n)))\nabla u_n\,dx + \int_\Omega b(x)\exp(-G(u_n)))T_k(u_n)^-dx$$

$$\ge \int_\Omega \mu_n \exp(-G(u_n)))T_k(u_n)^-dx.$$

Similarly we obtain

$$\int_{\{-k\le u_n\le 0\}} a(x,T_k(u_n),\nabla T_k(u_n))\nabla u_n\,dx \le \exp\big(\frac{\|\rho\|_{L^1(\mathbb{R})}}{\alpha}\big)k[\|\mu\|_{\mathcal{M}_b(\Omega)} + \|b\|_{L^1(\Omega)}]$$

$$= kC_4.$$

and by (10) we have

$$\int_\Omega \varphi(x,|\nabla T_k(u_n)^-|)dx \le kC_6.$$

We deduce respectively the results (19) and (18).

(2) Using (4) we have

$$\inf_{x\in\Omega}\varphi(x,\frac{k}{C_1})\mathrm{meas}\{|u_n|>k\} \le \int_{\{|u_n|>k\}} \varphi(x,\frac{|T_k(u_n)|}{C_1})\,dx$$

$$\le \int_\Omega \varphi(x,|\nabla T_k(u_n)|)dx \le kC_7.$$

Then

$$\text{meas}\{|u_n| > k\} \leq \frac{kC_7}{\inf_{x\in\Omega}\varphi(x, \frac{k}{C_1})},$$

for all n and for all k.

Assume that there exists a positive function M such that $\lim_{t\to\infty}\frac{M(t)}{t} = +\infty$ and $M(t) \leq ess\inf_{x\in\Omega}\varphi(x,t)$, $\forall t \geq 0$. Thus, we get

$$\lim_{k\to\infty} \text{meas}\{|u_n| > k\} = 0.$$

By the property (1) of Lemma 5.1, we deduce that $T_k(u_n)$ is bounded in $W_0^1L_\varphi(\Omega)$ and then there exists some $\tau_k \in W_0^1L_\varphi(\Omega)$ such that

$$T_k(u_n) \rightharpoonup \tau_k \quad \text{weakly in} \quad W_0^1L_\varphi(\Omega) \quad \text{for} \quad \sigma(\Pi L_\varphi, \Pi E_\psi),$$

$$\text{strongly in} \quad E_\varphi(\Omega) \quad \text{and a.e. in} \quad \Omega,$$

and by (2) of Lemma 5.1, the sequence $(u_n)_n$ converges almost everywhere to some measurable function u. Then we have $T_k(u_n) \rightharpoonup T_k(u)$ weakly in $W_0^1L_\varphi(\Omega)$ for $\sigma(\Pi L_\varphi, \Pi E_\psi)$, strongly in $E_\varphi(\Omega)$ and a.e. in Ω.

(3) We shall prove that $\{a(x, T_k(u_n), \nabla T_k(u_n))\}_n$ is bounded in $\big(L_\psi(\Omega)\big)^N$ for all $k > 0$. Let $w \in (E_\varphi(\Omega))^N$ be arbitrary. By (9) we have

$$(a(x, u_n, \nabla u_n) - a(x, u_n, w))(\nabla u_n - w) \geq 0.$$

Then

$$\int_{\{|u_n|\leq k\}} a(x, u_n, \nabla u_n) w dx \leq \int_{\{|u_n|\leq k\}} a(x, u_n, \nabla u_n)\nabla u_n dx + \int_{\{|u_n|\leq k\}} a(x, u_n, w)(w - \nabla u_n) dx.$$

By (8) and according to Remark 2.1 there exists $k' > 0$ such that $\gamma(x, \nu_1 k) \leq k'\varphi(x, 1)$ and for $\lambda > 0$ is large enough

$$\int_{\{|u_n|\leq k\}} \psi(\frac{a(x, u_n, \frac{w}{\nu_2})}{3\beta}) dx \leq \frac{1}{3}[\int_\Omega \psi(c(x))dx + \int_\Omega k'\varphi(x, 1)dx + \int_\Omega \varphi(x, w)dx] \leq C_7. \tag{24}$$

