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Consistent Lyapunov Methodology: Non-Differentiable Non-Linear Systems

Ly.T. Gruyitch

University of Technology Belfort - Montbeliard,
Site de Belfort, 90010 Belfort Cedex, France

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This invited paper is dedicated to late Professor Wolfgang Hahn and to late Professor Jose P. LaSalle, who contributed fundamentally to the stability theory and supported author's early research on stability domains.

Abstract: The consistent Lyapunov methodology enables us, after its single application, to solve completely the asymptotic (or, exponential) stability problem, to construct a system Lyapunov function and to determine accurately the domain of asymptotic stability. This is achieved in the paper for invariant sets of non-differentiable time-varying non-linear systems. The results (proved in details) presentthe necessary and sufficient conditions for: asymptotic stability, for a determination of a system Lyapunov function and for a set to be the asymptotic stability domain. They are not expressed in terms of existence of a system Lyapunov function. They determine well the procedure how to resolve all the relevant problems.

Keywords: *Asymptotic stability domains; Lyapunov method; Lyapunov functions; non-linear systems; sets; uniform asymptotic stability.*

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1 Introduction

The fundamental Lyapunov method [1] is based on two different methodologies, one for time-invariant linear systems and another one for all other systems. The former enables us to effectively apply the method and to get a definite result after its single application. The latter, which will be called the *classical Lyapunov methodology (for non-linear systems)*, does not. The latter lesses us to face two crucial problems unsolved: a) how to construct a system Lyapunov function and b) how to determine the exact asymptotic stability domain. The classical Lyapunov methodology (for non-linear systems) starts with a trial to guess a suitable choice of a positive definite function $v(.)$. Its application continues with the negative (semi-) definiteness test of the total time derivative of $v(.)$ along system motions. The theorems established for time-varying non-linear systems have been expressed only in terms of *existence* of a Lyapunov function $v(.)$ $[u(.)]$ rather than to clarify how to find it for a given non-linear system. If the weak inequlaity in the condition on the Lyapunov function derivative is replaced by the equality, then they do not provide any guideline how to chose a function $p(.)$ in $v^{(1)}(.) = -p(.)$ {or equivalently, in $u^{(1)} = -p(.)[1 - u(.)]$}. Once we understand this, it appears clear that we meet two subproblems: a) what are properties of the system and of the function $p(.)$ to garantee existence of a solution to the differential equation, and b) what are, relative to a selected $p(.)$, the necessary and sufficient conditions for a solution $v(.)$ $\{u(.)\}$, respectively, to guarantee uniform asymptotic stability of an invariant set and/or to determine accurately its domain of uniform asymptotic stability. The former problem is purely mathematical problem that is not related to the stability issue. However, the latter one is crucial for solving the stability problems.

Bhatia [2, 3], Bhatia and Lazer [4], Bhatia and Szegö [5], Corne and Rouche [6], Hajek [7, 8], Ladde *et al.* [9], Ladde and Leela [10, 11], Lakshmikantham and Leela [12, 13], LaSalle [14], Yoshizawa [15 – 18] and Zubov [19] extended the classical Lyapunov methodology from the analysis of stability properties of a state and of a motion to the analysis of various stability properties of sets.

A novel Lyapunov methodology for asymptotic stability analysis of the zero equilibrium state of non-linear time-invariant systems was discovered and established in [20 – 32]. It was extended to the asymptotic stability analysis of the zero equilibrium state of non-linear time-varying systems in [33 – 35], as well as of constant sets of non-linear time-invariant systems in [36] and of those time-varying in [37]. It has been aimed at solving the open stability problems. The methodology starts with a determination of a functional family $L(.)$ $[E(.)]$ of functions $p(.)$ that can be used to generate a function $v(.)$ [or, $u(.)$]. An important feature of the novel Lyapunov methodology and of the functional family as its tool is that it permits an *arbitrary selection* of a function $p(.)$ in the family in order to determine exactly a system Lyapunov function. Its another important characteristic is that it provides stability conditions that are not expressed in terms of existence of a system Lyapunov function. The methodology terminates by verifying the properties of $v(.)$ [or, $u(.)$], which are both necessary and sufficient for asymptotic stability of the zero state (or, of a time-invariant set), and/or for a set N to be the domain of its asymptotic stability. This methodolgy is consistent with Lyapunov's original methodlogy for time-invariant linear systems and has been called the *consistent Lyapunov methodology (for both linear and non-linear systems)* [37, 38].

The paper [38] further broadened the consistent Lyapunov methodology by presenting the complete solutions for uniform asymptotic stability of invariant sets of time-varying non-linear systems with differentiable motions. The class of systems will be enlarged in what follows by allowing for system motions to be non-differentiable.

The structure is the following: there are eight sections, an appendix and a list of references in the paper. A brief explanation of the notation is available in the next section. The relaxed smoothness properties of the systems are explained in Section 3 that is on the system description. Various stability domains are defined in Section 4. Functional families $L(.)$ and $E(.)$ are introduced in Section 5. The key part are Section 6, which contributes with new criteria for asymptotic stability domains of the sets, and Section 7 that presents the analogous conditions for uniform asymptotic stability of the sets. This order of the Sections 7 and 8 eases significantly their proofs. The criteria expose the consistent Lyapunov methodology. The conclusions compose Section 8. Appendix preceeds the list of references, which terminates the paper.

