Nonlinear Dynamics and Systems Theory, **19** (4) (2019) 523–536

Control, Stabilization and Synchronization of Fractional-Order Jerk System

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Received: May 31, 2019; Revised: October 14, 2019

Abstract: In this work we study the fractional-order jerk system stability by using the fractional Routh-Hurwitz conditions. These conditions have also been used to control the chaos of the proposed systems towards their equilibrium. It has been shown that the fractional-order systems are controlled at their equilibrium point unlike those of fractional order. The synchronization between two different coupled fractional systems is also achieved via the auxiliary system approach. The numerical simulation coincides with the theoretical analysis.

Keywords: *chaos; chaotic fractional-order system; Routh-Hurwitz criteria; chaos control; chaos synchronization.*

Mathematics Subject Classification (2010): 34C23, 34H10, 34H15, 34A34, 34D06, 37N35, 37C75, 37N30.

1 Introduction

Fractional calculus is a topic more than 300 years old. The idea of fractional calculus has been known since the regular calculus, with the first reference probably being associated with Leibniz and L'Hospital in 1695. Its applications to physics and engineering are just a recent focus of interest. It was found that many systems in interdisciplinary fields can be elegantly described with the help of fractional derivatives. In 1996, Hans Gottlieb thought,'What is the simplest jerk equation that gives chaos ?', by which he meant an equation of the form

$$\dddot{x} = f(\ddot{x}, \dot{x}, x).$$

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The term 'jerk' comes from the fact that in a mechanical system in which x is the displacement, $\dot{x}$ is the velocity, and $\ddot{x}$ is the acceleration, the quantity $\dddot{x}$ is called the 'jerk' (Schot, 1978). It is the lowest derivative for which an ODE with smooth continuous functions can give chaos.

In this paper, we investigate the chaotic behaviors of the fractional-order simple autonomous jerk system with cubic non-linearity. The system is a linear transformation of the MO4 and MO11 models introduced for the first time in [14]. We find that chaos exists in the fractional-order model MO4 and MO11 systems with an order less than 3. Many other studies on the dynamics in fractional-order systems are presented, in particular, in [13–15]. In addition, the auxiliary system method, generalized to the fractional-order, is also presented to synchronize the fractional chaotic order between MO4 and MO11. Both approaches, based on the theory of the stability of fractional order systems, are simple and theoretically rigorous. The results of the simulation are used to visualize and illustrate the effectiveness of the proposed synchronization methods.

2 Preliminaries

2.1 Fractional calculus

Fractional calculus is a generalization of integration and differentiation to the noninteger-order fundamental operator ${}_aD_\alpha^t$, where a and t are the bounds of the operation and $\alpha \in \mathbf{R}$. The continuous integro-differential operator is defined as

$${}_aD_\alpha^t = \begin{cases} \frac{d^\alpha}{dt^\alpha}, & \alpha > 0, \\ 1, & \alpha = 0, \\ \int_a^t (d\tau)^\alpha, & \alpha < 0. \end{cases}$$

In this paper, we will use the Caputo fractional derivatives defined by

$${}_aD_\alpha^t f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^n(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau \qquad \text{for } n-1 < \alpha < n.$$

2.2 Numerical method for solving fractional differential equations

For numerical simulation of the fractional-order system a predictor-corrector method has also been proposed [16]. It is suitable for Caputo's derivative because it just requires the initial conditions and for the unknown function it has a clear physical meaning. The method is based on the fact that the fractional differential equation

$$\begin{cases} D_\alpha^t x(t) = f(t, x(t)), & 0 \le t < T, \\ x^{(k)}(0) = x_0^{(k)}, & k = 0, 1, ..., n-1, \end{cases}$$

is equivalent to the Volterra integral equation

$$x(t) = \sum_{k=0}^{[\alpha]-1} x_0^{(k)} \frac{t^k}{k!} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, x(\tau))\, d\tau. \tag{1}$$

