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Kalman Filter Estimation of Identified Reduced Model Using Balanced Truncation: a Case Study of the Bengawan Solo River

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Abstract: In this paper, we compare the estimation results for the reduced model and original model of water level in a river. First, we compute a reduced model from the original model using the balanced truncation method, then we estimate the reduced model using the Kalman filter. Since the orders of the state variables in the reduced model and original model are different, we cannot compare them directly. Therefore, we need an identification of the state variables in the reduced model such that we can determine the corresponding state variables in the original model or the real data. The selected river flow model is the Bengawan Solo river in Indonesia. The Bengawan Solo river is the longest river in Indonesia and often causes floods in the area around the river. With the river length of 548 km, it is difficult to obtain complete data at each point, and this will lead to a large order river flow model. Since the Bengawan Solo river flow model is a large order model, we need to reduce the model using the balanced truncation method. Next, to obtain data on the water levels at each unknown point, we estimate the reduced model using the Kalman filter method. Based on the simulation results, we see that if more points are removed, the error value is larger. However, if fewer points are known, the computational time is less.

Keywords: *estimation; Kalman filter; model reduction; balanced truncation; Bengawan Solo river.*

Mathematics Subject Classification (2010): 93E10.

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1 Introduction

Indonesia is a maritime country with 2/3 of the area covered with water in the form of sea, lake, and river. River is one form of water that is useful for the life of Indonesian citizens. Furthermore, the river can also be disastrous if the volume of water in the river exceeds its capacity. Flood is a disaster in such event. One of the rivers in Indonesia that often causes floods is the Bengawan Solo river [1]. With the river length of 548 km, the Bengawan Solo river flows through 12 districts and is divided into 20 regions, i.e. upstream and downstream in Central Java and in East Java. The impact of flooding caused by the Bengawan Solo river is very large because it has a very long flow area. Therefore, the Bengawan Solo river water level is a system with a large order.

Thus, it is difficult to obtain complete data at each point. So, in anticipation of flooding due to the inability of the river to accommodate the increase in water volume, we estimate the river water level by taking into account the flow velocity using an estimator.

One of the well-known methods in estimation is the Kalman filter [2,3]. The Kalman filter was first introduced by Rudolph E. Kalman in 1960. There are some factors that cannot be modeled. Thus we added stochastic factors, such as the system noise and measurement noise. It follows that the system becomes a stochastic dynamical system. The Kalman filter consists of two processes: the time update and measurement update [4]. The time update is responsible for projecting forward in time the current state and error covariance estimates to obtain the a-priori estimates for the next time step. The measurement updates are responsible for the feedback for incorporating a new measurement into the a-priori estimate to obtain an improved a-posteriori estimate. After each time and measurement update, the process is repeated with the previous a-posteriori estimates used to project or predict the new a-priori estimates. This recursive nature is one of the very appealing features of the Kalman filter.

In the process of estimating the altitude of water level of the Bengawan Solo river, we use a shallow water equation, i.e. the Saint Venant equation. In this paper, the Saint Venant equation represents the original model which is widely used for the wave models in the atmosphere, rivers, lakes and oceans [5]. This equation is used to model the flow in open channels, such as the river flow. Since the Bengawan Solo river has many points of location denoted by states, the original model is a larger order model. Hence, in this paper, we also reduce the original model before we estimate the water level of the Bengawan Solo river using the Kalman filter.

Model reduction is used to simplify the size of realization in a model. This will reduce the computational time, and hopefully, the error is as small as possible [6]. Currently, there are many developed methods of model reduction such as the balanced truncation methods [7–9] and singular perturbation approximation [2, 10]. In [6], a Kalman filter algorithm has been developed in the reduced model and applied to the heat conduction distribution problem. The authors in [11] combine the Kalman filter estimation and model reduction without identification by using the balanced truncation method.

Since the orders of the state variables in the reduced and original models are different, we cannot compare them. Therefore, we need an identification of the state variables in the reduced model such that we can determine the corresponding state variables in the original model. In this paper, we want to determine a relationship between the state variables in the reduced and original model. We can compute the corresponding state variables in the original model using the reduced model [12]. The simulation results show that the Kalman filter estimation of the identified reduced model using balanced truncation can be carried out for several measurement points of the river water level.