2 Notation

Capital italic Roman letters are used for sets, lower case block Roman characters for vectors, Greek letters and lower case italic letters denote scalars except for the empty set $\emptyset$ and subscripts. The boundary, interior and closure of a set A are designated by ∂A, $\operatorname{In} A$ and $\operatorname{Cl} A$, respectively, where A is time-invariant set. If $A(.)\colon R \to 2^{R^n}$ is a set-valued function then its instantaneous set value $A(t)$ at an arbitrary time $t \in R$ will be called a time-varying set $A(t)$. Let $\|.\|\colon R^n \to R_+$ be Euclidean norm on R^n, where $R_+ = [0,\infty) = \{\xi\colon \xi \in R,\ 0 \le \xi < \infty\}$. An initial time $t_0 \in R_i$, where $R_i = (\sigma,\infty)$, $\sigma \in [-\infty,\infty)$. It determines $R_0 = [t_0,\infty)$. Let $R^+ = (0,\infty) = \{\xi\colon \xi \in R,\ 0 < \xi < \infty\}$.

A set J, $J \subset R^n$, will be a compact connected invariant set of the system with the boundary ∂J being also an invariant set. Its time-varying neighbourhood at time $t \in R$ will be denoted by $A(t;J)$, $M(t;J)$ or $S(t;J)$, and its δ-neighbourhood will be designated by $B_\delta(J)$, where $\delta \in R^+$ and $B_\delta(J) = \{x\colon \rho(x,J) < \delta\}$ with the distance function $\rho(.)\colon R \times 2^{R^n} \to R_+$ induced by $\|.\|$ as $\rho(x,J) = \inf\{\|x-y\|\colon y \in J\}$. Notice that $J \subset A(t;J)$, $\forall\, t \in R$, and $J \subset B_\delta(J)$. Besides, $M_m(R_i;J) = \cap[M(t;J)\colon t \in R_i]$, $M_M(R_i;J) = \cup[M(t;J)\colon t \in R_i]$ and $S(R_i;J) = \cap[S(t;J)\colon t \in R_i] = S_m(R_i;J)$. The distance between sets $M_1(t;J)$ and $M_2(t;J)$ at time t is the instantaneous value of a set-distance function $\rho(.)$ at time t, $\rho(.)\colon 2^{R^n} \times 2^{R^n} \to R_+$, where $\rho[M_1(t;J), M_2(t;J)] = \max\{\sup[\rho(x, M_1(t;J))\colon x \in M_2(t;J)],\ \sup[\rho(y, M_2(t;J))\colon y \in M_1(t;J)]\}$.

Let $t_k \to \tau$ as $k \to \infty$, where in special cases of an unbounded value of t:

$$\begin{aligned} t_k < \tau \quad &\text{if} \quad \tau = \infty, \\ t_k > \tau \quad &\text{if} \quad \tau = -\infty. \end{aligned}$$

A non-empty set-valued function $M(.)\colon R \times 2^{R^n} \to 2^{R^n}$ is continuous at $\tau \in R$ if and only if for every $\varepsilon \in R^+$ there is $L \in \{1,2,\dots\}$, $L = L(\varepsilon;\tau)$, such that $k > L$ implies $d\{M(t_k;J), M(\tau;J)\} < \varepsilon$. It is continuous on $R_{(.)}$ if and only if it is continuous at every $t \in R_{(.)}$, which is denoted by $M(t;J) \in C(R_{(.)})$. The time-varying set $M(t;J)$ is non-empty, connected and/or compact on $R_{(.)}$ if and only if it is non-empty, connected and/or compact at every $t \in R_{(.)}$, respectively.

$D_a(t;J)$, $D_s(t;J)$ and $D(t;J)$ will represent the (instantaneous) domain of attraction of the set J at time t, its domain of stability at time t and its domain of asymptotic stability at the same time t, respectively. Their definitions are given in Section 4.

Let $\boldsymbol{x}(.;t_0,x_0)$ be motion (solution) of a system through x_0 at an initial time t_0, and let its vector value at time t be $x(t)$, $x(t) = \boldsymbol{x}(t;t_0,x_0)$.

If a function $v(.)\colon R \times R^n \times 2^{R^n} \to R$ is continuous on $R \times R^n$ then we will use its right-hand Dini derivative $D^+v(t,x;J)$ taken along system motions and determined at $(t,x) \in R \times R^n$ with J being fixed:

$$D^+v(t,x;J) = \limsup\left\{\frac{v[t+\theta, \boldsymbol{x}(t+\theta;t,x);J] - v(t,x;J)}{\theta}\colon\ \theta \to 0^+\right\}.$$

Let $\zeta \in R^+$ and $p(.)$, $[v(.)]\colon R \times R^n \times 2^{R^n} \to R$. Then $P_\zeta(t;J)$, $[V_\zeta(t;J)]$ is the largest open connected neighbourhood of J at time $t \in R$ such that $p(t,x;J) < \zeta$, $[v(t,x;J) < \zeta]$ for every $x \in P_\zeta(t;J)$, $[V_\zeta(t;J)]$.