Set $h = \frac{T}{N}$, $t_n = nh$, $n = 0, 1, ..., N$, then (1) can be discredited as follows:

$$\begin{aligned} x_h\left(t_{n+1}\right) &= \sum_{k=0}^{[\alpha]-1} x_0^{(k)} \frac{t_{n+1}^k}{k!} + \frac{h^\alpha}{\Gamma\left(\alpha+2\right)} f\left(t_{n+1}, x_h^p\left(t_{n+1}\right)\right) \\ &+ \frac{h^\alpha}{\Gamma\left(\alpha+2\right)} \sum_{j=0}^{n} a_{j,n+1} f\left(t_j, x_h\left(t_j\right)\right), \end{aligned}$$

where

$$a_{j,n+1} = \begin{cases} n^{\alpha+1} - \left(n-\alpha\right)\left(n+1\right)^\alpha, & j = 0, \\ \left(n-j+2\right)^{\alpha+1} + \left(n-j\right)^{\alpha+1} - 2\left(n-j+1\right)^{\alpha+1}, & 1 \leq j \leq n, \\ 1, & j = 1, \end{cases}$$

$$x_h^p\left(t_{n+1}\right) = \sum_{k=0}^{[\alpha]-1} x_0^{(k)} \frac{t_{n+1}^k}{k!} + \frac{1}{\Gamma\left(\alpha\right)} \sum_{j=0}^{n} b_{j,n+1} f\left(t_j, x_h\left(t_j\right)\right),$$

$$b_{j,n+1} = \frac{h^\alpha}{\alpha}\left(\left(n+1-j\right)^\alpha + \left(n-j\right)^\alpha\right), \qquad 0 \leq j \leq n \quad .$$

This method, the error is estimated as

$$\varepsilon = \max_{j=0,1,\ldots,N} \left|x\left(t_j\right) - x_h\left(t_j\right)\right| = o\left(h^p\right),$$

where $p = \min\left(2, 1+\alpha\right)$.

2.3 Fractional-order Routh-Hurwitz stability conditions

Let us consider the following three-dimensional fractional-order commensurate system:

$$D^\alpha x = f\left(x\right),$$

where $\alpha \in \left]0, 1\right]$, $x \in \mathbf{R}^3$. We suppose that x_{eq} is an equilibrium point of this system, then its characteristic equation is given as

$$P(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0,$$

its discriminant is given by

$$D(P) = 18a_1a_2a_3 + (a_1a_2)^2 - 4a_3(a_1)^3 - 4(a_2)^3 - 27(a_3)^2.$$

We have the following fractional-order Routh–Hurwitz conditions:

1. If $D(P) > 0$, then the necessary and sufficient condition for the equilibrium point E to be locally asymptotically stable is $a_1 > 0, a_3 > 0$ and $a_1a_2 - a_3 > 0$.
2. If $D(P) < 0, a_1 \geq 0, a_2 \geq 0, a_3 > 0$, then E is locally asymptotically stable for $\alpha < 2/3$. However, if $D(P) < 0, a_1 < 0, a_2 < 0, \alpha > 2/3$, then E is unstable.
3. If $D(P) < 0, a_1 > 0, a_2 > 0, a_1a_2 - a_3 = 0$, then E is locally asymptotically stable for all $\alpha \in]0, 1[$.
4. The necessary condition for the equilibrium point E to be locally asymptotically stable is $a_3 > 0$.

3 Description and Analysis of the Models

3.1 First model

The mathematical model of the jerk system considered in this work is expressed by the following set of three coupled first-order nonlinear differential equations:

$$\begin{cases} \frac{d^\alpha x}{dt^\alpha} = y, \\ \frac{d^\alpha y}{dt^\alpha} = z, \\ \frac{d^\alpha z}{dt^\alpha} = -\mu z - y - \beta e^x + \delta, \end{cases} \tag{2}$$

where the parameters μ, β and δ are positive reals and μ is the fractional-order of system (2) which has the only equilibrium point, which is found by equating the right-hand sides of system (2) to zero and is given as follows: $E\left(\ln \frac{\delta}{\beta} 0, 0\right)$.

3.1.1 Stability of the equilibrium point

Proposition 3.1 *1. If $\mu < \sqrt{3}$, then E is asymptotically stable for $\alpha < 2/3$. In addition to this condition, if $\beta = \mu$, then E is locally asymptotically stable for all $\alpha \in]0, 1[$.*

2. If $\mu > \sqrt{3}$ and $\beta < \frac{1}{3}\mu - \frac{2}{27}\mu^3 + \frac{2}{27}\sqrt{(\mu^2-3)^3}$, then the first stability condition holds.