2 The Bengawan Solo River

Bengawan Solo is the longest river on the island of Java, Indonesia. The river is around 548.53 km long and flows through two provinces, Central Java and East Java [1]. Data on the Bengawan Solo river can be seen in Figure 1 [1].

No	POS TMA	LEVEL SIAGA (TTG)			←	12 Aug 18					→
					WAKTU (WIB)						
					TELEMETRI				MANUAL		
Wilayah Hulu		SH	SK	SM	06.00	12.00	18.00	00.00	06.00	12.00	18.00
1	Ngadipiro (Kab. Wonogiri)	151.00	152.00	153.00					146.52	146.51	146.50
2	Ngrembang (Kab. Wonogiri)	147.00	148.00	149.00		139.42	139.42		142.80	142.80	142.80
3	Colo Weir (Kab. Sukoharjo)	108.60	109.00	109.40	107.21	107.21	107.21		107.20	107.20	107.20
4	Jarum (Kab. Klaten)	94.00	95.00	95.50	89.80	89.87	89.82		89.94	89.94	89.94
5	Serenan (Kab. Sukoharjo)	92.00	93.00	94.00					86.24	86.23	86.22
6	Peren (Kab. Sukoharjo)	89.41	90.41	91.41					87.37	87.36	87.36
7	Jurug (Kota Surakarta)	82.73	83.73	84.73							
8	Kedungupit (Kab. Sragen)	74.00	75.00	76.00	69.11	69.11	69.11		69.11	69.11	
9	Wonogiri Dam (Kab. Wonogiri)	135.30	136.00	137.20	131.71	131.71	131.71		131.73	131.72	131.70
Wilayah Madiun		SH	SK	SM	06.00	12.00	18.00	00.00	06.00	12.00	18.00
10	Sekayu (Kab. Ponorogo)	97.50	98.00	98.50					0.00	0.00	0.00
11	AhmadYani (Kab. Madiun)	67.16	67.91	68.66					63.08	63.08	63.08
12	Napel (Kb. Ngawi)	43.50	44.50	45.50	38.36	38.35	38.36		38.28	38.27	38.27
13	Ketonggo Bridge (Kab. Ngawi)	47.00	48.00	49.00	37.45	37.88	37.59				
14	Jati Weir (Kab. Magetan)	96.45	97.05	97.65					74.00		74.00
15	Arjowinangun - Pacitan (Kab. Pacitan)	20.19	20.69	21.19					10.25	10.30	10.20
Wilayah Hilir		SH	SK	SM	06.00	12.00	18.00	00.00	06.00	12.00	18.00
16	Cepu (Kab. Bojonegoro)	34.87	35.87	36.87					-3.63	28.05	28.04
17	Bengkelo_lor	20.73	21.48	22.23							
18	Bojonegoro - Kali Kethek (Kab. Bojonegoro)	20.04	21.04	22.04							
19	Boboh Kali Lamong (Kab. Gresik)	14.48	14.73	14.98							
20	Karanggeneng (Kab. Lamongan)	6.50	7.50	8.50							

Figure 1: Data on the water level of the Bengawan Solo river.

Because of the length of the river, the recording of the river water level data is not easy. The river water level data are often not recorded properly as in Figure 1. So, it is necessary to estimate the river water level in order to anticipate floods. Since the BBWS data are not available at all for the water level data in the Karanggeneng area, we will estimate the water level at 19 points or locations. In this paper, we use the data on water level of the Bengawan Solo river for the period of June, 2018 – August, 2018 [1].

3 Model Representation

We discuss the shallow water equation that describes the flow of water in rivers [13]:

$$\begin{aligned} \frac{\partial h}{\partial t} + D\frac{\partial v}{\partial x} &= 0, \\ \frac{\partial v}{\partial t} + g\frac{\partial h}{\partial x} + C_f u &= 0, \end{aligned} \tag{1}$$

where the initial conditions are taken from the measurement data of water level in the Bengawan Solo river at $t = 1$ [1] and the boundary conditions are [2]

$$h(0,t) = h(x-1,t), \qquad h(L,t) = h(2,t), \tag{2}$$

where $h(x,t)$ is the water level above the reference plane at the position (or city) x and time t, t is the time variable, x is the position (or city) along the river, D is the water depth, g is the gravitational acceleration and C_f is a friction constant.