K is the family defined by Hahn [39] of all the comparison functions $\varphi(.)\colon R_+ \to R_+$ strictly increasing, continuous and vanishing at the origin:

$$\varphi(\zeta_1) < \varphi(\zeta_2), \quad 0 \le \zeta_1 < \zeta_2, \quad \varphi(\zeta) \in C(R_+), \quad \varphi(0) = 0.$$

3 System Description

Time-varying non-linear systems studied herein in general are described by (1),

$$\frac{dx(t)}{dt} = f(t, x(t)), \quad x(.)\colon R \to R^n, \quad f(.)\colon R \times R^n \to R^n, \tag{1}$$

and by one of the following features:

Weak smoothness property

(i) There is an open continuous connected neighbourhood $S(t;J)$ of J, $S(t;J) \subseteq R^n$, for every $t \in R_i$, such that $S(R_i;J) = \cap[S(t;J)\colon t \in R_i]$ is also open connected neighbourhood of J, and for every $(t_0, x_0) \in R_i \times S(t_0;J)$ the following holds:
   a) system (1) has a unique solution $\boldsymbol{x}(.;t_0,x_0)$ through x_0 at t_0 on the largest interval of its existence I_0, $I_0 = I_0(t_0,x_0)$, and
   b) $\boldsymbol{x}(t;t_0,x_0)$ is defined and continuous in (t,t_0,x_0).

(ii) For every $(t_0,x_0) \in R_i \times [R^n - \mathrm{Cl}\, S(t_0;J)]$ every motion $\boldsymbol{x}(.;t_0,x_0)$ of system (1) is continuous in $t \in I_0$.

Strong smoothness property

(i) System (1) obeys the weak smoothness property.

(ii) If the boundary $\partial S(t;J)$ of $S(t;J)$ is non-empty at any time $t \in R_i$ then every motion of system (1) passing through $x_0 \in \partial S(t_0;J)$ at $t_0 \in R_i$ satisfies $\inf\{\rho[\boldsymbol{x}(t;t_0,x_0),J]\colon t \in I_0\} > 0$ for every $(t_0,x_0) \in R_i \times \partial S(t_0;J)$.

Any of the above system smothness properties permits non-differentiability of system motions $\boldsymbol{x}(t;t_0,x_0)$ with respect to (t,t_0,x_0). This makes the difference between what follows and the results established in [38]. The smoothness properties are expressed directly in terms of smoothness of system motions rather than indirectly via smoothness of the function $f(.)$ for the following reasons. Dealing with physical systems we can often conclude on smoothness of their motions for physical reasons. We know only sufficient mathematical conditions on $f(.)$, which guarantee smoothness of system motions. Such conditions can be too conservative.

4 Asymptotic Stability Domains

The notions of various stability domains of states [19, 21 − 28, 32, 39 − 46], and of sets [36], of time-invariant systems were broadened to stability domains of sets of time-varying systems in [38] as follows.

Definition 4.1 A set J of system (1) has:

(a) *the domain of attraction at* $t_0 \in R$ denoted by $D_a(t_0; J)$, $D_a(t_0; J) \subseteq R^n$, if and only if:

1) for every $\zeta \in R^+$, there exists $\tau = \tau(t_0, x_0; \zeta; J) \in R_+$ such that

$$\rho[\boldsymbol{x}(t; t_0, x_0), J] < \zeta \quad \text{for all} \quad t \in (t_0 + \tau, \infty)$$

is valid provided only that $x_0 \in D_a(t_0; J)$,

2) the set $D_a(t_0; J)$ is a neighbourhood of J.

(b) *the domain* $D_a(R_i; J)$ *of uniform attraction on* R_i, $D_a(R_i; J) \subseteq R^n$, if and only if 1)−4) hold:

1) it has the domain $D_a(t_0; J)$ of attraction at every $t_0 \in R_i$,
2) $\cap[D_a(t; J) \colon t \in R_i]$ is a neighbourhood of J,
3) $D_a(R_i; J) = \cap[D_a(t; J) \colon t \in R_i]$,
4) the minimal $\tau(t_0, x_0; \zeta; J)$ obeying 1) of (a) and denoted by $\tau_m(t_0, x_0; \zeta; J)$ obeys

$$\sup[\tau_m(t_0, x_0; \zeta; J) \colon t_0 \in R_i] < +\infty \quad \text{for every} \quad (x_0, \zeta) \in D_a(R_i; J) \times R^+.$$

The expression "on R_i" is to be omitted if and only if $R_i = R$. Then and only then $D_a(R_i; J)$ will be denoted by $D_a(J)$, $D_a(J) = D_a(R; J)$.