Proof. The characteristic polynomial of the equilibrium point $E\left(\ln \frac{\delta}{\beta} 0, 0\right)$ is given by

$$\lambda^3 + \mu\lambda^2 + \lambda + \beta = 0,$$

so

$$a_1 = \mu > 0, a_2 = 1 > 0, a_3 = \beta > 0$$

and

$$D_E(p) = -4\mu^3\beta + \mu^2 + 18\mu\beta - 27\beta^2 - 4.$$

1. If $\mu < \sqrt{3}$, then $D_E(p) < 0$. Thus achieving the second of the stability conditions, therefore E is asymptotically stable for $\alpha < 2/3$. Moreover, if $\beta = \mu$ is verified, which means fulfilling the condition

   $$a_1 \times a_2 - a_3 = 0.$$

   From all of the foregoing, we arrive at the realization of the third stabilization conditions and from it, we conclude that the equilibrium point E is locally asymptotically stable for all $\alpha \in]0, 1[$.

2. If $\mu > \sqrt{3}$ and $\beta < \frac{1}{3}\mu - \frac{2}{27}\mu^3 + \frac{2}{27}\sqrt{(\mu^2-3)^3}$, both conditions result in satisfaction of $D(P) > 0$ and $a_1 > 0, a_3 > 0$ and $a_1 a_2 - a_3 > 0$, then the first stability condition holds.

3.1.2 Chaos

For the parameter values $\mu = 0.5$, $\beta = 1$ and $\delta = 5$, the integer-order form of the system (2) presents chaotic behavior, with the largest exponent of Lyapunov calculated numerically $LE = 0.035$, and its equilibrium $E(\ln 5, 0, 0)$ is locally asymptotically stable when $\alpha < 2/3$ and their eigenvalues are given as: $\lambda_1 = -1.6787$, $\lambda_{2,3} = 0.58933 \pm 1.6221i$. The equilibrium point is a saddle point of index 2, thus the necessary condition for the fractional-order system (2) to remain chaotic is $\alpha > \frac{2}{\pi} \arctan\left(\frac{|\lambda_{2,3}|}{\mathrm{Re}\,\lambda_{2,3}}\right)$. Consequently, the lowest fractional order α, for which the fractional-order system (2) demonstrates chaos at the above-mentioned parameters, is given by the inequality
$\alpha > 0.79051$, see Figs.1 and 2.

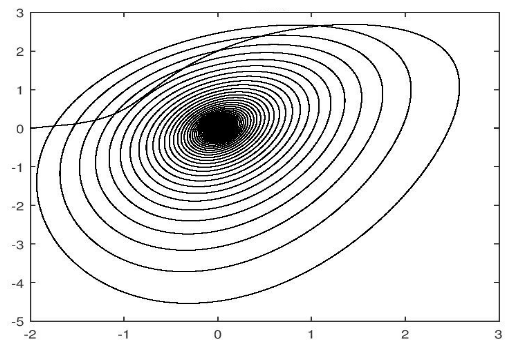

Figure 1: Phase plots of attractor generated by (2) y-z plane with $\mu = 0.5$, $\beta = 1$ and $\delta = 5$, at $\alpha = 0.77$.

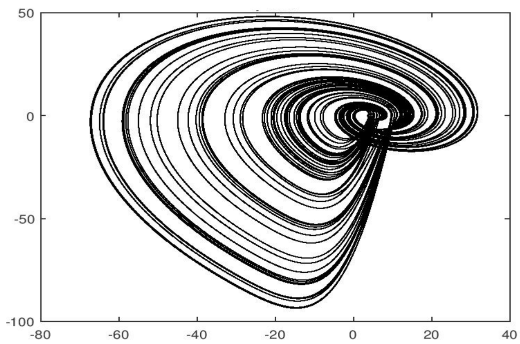

Figure 2: Phase plots of attractor generated by (2) y-z plane with $\mu = 0.5$, $\beta = 1$ and $\delta = 5$, at $\alpha = 0.97$.