Thus $\{a(x, T_k(u_n), \frac{w}{\nu_2})\}$ is bounded in $(L_\psi(\Omega))^N$, by (24),(18) and in view of the Banach-Steinhaus theorem, the sequence $\{a(x, T_k(u_n), \nabla T_k(u_n))\}$ remains bounded in $(L_\psi(\Omega))^N$ and for a subsequence

$$a(x, T_k(u_n), \nabla T_k(u_n)) \rightharpoonup \varpi_k \quad \text{weakly in} \quad \big(L_\psi(\Omega)\big)^N \quad \text{for} \quad \sigma(\Pi L_\psi, \Pi E_\varphi).$$

Step 3: Almost everywhere convergence of the gradients.

To have that the gradient converges almost everywhere, we need to prove the following proposition.

Proposition 5.2 *Let $\{u_n\}_n$ be a solution of the approximate problem(15), then*

1.

$$\lim_{m\to\infty}\limsup_{n\to\infty}\int_{\{-(m+1)\leq u_n\leq -m\}} a(x, u_n, \nabla u_n)\nabla u_n\, dx = 0; \tag{25}$$

2. for a subsequence as $n \to \infty$

$$\nabla u_n \to \nabla u \quad a.e.\ in \quad \Omega. \tag{26}$$

Proof. (1) Take the function $v_0 = T_1(u_n - T_m(u_n))^-$ in (23), this function is admissible since $v_0 \in W_0^1 L_\varphi(\Omega) \cap L^\infty(\Omega)$, and $v_0 \geq 0$, then we have

$$-\int_\Omega a(x,u_n,\nabla u_n)\exp(-G(u_n))\nabla T_1(u_n - T_m(u_n))^- \,dx$$
$$\leq \int_\Omega b(x)\exp(-G(u_n))T_1(u_n - T_m(u_n))^- dx.$$

Since μ is nonnegative, we get

$$\int_{\{-(m+1)\leq u_n \leq -m\}} a(x,u_n,\nabla u_n)\exp(-G(u_n)))\nabla u_n \,dx$$
$$\leq \int_\Omega b(x)\exp(-G(u_n)))T_1(u_n - T_m(u_n))^- dx$$
$$\leq \exp(\frac{\|\rho\|_{L^1_{\mathbb{R}}}}{\alpha})\int_\Omega |b(x)|T_1(u_n - T_m(u_n))^- \,dx.$$

By Lebesgue's theorem, we conclude the result (25).

(2) To show that $\nabla u_n \to \nabla u$ a.e. in Ω is true, simply adapt the proof from [3] and follow the same steps by taking $\Phi = 0$.

Step 4: Equi-integrability of the nonlinearity sequence.

We shall prove that

$$K_n(x,u_n,\nabla u_n) \to K(x,u,\nabla u) \quad \text{strongly in} \quad L^1(\Omega). \tag{27}$$

Consider $v_0 = \int_0^{u_n} \rho(s)\chi_{\{s>h\}}dx$ in (22), we get

$$\int_\Omega a(x,u_n,\nabla u_n)\exp(G(u_n))\nabla v_0 dx \leq \int_\Omega \mu_n \exp(G(u_n))v_0 dx + \int_\Omega b(x)\exp(G(u_n))v_0 dx.$$

Then using (10) and (11) we have

$$\alpha\int_{\{u_n>h\}} \rho(u_n)\varphi(x,\nabla u_n)dx \leq (\int_h^{+\infty}\rho(s)dx)\exp\big(\frac{\|\rho\|_{L^1(\mathbb{R})}}{\alpha}\big)[\|\mu\|_{\mathcal{M}_b(\Omega)} + \|b\|_{L^1(\Omega)}]$$
$$\int_{\{u_n>h\}} \rho(u_n)\varphi(x,\nabla u_n)dx \leq \frac{C_4}{\alpha}(\int_h^{+\infty}\rho(s)dx).$$