Definition 4.2 A set J of system (1) has:

(a) *the domain of stability at* $t_0 \in R_i$ denoted by $D_s(t_0; J)$, $D_s(t_0; J) \subseteq R^n$, if and only if:

1) for every $\varepsilon \in R^+$ the motion $\boldsymbol{x}(.; t_0, x_0)$ satisfies $\rho[\boldsymbol{x}(t; t_0, x_0), J] < \varepsilon$ for all $t \in R_0$ provided only that $x_0 \in D_s(t_0, \varepsilon; J)$,
2) the set $D_s(t_0, \varepsilon; J)$ is a neighbourhood of J for every $\varepsilon \in R^+$,
3) the set $D_s(t_0; J)$ is the union of all the sets $D_s(t_0, \varepsilon; J)$ over $\varepsilon \in R^+$:

$$D_s(t_0; J) = \cup[D_s(t_0, \varepsilon; J) \colon \varepsilon \in R^+].$$

(b) *the domain* $D_s(R_i; J)$ *of uniform stability on* R_i if and only if:

1) J has the domain of stability $D_s(t_0; J)$ at every $t_0 \in R_i$,
2) $\cap[D_s(t, \varepsilon; J) \colon t \in R_i]$ is a neighbourhood of J for any $\varepsilon \in R^+$,
3) $D_s(R_i; J) = \cap[D_s(t; J) \colon t \in R_i]$.

The expression "on R_i" is to be omitted if and only if $R_i = R$. Then and only then $D_s(R_i; J)$ will be denoted by $D_s(J)$, $D_s(J) = D_s(R; J)$.

Definition 4.3 A set J of system (1) has:

(a) *the domain of asymptotic stability at* $t_0 \in R_i$ denoted by $D(t_0; J)$, $D(t_0; J) \subseteq R^n$, if and only if it has both $D_a(t_0; J)$ and $D_s(t_0; J)$, and $D(t_0; J) = D_a(t_0; J) \cap D_s(t_0; J)$.

(b) *the domain* $D(R_i; J)$ *of uniform asymptotic stability on* R_i if and only if it has both $D_a(R_i; J)$ and $D_s(R_i; J)$, and $D(R_i; J) = D_a(R_i; J) \cap D_s(R_i; J)$.

The expression "on R_i" is to be omitted if and only if $R_i = R$. Then and only then $D(R_i; J)$ will be denoted by $D(J)$, $D(J) = D(R; J)$.

Qualitative features of the stability domains of an invariant set J of system (1) are discovered in Appendix. They are used for the proofs of the results of the paper (Section 6).

5 Families $L(.)$ and $E(.)$ of Functions $p(.)$

A fundamental problem which has not been solved in the classical Lyapunov methodology is that of the generation of a system Lyapunov function. The theorems based on the classical Lyapunov methodology (including also the converse theorems) express conditions on the Lyapunov function derivative in the inequality form: $v^{(1)}(.) \le -p(.)$. They do not specify how to select the function $p(.)$ in order to get a system Lyapunov function obeying the weak inequality that may be replaced by the equality in order to ease the function generation. Various forms of families $P(.)$ and $P^1(.)$ of functions $p(.)$ were introduced in [21 − 33, 36, 37] in order to generate Lyapunov functions $v(.)$ from $v^{(1)}(.) = -p(.)$ {or, to determine Lyapunov functions $u(.)$ as solutions of $u^{(1)}(.) = -[1-u(.)]p(.)$} in the framework of time invariant systems, and for time-varying systems in [34, 35, 38]. They will be replaced by families $L(.)$ and $E(.)$ of functions $p(.)$ in the sequel. One role of these families is to separate the problem of existence of the differential equation solution from the stability problem. Another their role is to enable an exact determination of a family of system Lyapunov functions [47, 48].

Definition 5.1 A *function* $p(.)\colon R_i \times R^n \times 2^{R^n} \to R$ *belongs to the family* $L(R_i, S; f; J)$ if and only if:

1) $p(.)$ is continuous on $R_i \times S(t;J)\colon p(t,x;J) \in C[R_i \times S(t;J)]$;
2) the equations (2) with (2a) taken along motions of system (1),

$$D^+ v(t,x;J) = -p(t,x;J), \tag{2a}$$

$$v(t,x;J) = 0, \quad \forall\, x \in \partial J, \quad \forall\, t \in R_i, \tag{2b}$$

have a solution $v(.)\colon R_i \times R^n \times 2^{R^n} \to R$ that is continuous in $(t,x) \in R_i \times \operatorname{Cl} B_\mu(J)$ for an arbitrarily small $\mu \in R^+$, $\mu = \mu(f,p;J)$, and which obeys (3) for some $w_\mu(x;J) \in C[\operatorname{Cl} B_\mu(J)]$:

$$v(t,x;J) \le w_\mu(x;J), \quad \forall\, (t,x) \in R_i \times [\operatorname{Cl} B_\mu(J) - \operatorname{In} J]; \tag{3}$$

3) the following holds for any $\zeta \in R^+$ fulfilling $\operatorname{Cl} P_\zeta(t;J) \subset S(t;J)$ for all $t \in R_i$:

$$\min\{p(t,x;J)\colon (t,x) \in R_i \times [S(t;J) - P_\zeta(t;J)]\} = \alpha, \qquad \alpha = \alpha(\zeta;p) \in R^+.$$

Definition 5.2 A *function* $p(.)\colon R_i \times R^n \times 2^{R^n} \to R$ *belongs to the family* $E(R_i, S; f; J)$ if and only if:

1) $p(.)$ is continuous on $R_i \times S(t;J)\colon p(t,x;J) \in C[R_i \times S(t;J)]$;
2) the equations (4) with (4a) taken along motions of system (1),

$$D^+ u(t,x;J) = -[1-u(t,x;J)]p(t,x;J), \tag{4a}$$

$$u(t,x;J) = 0, \quad \forall\, x \in \partial J, \quad \forall\, t \in R_i, \tag{4b}$$

have a solution $u(.)\colon R_i \times R^n \times 2^{R^n} \to R$ that is continuous in $(t,x) \in R_i \times \mathrm{Cl}\,B_\mu(J)$ for an arbitrarily small $\mu \in R^+$, $\mu = \mu(f,p;J)$, and which obeys (5) for some $w_\mu(x;J) \in C[\mathrm{Cl}\,B_\mu(J)]$:

$$u(t,x;J) \le w_\mu(x;J), \quad \forall\,(t,x) \in R_i \times [\mathrm{Cl}\,B_\mu(J) - \mathrm{In}\,J]; \tag{5}$$

3) the following holds for any $\zeta \in R^+$ fulfilling $\mathrm{Cl}\,P_\zeta(t;J) \subset S(t;J)$ for all $t \in R_i$:

$$\min\{p(t,x;J)\colon (t,x) \in R_i \times [S(t;J) - P_\zeta(t;J)]\} = \alpha, \qquad \alpha = \alpha(\zeta;p) \in R^+.$$

Comment 5.1 Notice that $p(.) \in L(R_i,S;f;J)$ if and only if $p(.) \in E(R_i,S;f;J)$. If $p(.) \in L(R_i,S;f;J)$, hence $p(.) \in E(R_i,S;f;J)$, then solutions $v(.)$ and $u(.)$ to (2) and (4), respectively, are interrelated by (6),

$$u(t,x;J) = 1 - \exp[-v(t,x;J)], \tag{6}$$

which was pointed out by Vanelli and Vidyasagar [45]. Besides, $u(t,x;J) = 0$ if and only if $v(t,x;J) = 0$, and $u(t,x;J) \to 1$ if and only if $v(t,x;J) \to \infty$.

Comment 5.2 No stability condition is imposed on the system and no definiteness requirement is imposed on $p(.)$, $v(.)$ and $u(.)$ in Definition 5.1 and Definition 5.2. Therefore, $L(R_i,S;f;J)$ and $E(R_i,S;f;J)$ are not dependent on a stability property of the system.

6 Domains of Asymptotic Stability Properties of Invariant Sets

The notions of a positive definite function and of a decrescent function relative to a set will be used in the usual sense (c.f. Lyapunov [1], Bhatia and Szegö [5], Yoshizawa [18], Zubov [19], Gruyitch [38], Hahn [39], Krasovskii [49], Rouche *et al.* [50]). Let $\varphi_i(.)$ be a comparison function from the class K defined by Hahn [39]: $\varphi_i(.) \in K$, $i = 1,2$.

Definition 6.1 A *function* $v(.)\colon R \times R^n \times 2^{R^n} \to R$

(a) *is positive definite on* $R_i \times M(t;J)$ *with respect to* J if and only if $M(t;J)$ is open connected neighbourhood of J for all $t \in R_i$ such that there exist $w_1(.)\colon R^n \times 2^{R^n} \to R$ and $\varphi_1(.) \in K$ obeying the following:
   1) $v(t,x;J)$ and $w_1(x;J)$ are uniquely determined by $(t,x) \in R_i \times M(t;J)$ and continuous on $R_i \times M(t;J)$; that is that $v(t,x;J) \in C[R_i \times M(t;J)]$ and $w_1(x;J) \in C[M_M(R_i;J)]$,
   2) $v(t,x;J) = 0$ and $w_1(x;J) = 0$ for all $(t,x) \in R_i \times \partial J$,
   3) $v(t,x;J) \le 0$ for all $(t,x) \in R_i \times \mathrm{In}\,J$,
   4) $v(t,x;J) \ge w_1(x;J) \ge \varphi_1[\rho(x,J)]$ for all $(t,x) \in R_i \times [M(t;J) - \mathrm{In}\,J]$.

(b) *is decrescent on* $R_i \times M(t;J)$ *with respect to* J if and only if $M_m(R_i;J)$ is open connected neighbourhood of J, and there exist $w_2(.)\colon R^n \times 2^{R^n} \to R$ and $\varphi_2(.) \in K$ obeying the following:
   1) $v(t,x;J)$ and $w_2(x;J)$ are uniquely determined by $(t,x) \in R_i \times M(t;J)$ and continuous on $R_i \times M(t;J)$, that is that $v(t,x;J) \in C[R_i \times M(t;J)]$ and $w_2(x;J) \in C[M_M(R_i;J)]$, hence $w_2(x;J) \in C[M_m(R_i;J)]$,
   2) $v(t,x;J) \le w_2(x;J) \le \varphi_2[\rho(x,J)]$ for all $(t,x) \in R_i \times [M_m(R_i;J) - \mathrm{In}\,J]$.

The expression "$R_i\times$" is to be omitted if and only if $R_i = R$, and the expression "$\times M(t;J)$" is to be omitted if and only if $M(t;J)$ is an arbitrarily small open connected neighbourhood of J for all $t\in R_i$.

The functions $w_i(.)$, $i=1,2$, can have the following form: $w_i(x;J)=\varphi_i[\rho(x,J)]$.