3.1.3 Chaos control of the fractional-order systems

A three-dimensional fractional-order chaotic system and the control of chaos are described as follows:

$$\begin{cases} \frac{d^\alpha X}{dt^\alpha} = F(X), \\ \frac{d^\alpha X}{dt^\alpha} = F(X) - K(X - X^*), \end{cases}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbf{R}^3$, $\alpha_i > 0$, is the fractional order and the systems are chaotic. K is a coupling matrix. For simplicity, let K have the form $K = diag(k_1, k_2, k_3)$, where $k_i \geq 0$. The error is $e = X - X^*$ and the aim of the control is to design the coupling matrix so that $||e(t)|| \to 0$ as $t \to +\infty$. Let us consider the system (2). The controlled fractional-order system associated with the system (2) is given by

$$\begin{cases} \frac{d^\alpha x}{dt^\alpha} = y - k_1(x - x^*), \\ \frac{d^\alpha y}{dt^\alpha} = z - k_2(y - y^*), \\ \frac{d^\alpha z}{dt^\alpha} = -\alpha z - y - \beta e^x + \delta - k_3(z - z^*), \end{cases} \tag{3}$$

where (x^*, y^*, z^*) represents an arbitrary equilibrium point of system (2). The goal is to drive the system trajectories to any of the three unstable equilibrium point E. For simplicity, we are going to choose the feedback gains $K = diag(0, k_2, 0)$.

3.1.4 Stabilizing the equilibrium point

Sufficient conditions for the stabilization of the controlled systems (3) are given in the following proposition.

Proposition 3.2 *If* $k_2 = -\frac{1}{2\mu}\left(-\sqrt{-2\mu^2 + \mu^4 + 4\mu\beta + 1} + \mu^2 + 1\right)$ *and the parameter* β *satisfies* $\beta > 0$*, then the trajectories of the controlled system* (3) *are driven to the unstable equilibrium point* E*.*

Proof. The characteristic equation of the controlled system (3) at E is given as

$$\lambda^3 + (k_2 + \mu)\lambda^2 + (\mu k_2 + 1)\lambda + \beta.$$

Choose the parameter $\beta > \mu$ and the feedback control gain

$$k_2 = -\frac{1}{2\mu}\left(-\sqrt{-2\mu^2 + \mu^4 + 4\mu\beta + 1} + \mu^2 + 1\right).$$

If $D(p) < 0$ for the found value of the parameter k_2, then the stability condition (3) holds and the trajectories of the controlled system (3) are driven to the stable equilibrium point E for all $\alpha \in]0, 1[$.

3.1.5 Numerical results

In this section, we apply the result in the previous system (2) for the purpose of control chaos, we take $\mu = 0.5$, $\beta = 1, \delta = 5$ and the fractional order $q = 0.97$, by Proposition 3.2 we have $k_2 = 0.35078$, $k_1 = k_3 = 0$. It follows that $D(p)$=$-22.25 < 0$, $a_1 > 0$, $a_2 > 0$, $a_1 a_2 = a_3$. Therefore, the stability conditions (3) and (4) are checked. This implies that the trajectories of the controlled fractional-order system (3) converge to the equilibrium point as shown in Fig. 3. But in the integer-order case, there are two pure imaginary eigenvalues of the characteristic equation. This means that the integer-order form of the controlled system (3) is not stabilized to the same equilibrium point when choosing the above-mentioned parameter values and feedback control gains see Fig. 4.

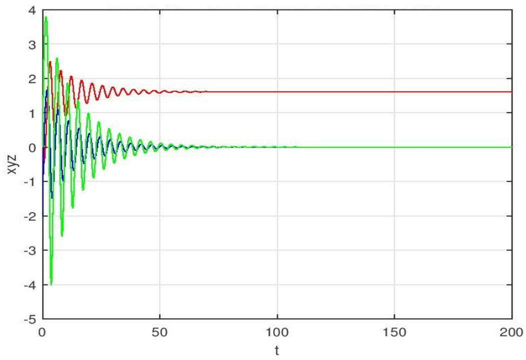

Figure 3: The trajectories of the controlled system (3), $\mu = 0.5$,$\beta = 1, \delta = 5$ and the controllers $k_2 = 0.35078$, $k_1 = k_3 = 0$. Stabilized to the equilibrium point E for $\alpha = 0.97$.