Since $\rho \in L^1(\mathbb{R})$, we get

$$\lim_{h\to+\infty}\sup_{n\in\mathbb{N}}\int_{\{u_n>h\}} \rho(u_n)\varphi(x,\nabla u_n)dx = 0.$$

Similarly, let $v_0 = \int_{u_n}^0 \rho(s)\chi_{\{s<-h\}}dx$ in (23), we have also

$$\lim_{h\to+\infty}\sup_{n\in\mathbb{N}}\int_{\{u_n<-h\}} \rho(u_n)\varphi(x,\nabla u_n)dx = 0.$$

We conclude that
$$\lim_{h\to+\infty}\sup_{n\in\mathbb{N}}\int_{\{|u_n|>h\}}\rho(u_n)\varphi(x,\nabla u_n)dx=0. \tag{28}$$
Let $D\subset\Omega$, then
$$\int_D\rho(u_n)\varphi(x,\nabla u_n)dx\leq\max_{\{|u_n|\leq h\}}(\rho(x))\int_{D\cap\{|u_n|\leq h\}}\varphi(x,\nabla u_n)dx$$
$$+\int_{D\cap\{|u_n|>h\}}\rho(u_n)\varphi(x,\nabla u_n)dx.$$
Consequently, $\rho(u_n)\varphi(x,\nabla u_n)$ is equi-integrable, and since $\rho(u_n)\varphi(x,\nabla u_n)$ converges to $\rho(u)\varphi(x,\nabla u)$ strongly in $L^1(\mathbb{R})$, we get our result.

Step 5: We show that u **satisfies** (14)**.**

- $\{u_n\}$ is bounded $W_0^1L_\phi(\Omega)$ and converges to u strongly in $L_\phi(\Omega)$, where $\phi\in\mathcal{A}_\varphi$.

Firstly, we can take $T_\epsilon(u_n-T_t(u_n)),\epsilon>0,t>0$ as a test function in (17), from (11) and (28) we have
$$\int_{\{t\leq|u_n|\leq t+\epsilon\}}a(x,u_n,\nabla u_n)\nabla u_n dx\leq\epsilon C_{10}.$$
The constant C_{10} is independent of n,ϵ and t, then
$$\frac{1}{\epsilon}\int_{\{t\leq|u_n|\leq t+\epsilon\}}\varphi(x,\nabla u_n)dx\leq\frac{C_{10}}{\alpha}.$$
Let now $\epsilon\to0$, we have
$$-\frac{d}{dt}\int_{\{t\leq|u_n|\}}\varphi(x,\nabla u_n)dx\leq\frac{C_{10}}{\alpha}. \tag{29}$$

Secondly, let $\phi\in\mathcal{A}_\varphi$ and $\phi\sim\varphi$. Using Lemma 3.2, Lemma3.3, the equation (29) and the same techniques as in [10], we deduce that ∇u_n is bounded in $L_\phi(\Omega)$ for each $\phi\in\mathcal{A}_\varphi$, then u_n is bounded in $W_0^1L_\phi(\Omega)$ for each $\phi\in\mathcal{A}_\varphi$.

- $a(x,u_n,\nabla u_n)\rightharpoonup a(x,u,\nabla u)$ weakly for $\sigma(\Pi L_{\phi\circ H^{-1}},\Pi E_{\varpi})$, where ϖ and $\phi\circ H^{-1}$ are two complementary Musielak-Orlicz functions.
  The first time, using (8) and Remark 2.1, we have
$$\int_\Omega\phi\circ H_x^{-1}\big(\frac{|a(x,u_n,\nabla u_n)|}{6\beta}\big)dx\leq\int_\Omega\phi\circ H_x^{-1}\big(\frac{1}{6}[c(x)+k(\nu_1)\psi_x^{-1}(\varphi(x,|u_n|))$$
$$+\psi_x^{-1}(\varphi(x,\nu_2|\nabla u_n|))]\big)dx.$$
  Since $\phi\circ H_x^{-1}$ is a Musielak-Orlicz function, we get
$$\int_\Omega\phi\circ H_x^{-1}\big(\frac{|a(x,u_n,\nabla u_n)|}{6\beta}\big)dx\leq\frac{1}{3}\int_\Omega[\phi\circ H_x^{-1}\big(\frac{1}{2}(c(x))$$
$$+\phi\circ H_x^{-1}\big(\frac{1}{2}(k(\nu_1)\psi_x^{-1}(\varphi(x,|u_n|)))+\phi\circ H_x^{-1}\big(\frac{1}{2}\psi_x^{-1}(\varphi(x,\nu_2|\nabla u_n|)))]dx. \tag{30}$$