The form of problem solutions to be established depends on the smoothness properties of system (1) as well as whether a function $p(.)$ generating a system Lyapunov function is selected from $L(R_i,S;f;J)$ or from $E(R_i,S;f;J)$.

Theorem 6.1 *In order for a compact connected invariant set J of system (1) with the strong smoothness property to have the domain $D(R_i;J)$ of uniform asymptotic stability on R_i, for a set $N(t_0)$, $N(t_0)\subseteq R^n$, to be the domain of its asymptotic stability at any $t_0\in R_i$: $N(t_0)\equiv D(t_0;J)$, and for a set N, $N\subseteq R^n$, to be the domain of its uniform asymptotic stability on R_i, $N=D(R_i;J)$, it is both necessary and sufficient that:*

1) *the set $N(t)$ is open continuous neighbourhood of J and $N(t)\subseteq S(t;J)$ for every $t\in R_i$,*
2) *the set N is a connected neighbourhood of J such that $N=\cap[N(t)\colon t\in R_i]=N_m(R_i;J)\subseteq S(R_i;J)$,*
3) *$f(t,x)=0$ for all $t\in R_i$ is possible only for $x\notin[N(t)-J]$,*
   and
4) *for any decrescent positive definite function $p(.)$ on $R_i\times S(t;J)$ with respect to J, which obeys:*
   - (a) *$p(.)\in L(R_i,S;f;J)$, the equations (2) have the unique solution function $v(.)$ with the following properties:*
     - (i) *$v(.)$ is decrescent positive definite function on $R_i\times N(t)$ with respect to J, and*
     - (ii) *if the boundary $\partial N(t)$ of $N(t)$ is nonempty then $x\to\partial N(t)$, $x\in N(t)$, implies $v(t,x;J)\to\infty$ for every $t\in R_i$,*

   or obeying:
   - (b) *$p(.)\in E(R_i,S;f;J)$, the equations (4) have the unique solution function $u(.)$ with the following properties:*
     - (i) *$u(.)$ is decrescent positive definite function on $R_i\times N(t)$ with respect to J, and*
     - (ii) *if the boundary $\partial N(t)$ of $N(t)$ is nonempty then $x\to\partial N(t)$, $x\in N(t)$, implies $u(t,x;J)\to 1$ for every $t\in R_i$.*

Proof The proof will be a modification and generalization of the proof of Theorem 1 in [38]. The modification results from non-differentiability of system motions, which was requested in [38].

Necessity. Let the invariant set J of system (1) possessing the strong smoothness property have the uniform asymptotic stability domain $D(R_i;J)$ on R_i. Hence, it has also the asymptotic stability domain $D(t_0;J)$ at every $t_0\in R_i$ (Definition 4.1). Definitions 4.1 and 4.3 show that it has also the uniform attraction domain $D_a(R_i;J)$ and the instantaneous attraction domain $D_a(t_0;J)$ at every $t_0\in R_i$. By the definition (Definition 4.3), $D_a(t_0;J)\supseteq D(t_0;J)$ for all $t_0\in R_i$ and $D_a(R_i;J)\supseteq D(R_i;J)$. Besides, $D_a(t_0;J)$ is a neighbourhood of J at every $t_0\in R_i$ and $D_a(R_i;J)$ is also a neighbourhood of J (Definition 4.1). The set $S(t_0;J)$ is a neighbourhood of J at every $t_0\in R_i$ and $S(R_i;J)$ is also a neighbourhood of J (the weak smoothness property).