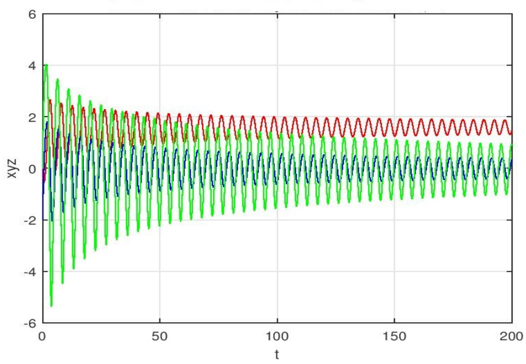

Figure 4: The trajectories of the controlled system (3), $\mu = 0.5$, $\beta = 1, \delta = 5$ and the controllers $k_2 = 0.35078$, $k_1 = k_3 = 0$. Not stabilized to the equilibrium point E for $\alpha = 1$.

3.2 Second model

The mathematical model of the jerk system considered in this work is expressed by the following set of three coupled first-order nonlinear differential equations:

$$\begin{cases} \frac{d^\alpha x}{dt^\alpha} = y, \\ \frac{d^\alpha y}{dt^\alpha} = z, \\ \frac{d^\alpha z}{dt^\alpha} = -\mu z - y - \sigma x\left(x-1\right), \end{cases} \tag{4}$$

where the parameters μ and σ are positive reals and α is the fractional-order. The system (4) has two equilibrium points which are found by equating the right-hand sides of (4) to zero and are given as follows: $E_1\,(0,0,0)\,, E_2\,(1,0,0)\,.$

3.2.1 Stability of the equilibrium points

The characteristic polynomial of the equilibrium point E_1 is given by

$$\lambda^3 + \mu\lambda^2 + \lambda - \sigma = 0.$$

So $a_3 = -\sigma < 0$, then E_2 is unstable. The characteristic polynomial of the equilibrium point E_2 is given by

$$\lambda^3 + \mu\lambda^2 + \lambda + \sigma = 0.$$

So $a_1 = \mu > 0$, $a_2 = 1 > 0$, $a_3 = \sigma > 0$ and $\mu^2 + 18\mu\sigma - 27\sigma^2 - 4$.

If $\mu < \sqrt{3}$, then $D_E\,(p) < 0$ and E_2 is asymptotically stable for $\alpha < 2/3$. However, if $\sigma = \mu$, then E_2 is locally asymptotically stable for all $\alpha \in]0,1[$ according to the third condition of the Routh-Hurwitz criterion.

If $\mu > \sqrt{3}$ and $\sigma < \frac{1}{3}\mu - \frac{2}{27}\mu^3 + \frac{2}{27}\sqrt{(\mu^2-3)^3}$, then the first condition of the Routh-Hurwitz criterion holds. From which stability is achieved.

3.2.2 Chaos

For the parameter values $\mu = 0.5$ and $\sigma = 1$, the integer-order form of the system (4) presents chaotic behavior, with the largest exponent of Lyapunov calculated numerically $LE = 0.094$, and its equilibrium E_1 is unstable and $E_2(1,0,0)$ is locally asymptotically stable when $\alpha < 2/3$ and their eigenvalues are given as E_2:$\lambda_1 = -0.803\,76$, $\lambda_{2,3} = 0.151\,88 \pm 1.105i$ $E_2 : \lambda_1 = 0.601\,49$, $\lambda_{2,3} = -0.550\,75 \pm 1.165\,9i$. The equilibrium point E_2 is a saddle point of index 2, thus the necessary condition for the fractional-order system (4) to remain chaotic is $\alpha > \frac{2}{\pi}\arctan\left(\frac{|\lambda_{2,3}|}{\mathrm{Re}\,\lambda_{2,3}}\right)$. Consequently, the lowest fractional order α, for which the fractional-order system (4) demonstrates chaos at the above-mentioned parameters, is given by the inequality $\alpha > 0.913\,84$, see fig. 5 and 6.