On the other hand, due to the definition of Musielak-Orlicz function, we can easily deduce

$$\frac{1}{2}\psi_x^{-1}(\varphi(x,t)) \leq \frac{\varphi(x,t)}{t},$$

by definition of H we have

$$\varphi(x, H_x^{-1}(\frac{t}{2})) \leq \psi(x,t),$$

and hence, by Remark 2.1,

$$\phi \circ H_x^{-1}\big(\frac{1}{2}c(x)) \leq k_1\varphi(x, H_x^{-1}(\frac{c(x)}{2})) \leq k_1\psi(x, c(x)), \tag{31}$$

also we have

$$\phi \circ H^{-1}\big(\frac{1}{2}\psi_x^{-1}(k(\nu_1)\varphi(x,|u_n|))) \leq \phi \circ H_x^{-1}\big(\frac{1}{2}\psi_x^{-1}\varphi(x, k_2|u_n|))$$

$$\leq \phi \circ H_x^{-1}\big(\frac{\varphi(x, k_2|\nabla u_n|)}{k_2|\nabla u_n|}),$$

where $k_2 = \max(1, k(\nu_1))$, then

$$\phi \circ H^{-1}\big(\frac{1}{2}\psi_x^{-1}(k(\nu_1)\varphi(x,|u_n|))) \leq k_3\varphi(x,|u_n|), \tag{32}$$

and

$$\phi \circ H_x^{-1}\big(\psi_x^{-1}(\frac{1}{2}\varphi(x,\nu_2|\nabla u_n|))) \leq \phi \circ H_x^{-1}\big(\frac{\varphi(x,\nu_2|\nabla u_n|)}{\nu_2|\nabla u_n|)}).$$

$$= \phi(x, \nu_2|\nabla u_n|).$$

Using Remark 2.1 we get

$$\phi \circ H_x^{-1}\big(\psi_x^{-1}(\frac{1}{2}\varphi(x,\nu_2|\nabla u_n|))) \leq k_4\varphi(x,|\nabla u_n|), \tag{33}$$

applying (31), (32) and (33) in (30) we obtain

$$\int_\Omega \frac{1}{3}\phi \circ H_x^{-1}\big(\frac{|a(x,u_n,\nabla u_n)|}{6\beta}\big)dx$$

$$\leq \int_\Omega [k_1\psi(x,c(x))dx + k_3\varphi(x,|u_n|) + k_4\varphi(x,|\nabla u_n|)]dx < C_{11}.$$

Consequently, $a(x,u_n,\nabla u_n) \rightharpoonup a(x,u,\nabla u)$ weakly for $\sigma(\Pi L_{\phi\circ H^{-1}}, \Pi E_{\varpi})$.

- Take now $v \in \mathcal{D}(\Omega)$ as a test function in approximate equation (15), one has

$$\int_\Omega a(x,u_n,\nabla u_n)v dx + \int_\Omega K(x,u_n,\nabla u_n)v dx = \int_\Omega \mu_n v dx,$$

since we have $u_n \to u$ strongly in $(E_\kappa(\Omega))^N$, for every $\kappa \prec\prec \phi$, $\forall \phi \in \mathcal{A}_\varphi$.

Using (27) and (16) we can pass to the limit as $n \to +\infty$ to end the proof of Theorem 5.1.

References

[1] R.A. Adams. *Sobolev Spaces*. New York, Academic Press, 1975.