Hence, $D_a(t_0;J) \cap S(t_0;J) \neq \emptyset$ for all $t_0 \in R_i$ and $D_a(R_i;J) \cap S(R_i;J) \neq \emptyset$. Let us prove $S(t_0;J) \supseteq D_a(t_0;J)$ for every $t_0 \in R_i$. If $S(t_0;J) \supseteq D_a(t_0;J)$ were not true for all $t_0 \in R_i$ then there would exist $t_0 \in R_i$ and $z \in [D_a(t_0;J) - S(t_0;J)]$, which would mean either $z \in [D_a(t_0;J) \cap \partial S(t_0;J)]$ or $z \in [D_a(t_0;J) - \operatorname{Cl} S(t_0;J)]$ due to $D_a(t_0;J) \cap S(t_0;J) \neq \emptyset$ and the fact that $S(t_0;J)$ is open [(i) of the weak smoothness property and (i) of the strong smoothness property]. If $z \in [D_a(t_0;J) \cap \partial S(t_0;J)]$ then $\inf\{\rho[\boldsymbol{x}(t;t_0,z),J]\colon t \in I_0\} > 0$ due to (ii) of the strong smoothness property, which would mean $z \notin D_a(t_0;J)$ and would contradict $z \in [D_a(t_0;J) \cap \partial S(t_0;J)]$. Hence, $z \notin [D_a(t_0;J) \cap \partial S(t_0;J)]$ and $D_a(t_0;J) \cap \partial S(t_0;J) = \emptyset$. If $z \in [D_a(t_0;J) - \operatorname{Cl} S(t_0;J)]$ then $\lim\{\rho[\boldsymbol{x}(t;t_0,z),J]\colon t \to \infty\} = 0$, which together with (i) of the strong smoothness property, (ii) of the weak smoothness property and with $S(t;J) \in C(R_i)$ would imply existence of $t^* \in (t_0,\infty)$ such that $\boldsymbol{x}(t^*;t_0,z) \in \partial S(t^*;J)$. This is impossible as shown above. Assumed t^* does not exist. Hence, $[D_a(t_0;J) - \operatorname{Cl} S(t_0;J)] = \emptyset$, which together with $D_a(t_0;J) \cap \partial S(t_0;J) = \emptyset$ and $D_a(t_0;J) \cap S(t_0;J) \neq \emptyset$ implies $S(t_0;J) \supseteq D_a(t_0;J)$ by having in mind that both $D_a(t_0;J)$ and $S(t_0;J)$ are open neighbourhoods of J [a-1) of Lemma A.1 (Appendix), (i) of the weak smoothness property and (i) of the strong smoothness property]. Therefore, $S(t_0;J) \supseteq D(t_0;J)$ due to $D_a(t_0;J) \equiv D(t_0;J)$ (Lemma A.2, Appendix). Let $N(t_0) \equiv D(t_0;J)$ so that $S(t_0;J) \supseteq N(t_0)$. Hence, $N(t)$ is open continuous neighbourhood of J for all $t \in R_i$ [a-2) of Lemma A.1] and $N = D(R_i;J)$ is connected neighbourhood of J [a-3) of Lemma A.1]. Besides, $N = \cap[N(t)\colon t \in R_i]$ because of $N(t) \equiv D(t;J)$ and $N = D(R_i;J) = \cap[D(t;J)\colon t \in R_i]$. They prove necessity of the conditions 1) and 2). From $D_s(t;J) \supseteq D_a(t;J) \equiv D(t;J) \equiv N(t)$ [a) of Lemma A.2] and Definitions 4.1 − 4.3 it results that there is not an equilibrium state of system (1) in $[N(t) - J]$, $\forall\, t \in R_i$, which implies that $f(t,x) = 0$ for all $t \in R_i$ is possible only for $x \notin [N(t) - J]$ (Proposition 7 in [44]). This proves necessity of the condition 3). From $N(t_0) \equiv D(t_0;J)$ it follows that the interval I_0 of existence of $\boldsymbol{x}(.;t_0,x_0)$ satisfies $I_0 \supseteq R_0$, $\forall\,(t_0,x_0) \in R_i \times N(t_0)$ due to Definitions 4.1 through 4.3. Let $p(.) \in L(R_i,S;f;J)$ be arbitrarily selected positive definite decrescent function on $R_i \times S(t;J)$ with respect to J. Hence, there is $\mu > 0$ such that there exists a solution function $v(.)$ to the equations (2), which is continuous in $(t,x) \in R_i \times B_\mu(J)$ and satisfies (3). Therefore,

$$|v(t,x;J)| < \infty, \quad \forall\,(t,x) \in R_i \times \operatorname{Cl} B_\mu(J). \tag{7}$$

Let $\beta \in (1,\infty)$ and $\zeta \in R^+$ be such that

$$\operatorname{Cl} B_\beta(J) \cap \operatorname{Cl} B_\mu(J) \cap S(t;J) \supset P_\zeta(t;J), \quad \forall\, t \in R_i. \tag{8}$$

Existence of such β and ζ is guaranteed by positive definiteness of $p(.)$ on $R_i \times S(t;J)$ and by the fact that $S(R_i;J)$ is a neighbourhood of J. Let $t_0 \in R_i$ be arbitrary and $\tau \in R_+$, $\tau = \tau(t_0,x_0;\zeta;J;p)$, be such that for any $x_0 \in N(t_0)$ the condition (9) holds,

$$\boldsymbol{x}(t;t_0,x_0) \in \operatorname{Cl} P_\zeta(t;J), \quad \forall\, t \in [t_0+\tau,\infty). \tag{9}$$

Such τ exists due to Definitions 4.1 and 4.3, $x_0 \in N(t_0)$ and $D_a(t_0;J) \equiv D(t_0;J) \equiv N(t_0)$. Notice that $x_0 \in N(t_0)$ yields also

$$\rho[\boldsymbol{x}(\infty;t_0,x_0),J] = 0. \tag{10}$$

Let (2a) be integrated from $t \in R_0$ to ∞,

$$v[\infty, \boldsymbol{x}(\infty; t_0, x_0); J] - v[t, \boldsymbol{x}(t; t_0, x_0); J] = -\int_t^\infty p[\sigma, \boldsymbol{x}(\sigma; t_0, x_0); J]\, d\sigma, \quad \forall (t, x_0) \in R_0 \times N(t_0). \tag{11}$$

Now, invariance of ∂J (by the definition, Section 2), (2b) and (10) enable the transformation of (11) to the next form,

$$v[t, \boldsymbol{x}(t; t_0, x_0); J] = \int_t^{t_0+\tau} p[\sigma, \boldsymbol{x}(\sigma; t_0, x_0); J]\, d\sigma + \int_{t_0+\tau}^\infty p[\sigma, \boldsymbol{x}(\sigma; t_0, x_0); J]\, d\sigma, \quad \forall (t, x_0) \in R_0 \times N(t_0). \tag{12}$$