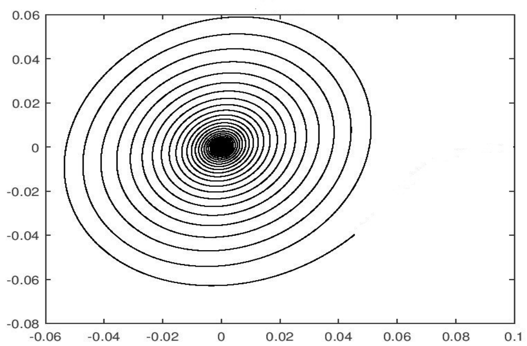

Figure 5: Phase plots of attractor generated by (3) y-z plane with $\mu = 0.5$ and $\sigma = 1$, at $\alpha = 0.99$.

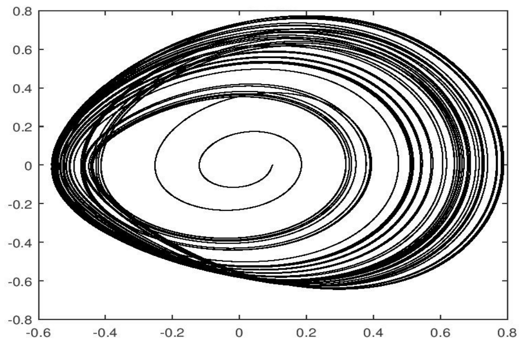

Figure 6: Phase plots of attractor generated by (3) y-z plane with $\mu = 0.5$ and $\sigma = 1$, at $\alpha = 0.99$.

3.2.3 Chaos control of the fractional-order systems

The controlled fractional-order system assisted with system (4) is given by

$$\begin{cases} \frac{d^\alpha x}{dt^\alpha} = y - k_1(x - x^*), \\ \frac{d^\alpha y}{dt^\alpha} = z - k_2(y - y^*), \\ \frac{d^\alpha z}{dt^\alpha} = -\mu z - y - \sigma x\,(x-1) - k_3(z - z^*), \end{cases} \tag{5}$$

where (x^*, y^*, z^*) represents an arbitrary equilibrium point of system (4). The goal is to drive the system trajectories to any of the two unstable equilibrium points E_1 and E_2. As in the previous model we chose the feedback gains $K = diag(0, k_2, 0)$.

3.2.4 Stabilizing the equilibrium points

Sufficient conditions for the stabilization of the controlled systems (5) are given in the following proposition.

Proposition 3.3 • *The trajectories of the system* (5) *are not driven to the unstable equilibrium point* E_1.

- *If* $k_1 = -\frac{1}{2\mu}\left(-\sqrt{-2\mu^2 + \mu^4 + 4\mu\sigma + 1} + \mu^2 + 1\right)$ *and the parameter* σ *satisfies* $\sigma > 0$, *then the trajectories of the controlled system* (5) *are driven to the stable equilibrium point* E_1 *for all* $q \in]0, 1[$.

Proof. • The characteristic equation of the controlled system at E_1 is given as

$$\lambda^3 + (k_2 + \mu)\,\lambda^2 + (k_2\mu + 1)\,\lambda - \sigma = 0.$$

We have $a_3 = -\sigma$, according to the fourth condition of the Routh -Hurwitz criterion, the system (5) can not be stable.

• By choosing the parameter $\sigma > \alpha$ and the feedback control gain

$$k_2 = -\frac{1}{2\mu}\left(-\sqrt{-2\mu^2 + \mu^4 + 4\mu\sigma + 1} + \mu^2 + 1\right)$$

and assuming that $D(p) < 0$, the stability condition (3) is satisfied and the trajectories of the controlled system (5) are driven to the stable equilibrium point E_2 for all $\alpha \in]0, 1[$.

3.2.5 Numerical results

In this section, we take $\alpha = 0.5$, $\sigma = 1$ and the fractional-order $\alpha = 0.98$, by Proposition 3.3 we have $k_2 = 0.35078$, $k_1 = k_3 = 0$. It follows that $D(p) < 0, a_1 > 0, a_2 > 0, a_1 a_2 = a_3$. Therefore, the stability conditions (3) and (4) are checked. This implies that the trajectories of the controlled fractional-order system (5) converge to the equilibrium point E_2 as shown in Fig. 7. But in the integer-order case, there are two pure imaginary eigenvalues of the characteristic equation. This means that the integer-order form of the controlled system (5) is not stabilized to the same equilibrium point when choosing the above-mentioned parameter values and feedback control gains, Fig. 8.