[2] A. Aberqi, J. Bennouna and M. Hammoumi. Existence Result for Nonlinear Degenerated Parabolic Systems. *Nonlinear Dynamics and Systems Theory.* **17**(3) (2017) 217–229.

[3] A. Aberqi, J. Bennouna and M. Elmassoudi. Nonlinear elliptic equations with measure data in Musielak-Orlicz spaces. *Gulf Journal of Mathematics.* **6** (4) (2018) 79–100.

[4] T. Ahmedatt, A. Aberqi, A. Touzani and C. Yazough. On some nonlinear hyperbolic $p(x,t)$-Laplacian equations. *J. Appl. Anal.* **24**(1) (2018) 55–69.

[5] M. Ait Khellou and A. Benkirane. Elliptic inequalities with L^1 data in Musielak-Orlicz spaces. *Monatsh Math.* **183** (1) (2017) 1–33.

[6] M. Ait Khellou, A. Benkirane and S. M. Douiri. Strongly non-linear elliptic problems in Musielak spaces with L^1 data. *Nonlinear Studies.* **23** (3) (2016) 491–510.

[7] M. Ait Kellou, A. Benkirane and S. M. Douiri. An inequality of type Poincaré in Musielak spaces and applications to some nonlinear elliptic problems with L^1-data. *Complex Variables and Elliptic Equations.* **60** (2015) 1217–1242.

[8] Y. Akdim, N. El gorch and M. Mekkour. Existence of renormalized solutions for $p(x)$-parabolic equations with three unbounded non-linearities. *Bol. Soc. Parana. Mat.* **34** (1) (2016) 225–252.

[9] S. Antontsev and S. Shmarev. *Evolution PDEs with Nonstandard Growth Conditions, Existence, Uniqueness, Localization, Blow-up.* Atlantis Studies in Differential Equations, v. 4. Series editor Michel Chipot. Zürich, Switzerland, 2015.

[10] A. Benkirane and J. Bennouna. Existence of entropy solutions for some nonlinear problems in Orlicz spaces. *Abstract and Applied Analysis.* **7** (2) (2002) 85–102.

[11] A. Benkirane and M. Sidi El Vally. Some approximation properties in Musielak-Orlicz-Sobolev spaces. *Thai. J. Math.* (2012) 371-381.

[12] A. Benkirane and M. Sidi El Vally. Variational inequalities in Musielak-Orlicz-Sobolev spaces. *Bull. Belg. Math. Soc.* Simon Stevin, (2014) 787–811.

[13] G. Dong and X. Fang. Existence results for some nonlinear elliptic equations with measure data in Orlicz-Sobolev spaces. *Boundary Value Problems.* **2015** (2015) 18.

[14] A. Elmahi and D. Meskine. Existence of solutions for elliptic equations having natural growth terms in Orlicz spaces. *Abstr. Appl. Anal.* **12** (2004) 1031–1045.

[15] M. Elmassoudi, A. Aberqi and J. Bennouna. Existence of Entropy Solutions in Musielak-Orlicz Spaces Via a Sequence of Penalized Equations. *Bol. Soc. Paran. Mat.* **38** (6) (2020) 203–238.

[16] W. Fleming and R. Rishel. An integral formula for total gradient variation. *Arch. Math.* **11** (1960).

[17] J.P. Gossez. A strongly nonlinear elliptic problem in Orlicz Sobolev spaces. *Proc. Am. Math. Soc. Symp. Pure Math.* **45** (1986) 455–462.

[18] M. C. Hassib, Y. Akdim, A. Benkirane and N. Aissaoui. Capacity, Theorem of H. Brezis and F.E. Browder Type in Musielak Orlicz Sobolev Spaces and Application. *Nonlinear Dynamics and Systems Theory.* **17** (2) (2017) 175–192.

[19] J. Musielak. *Modular Spaces and Orlicz Spaces. Lecture Notes in Math.* 1983.

[20] G. Talenti. Nonlinear elliptic equations, Rearrangements of functions and Orlicz Spaces. *Annali di Matematica.* (1976) 159–184.