4 Discretization

The shallow water equation in (1) will be discretized using the Lax-Wendroff scheme. We can obtain a discrete-time system that is suitable for the Kalman filter and model reduction. The result of discretization in (1) is as follows [2]:

$$\begin{aligned} h_i^{k+1} &= \frac{1}{2}(h_{i+1}^k + h_{i-1}^k) - \frac{D\Delta t}{2\Delta x}(u_{i+1}^k - u_{i-1}^k), \\ u_i^{k+1} &= \frac{(1 - C_f\Delta t)}{2}(u_{i+1}^k + u_{i-1}^k) - \frac{g\Delta t}{2\Delta x}(h_{i+1}^k - h_{i-1}^k), \end{aligned} \tag{3}$$

where h represents the water level and u represents the water velocity. The Lax-Wendroff scheme is a combination of the Lax-Friedrichs scheme and Leapfrog scheme [2]. The Leapfrog scheme works by replacing Δt with $2\Delta t$ such that $g\Delta t$ or $D\Delta t$ has smaller value than Δx in order to achieve the desired accuracy. The result of discretization of h_i^{k+1} and u_i^{k+1} is as follow:

$$\begin{aligned} h_i^{k+1} &= h_i^k - a(u_{i+1}^k - u_{i-1}^k) + c(h_{i+1}^k - 2h_i^k + h_{i-1}^k), \\ u_i^{k+1} &= du_i^k - b(h_{i+1}^k - h_{i-1}^k) + c(u_{i+1}^k - 2u_i^k + u_{i-1}^k), \end{aligned} \tag{4}$$

where

$$a = \frac{D\Delta t}{\Delta x}(1 - C_f\Delta t),\ b = \frac{g\Delta t}{\Delta x},\ c = \frac{Dg\Delta t^2}{2\Delta x^2},\ d = (1 - 2C_f\Delta t).$$

Thus, we can write (4) in matrix realization as follows:

$$\begin{cases} x_{k+1} &= Ax_k + Bu_k, \\ y_k &= Cx_k + Du_k, \end{cases} \tag{5}$$

where

$$x_{k+1} = \begin{bmatrix} h_1^{k+1} \\ u_1^{k+1} \\ h_2^{k+1} \\ u_2^{k+1} \\ h_3^{k+1} \\ u_3^{k+1} \\ h_4^{k+1} \\ u_4^{k+1} \\ \vdots \\ h_{N-1}^{k+1} \\ u_{N-1}^{k+1} \end{bmatrix}, \qquad x_k = \begin{bmatrix} h_1^k \\ u_1^k \\ h_2^k \\ u_2^k \\ h_3^k \\ u_3^k \\ h_4^k \\ u_4^k \\ \vdots \\ h_{N-1}^k \\ u_{N-1}^k \end{bmatrix}, \qquad u_k = \begin{bmatrix} h_0^k \\ u_0^k \\ h_N^k \\ u_N^k \end{bmatrix}.$$

For the measurement matrix C, we use the number of the Bengawan Solo river elevation points which do not have the real data, and for the matrix D, it is assumed that 0 is the adjusted size.

5 Estimation and Identification of Model Reduction

The Kalman filter is one of the data assimilation methods, i.e. estimation of state variables based on the noisy model and measurement systems [14]. The Kalman filter is divided into two processes: the time update and measurement update [4]. The time updates are responsible for projecting forward in time the current state and error covariance estimates to obtain the a-priori estimates for the next time step. The measurement updates are responsible for the feedback for incorporating a new measurement into the a-priori estimate to obtain an improved a-posteriori estimate. The estimation of large-order model needs a long computational time, so in this case we use the model reduction to simplify the model without any significant error.