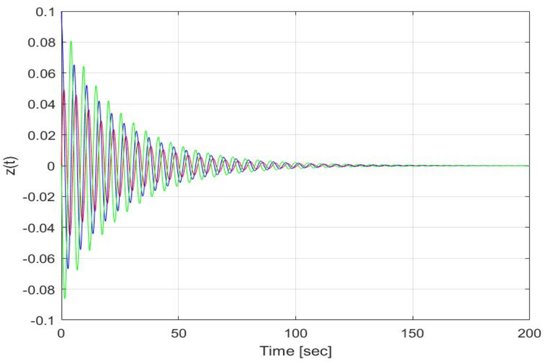

Figure 7: The trajectories of the controlled system (5) for $\mu = 0.5$, $\sigma = 1$, $k_2 = 0.35078$ and $k_1 = k_3 = 0$. Stabilized to the equilibrium point E_2 for $\alpha = 0.98$.

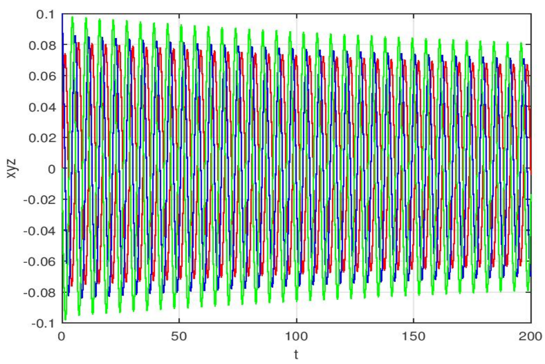

Figure 8: The trajectories of the controlled system (5) for $\mu = 0.5$, $\sigma = 1$, $k_2 = 0.35078$ and $k_1 = k_3 = 0$. Not stabilized to the equilibrium point E_2 for $\alpha = 1$.

4 Chaos Synchronization

In this section, we realize the synchronization between two different fractional-order systems via the auxiliary system approach. We choose as a master system the following system:

$$\begin{cases} \frac{d^\alpha x_1}{dt^\alpha} = y_1, \\ \frac{d^\alpha y_1}{dt^\alpha} = z_1, \\ \frac{d^\alpha z_1}{dt^\alpha} = -\mu z_1 - y_1 - \beta e^{x_1} + \delta, \end{cases} \tag{6}$$

and the slave system is

$$\begin{cases} \frac{d^\alpha x_2}{dt^\alpha} = y_2 - k_1(x_2 - x_1), \\ \frac{d^\alpha y_2}{dt^\alpha} = z_2 - k_2(y_2 - y_1), \\ \frac{d^\alpha z_2}{dt^\alpha} = -\mu z_2 - y_2 - \sigma x_2 (x_2 - 1) - k_3(z_2 - z_1). \end{cases} \tag{7}$$

The master system is coupled with the slave system only by the scalar $x(t)$. We choose the auxiliary system that is identical to the slave system (7) (with different initial conditions)

$$\begin{cases} \frac{d^\alpha x_3}{dt^\alpha} = y_3 - k_1(x_3 - x_1), \\ \frac{d^\alpha y_3}{dt^\alpha} = z_3 - k_2(y_3 - y_1), \\ \frac{d^\alpha z_3}{dt^\alpha} = -\mu z_3 - y_3 - \sigma x_3 (x_3 - 1) - k_3(z_3 - z_1). \end{cases} \tag{8}$$

The substraction of two systems (7) and (8) yields the fractional-order synchronization error system which can be written as follows:

$$\begin{cases} \frac{d^q e_1}{dt^q} = e_2 - k_1 e_1, \\ \frac{d^q e_2}{dt^q} = e_3 - k_2 e_2, \\ \frac{d^q e_3}{dt^q} = -\alpha e_3 - e_2 - \sigma e_1 x_3 - \sigma e_1 x_2 - e_{1-} k_3 e_3, \end{cases} \tag{9}$$

where $e_1 = x_3 - x_2$, $e_2 = y_3 - y_2$ and $e_3 = z_3 - z_2$. Further (9) can be written as

$$\frac{d^\alpha e}{dt^\alpha} = Ae + \varphi(x_{2,3}, y_{2,3}, z_{2,3}), \tag{10}$$

where $e = [e_1, e_2, e_3]^T$

$$A = \begin{bmatrix} -k_1 & 1 & 0 \\ 0 & -k_2 & 1 \\ -1 & -1 & -\mu - k_3 \end{bmatrix}, \quad \varphi(x_{2,3}, y_{2,3}, z_{2,3}) = \begin{pmatrix} 0 \\ 0 \\ -\sigma e_1 (x_2 + x_3) \end{pmatrix},$$

$\varphi(x_{2,3}, y_{2,3}, z_{2,3})$ is a nonlinear function satisfying the Lipschitz condition, so, near to zero, it converges to zero. To study the stability of the system (10), we use the conditions of the Routh-Hurwitz criterion generalized in fractional order. The characteristic polynomial of matrix A is given by

$\lambda^3 + (\mu + k_1 + k_2 + k_3)\lambda^2 + ((\mu + k_3)(k_1 + k_2) + k_1 k_2 + 1)\lambda + (k_1 + k_1 k_2(\mu + k_3) + 1)$.

For simplicity, we choose the feedback gains $k_1 = k_2 = 0$ and $k_3 = k$. The characteristic polynomial becomes

$$P(\lambda) = \lambda^3 + (k + \mu)\lambda^2 + \lambda + 1. \tag{11}$$

Its discriminant is as follows:

$$D(p) = -3k^2 + (18 - 6\mu)k - 3\mu^2 + 18\mu - 31,$$

which is always negative for all values of k and μ, now for the condition $a_1 \times a_2 - a_3 = 0$ to be satisfied, it is enough that $k = 1 - \mu$. Therefore the zero solution of the system (9) is locally asymptotically stable for all $\alpha \in]0; 1[$. In this case, the fractional-order drive and response systems (6) and (7) are synchronized.

4.1 Numerical results

In numerical simulations, we set the parameters of the drive system as $\mu = 0.5$, $\beta = 1$ and $\delta = 5$, the parameters of the response and auxiliary systems as $\mu = 0.5$ and $\sigma = 1$ with the fractional-order $\alpha = 0.98$ and the coefficient of control function $k = 0.5$. We also have the initial conditions $x_1(0) = 1, y_1(0) = 2, z_1(0) = 5$ for the drive system, the initial conditions $x_2(0) = 10, y_2(0) = 32, z_2(0) = 7$ for the slave system, and $x_3(0) = 9, y_3(0) = 28, z_3(0) = 8$ for the auxiliary system. Numerical results show that the synchronization of two different fractional-order systems is achieved, see Fig. 9.

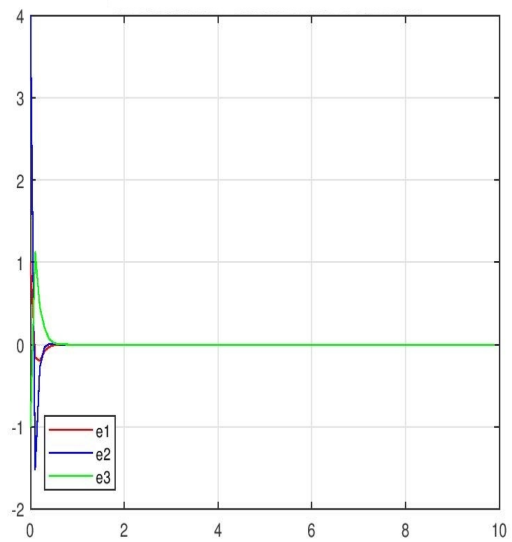

Figure 9: Synchronization error of the coupled systems.

5 Conclusion

In this study, we examined the local stability of the equilibrium in fractional system by using the fractional Routh-Hurwitz conditions which are also used to control chaos in the proposed systems towards their equilibrium by choosing some specific linear controllers. We showed that the fractional-order systems are controlled to their equilibrium points, however, their integer-order counterparts are not. This fact gives an advantage to the fractional-order systems compared with their integer-order counterparts, the effect of the fractional system on the synchronization of the chaos of these systems was also presented. And the numerical simulation matches with the theoretical analysis.

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