Model reduction is used to simplify the large order system without any significant error. The behavior of the reduced system is almost the same as that of the original system [10]. There are many methods of model reduction and one of them is the balanced truncation method. Before we apply the balanced truncation method [7–9], the realization of the system has to be balanced, i.e. the controllability Gramian is the same as the observability Gramian [10]. In order to do so, we apply a transformation matrix T to the original system (A, B, C, D)

$$\tilde{A} = T^{-1}AT, \qquad \tilde{B} = T^{-1}B, \qquad \tilde{C} = CT, \qquad \tilde{D} = D.$$

The balanced system $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ can be written as

$$\begin{cases} \tilde{x}_{k+1} &= \tilde{A}\tilde{x}_k + \tilde{B}\tilde{u}_k, \\ \tilde{y}_k &= \tilde{C}\tilde{x}_k + \tilde{D}\tilde{u}_k. \end{cases} \tag{6}$$

After we obtain the balanced system in (6), we partition the Gramian Σ such that $\Sigma = diag(\Sigma_1, \Sigma_2)$, where $\Sigma_1 = diag(\sigma_1, \sigma_2, ..., \sigma_r)$ and $\Sigma_2 = diag(\sigma_{r+1}, \sigma_{r+2}, ..., \sigma_n)$. Then the balanced system is partitioned into

$$\begin{bmatrix} \tilde{x}_1(k+1) \\ \tilde{x}_2(k+1) \end{bmatrix} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} \begin{bmatrix} \tilde{x}_1(k) \\ \tilde{x}_2(k) \end{bmatrix} + \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix} u(k), \tag{7}$$

$$\tilde{y}(k) = \begin{bmatrix} \tilde{C}_1 & \tilde{C}_2 \end{bmatrix} \begin{bmatrix} \tilde{x}_1(k) \\ \tilde{x}_2(k) \end{bmatrix} + \tilde{D}u(k), \tag{8}$$

where, $\tilde{x}_1(k) \in \mathbb{R}^r$ corresponds to Σ_1 and $\tilde{x}_2(k) \in \mathbb{R}^{n-r}$ corresponds to Σ_2.

Model reduction by using the balanced truncation method is done by assuming $\tilde{x}_2(k+1) = 0$. The reduced system can be written as [9]

$$\begin{cases} \tilde{x}_{rk+1} &= \tilde{A}_r\tilde{x}_{rk} + \tilde{B}_r\tilde{u}_k, \\ \tilde{y}_{rk} &= \tilde{C}_r\tilde{x}_{rk} + \tilde{D}_r\tilde{u}_k. \end{cases} \tag{9}$$

Because there are differences in the size of the original system matrix and the reduced system, the results cannot be compared directly. In order to produce a reduced system that corresponds to the original system, it is necessary to identify the relationship between the states of the two systems. The identification can be obtained from the transformation matrix T [12]

$$x_k = T\tilde{x}_k. \quad (10)$$

Equation (10) can be written as

$$x_{id} = T_r \tilde{x}_{rk}, \quad (11)$$

where x_{id} is the state of the identified reduced model with size $n \times 1$, T_r is obtained by reducing the first part of the inverse transformation matrix T of size $n \times r$, and x_{rk} is the reduced model of size $r \times 1$.

6 Simulation Results

The shallow water equation (1) describes the relationship between the water level h and water debit u. In this paper, we focus on the estimation of water level in the Bengawan Solo river. Due to the unavailability of water debit data, the initial value of u is defined as 0. We use the following values for the parameters in shallow water equations:

$$D = 150m, \quad C_f = 0.0002, \quad \Delta x = 548km, \quad \Delta t = 100, \quad g = 9.8m/s^2.$$

With the parameters above, we estimate the water level h using the Kalman filter by using the real data for the period of June, 2018 - August, 2018 [1]. First, we reduce the number of state variables in the model. Each state represents the water level point. Thus, we will reduce the number of the water level points. The original model has 19 states, it means that the number of the water level points is 19. In the second simulation, the states in the original model will be reduced to 4-18 states (or points). The simulation results of the Kalman filter estimation of the identified reduced model using balanced truncation for 10 and 15 water level points are shown in Figure 2-3.

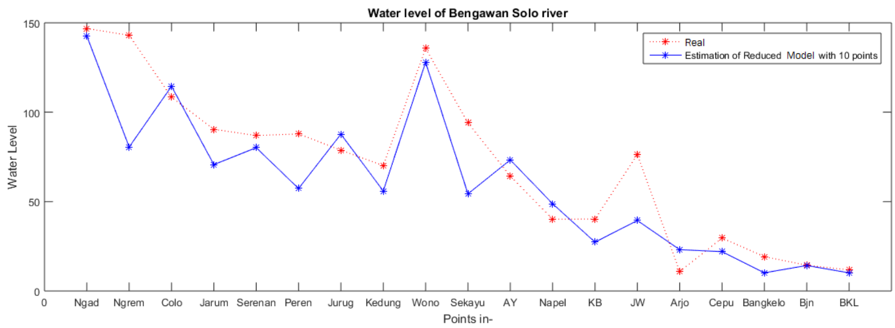

Figure 2: Estimation of the reduced system with 10 water level points.

From Figures 2-3, we can see that the simulation results of the Kalman filter estimation of the identified reduced model using balanced truncation are quite accurate or almost the same as those for the original model for several points. For more detailed values, we describe the relative error value and computational time for each known point in Table 1.

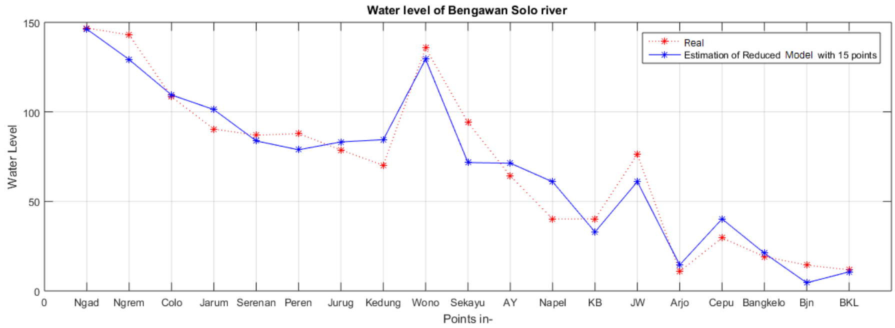

Figure 3: Estimation of the reduced system with 15 water level points.

Known points	Relative Error		Computational time	
	Original model	Reduced model	Original model	Reduced model
2	1.11E-04	1.1660	0.125465	0.024951
3	1.77E-04	1.1269	0.130171	0.025277
4	1.14E-04	1.0989	0.129215	0.032373
5	8.63E-05	1.0398	0.212791	0.032134
6	7.02E-05	1.0348	0.134526	0.032752
7	1.32E-04	1.0298	0.134507	0.035429
8	1.73E-04	0.6755	0.133507	0.025639
9	1.51E-04	0.4274	0.133474	0.047240
10	1.31E-04	0.4239	0.133721	0.029778
11	9.37E-05	0.4139	0.131884	0.033390
12	1.23E-04	0.3837	0.135993	0.031910
13	9.26E-05	0.3357	0.128158	0.035152
14	1.33E-04	0.3018	0.134753	0.035960
15	1.49E-04	0.2321	0.127770	0.042461
16	1.09E-04	0.1819	0.129831	0.041342
17	8.10E-05	0.1526	0.132840	0.048576
18	8.62E-05	0.0085	0.137477	0.059243

Table 1: Comparison between the error and computational time for estimation of the original and reduced models.

Based on Table 1, we conclude that the Kalman filter estimation of the original model is better than the Kalman filter estimation of the identified reduced model using balanced truncation. This result is reasonable, because the reduced model cannot achieve better performance than the best estimation of the original model. In terms of the computational time, the Kalman filter estimation of the identified reduced model using balanced truncation is faster than that of the original one. This result is also reasonable, because the order of the reduced model is smaller than that of the original model.

Based on Table 1, we can see that the error of 18 (from 19) water level points is 0.0085 and the computational time is 0.059243 seconds. On the other hand, the error of 2 (from 19) water level points is 1.1660 and the computational time is 0.024951 seconds. If the order of the reduced model is smaller, the error value is larger and inversely proportional to the computational time. We can see in Table 1 that the computational time for the Kalman filter estimation of the identified reduced model using balanced truncation is less than that for the Kalman filter estimation of the original model.

7 Conclusions

In this work, we estimate the water level in the reduced model using balanced truncation. Since the orders of the state variables in the reduced and original models are different, we cannot compare them directly. Therefore, we need an identification of the state variables in the reduced model such that we can determine the corresponding state variables in the original model. The simulation result shows that the Kalman filter estimation of the identified reduced model using balanced truncation has an error larger than that of the original model, but the average computational time to estimate the reduced system is 26% less compared to the estimation of the original model. Thus, for the model reduction, we can choose the number of water level points based on our needs.

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