Invariance of $D_a(t; J)$ with respect to system motions on R_i [a-1) of Lemma A.1], $S(t; J) \supseteq D(t; J) \equiv D_a(t; J) \equiv N(t)$, continuity of $\boldsymbol{x}(t; t_0, x_0)$ in $(t; t_0, x_0) \in I_0 \times R_i \times S(t_0; J)$ [(i-b) of the weak smoothness property], continuity, positive definiteness and decrescency of $p(.)$ on $R_i \times S(t; J)$, the definition of τ, (9), and compactness of $[t, t_0 + \alpha]$ for any $\alpha \in R_+$ imply

$$\left| \int_t^{t_0+\alpha} p[\sigma, \boldsymbol{x}(\sigma; t_0, x_0); J]\, d\sigma \right| < \infty, \quad \forall (\alpha, t, t_0, x_0) \in R_+ \times R_0 \times R_i \times N(t_0). \tag{13}$$

Now, (7) − (9), (12) and (13) for $\alpha = \tau$ yield

$$|v[t, \boldsymbol{x}(t; t_0, x_0); J]| < \infty, \quad \forall (t, t_0, x_0) \in R_0 \times R_i \times N(t_0). \tag{14}$$

Let $t = t_0$ and $x = x_0$ be set in (14). Then,

$$|v(t, x; J)| < \infty, \quad \forall (t, x) \in R_i \times N(t). \tag{15}$$

Continuity of $p(.)$ on $R_i \times S(t; J)$, $p(.) \in L(R_i, S; f; J)$, Definition 5.1, $S(t; J) \supseteq N(t)$, (12) and (15) prove

$$v(t, x; J) \in C[R_i \times N(t)] = C[R_i \times D(t)]. \tag{16}$$

Invariance of $D_a(t; J)$ [a-1) of Lemma A.1], $D_a(t; J) \equiv D(t; J) \equiv N(t)$, continuity of $\boldsymbol{x}(t; t_0, x_0)$ in $(t; t_0, x_0) \in I_0 \times R_i \times D(t_0; J)$, positive definiteness and decrescency of $p(.)$ on $R_i \times N(t)$, $p(.) \in L(R_i, S; f; J)$, (3), the definition of τ and compactness of $[t, t_0 + \tau]$ guarantee existence of $\zeta_k(.) \colon R^n \times 2^{R^n} \to R$, $k = 1, 2$, $\zeta_1(x; J) \in C[N_M(R_i; J)]$ and $\zeta_2(x; J) \in C[N_m(R_i; J)]$, where $N_M(R_i; J) = \cup[N(t; J) \colon t \in R_i]$, $N_m(R_i; J) = \cap[N(t; J) \colon t \in R_i]$ and $\psi_k(.) \colon R^n \times 2^{R^n} \to R$, $k = 1, 2$, such that

$$0 < \varsigma_1(x_0, J) \le \int_t^{t_0+\tau} \psi_1[\boldsymbol{x}(\sigma; t_0, x_0); J]\, d\sigma, \quad \forall (t, t_0, x_0) \in R_0 \times R_i \times [N(t_0) - \mathrm{Cl}\, B_\mu(J)], \tag{17a}$$

$$\infty > \varsigma_2(x_0, J) \ge \int_t^{t_0+\tau} \psi_2[\boldsymbol{x}(\sigma; t_0, x_0); J]\, d\sigma, \quad \forall (t, t_0, x_0) \in R_0 \times R_i \times [N_m(R_i; J) - \mathrm{Cl}\, B_\mu(J)], \tag{17b}$$

and

$$\psi_1(x;J) \in C[N_M(R_i;J)], \qquad \psi_2(x;J) \in C[N_m(R_i;J)], \tag{18a}$$

$$\psi_k(x;J) = 0, \quad \forall\, x \in \partial J, \quad k = 1,2, \tag{18b}$$

$$\begin{aligned} &\psi_1(x;J) > 0, \quad \forall\, x \in [N_M(R_i;J) - J], \\ &\psi_2(x;J) > 0, \quad \forall\, x \in [N_m(R_i;J) - J], \end{aligned} \tag{18c}$$

$$\psi_1(x;J) \le p(t,x), \quad \forall\,(t,x) \in R_i \times [N_M(R_i;J) - \mathrm{In}\, J], \tag{18d}$$

$$p(t,x) \le \psi_2(x;J), \quad \forall\,(t,x) \in R_i \times [N_m(R_i;J) - \mathrm{In}\, J]. \tag{18e}$$

Such functions $\psi_k(.)$ exist due to decrescency and positive definiteness of $p(.)$ on $R_i \times S(t;J)$, $p(.) \in L(R_i,S;f;J)$ and $S(t;J) \supseteq N(t)$. They can be of the form $\psi_k(x;J) = g_k[\rho(x,J)]$, $k = 1,2$, together with $g_k(.)$ in the class K: $g_k(.) \in K$. Let $w_k(.)\colon R^n \times 2^{R^n} \to R$, $k = 1,2$, obey (19),

$$w_k(x;J) \in C(R^n) \quad \text{and} \quad w_k(x;J) = 0, \quad \forall\, x \in \partial J, \quad k = 1,2, \tag{19a}$$

$$0 < w_1(x;J) \le \begin{cases} \varsigma_1(x;J), & \forall\, x \in [N_M(R_i;J) - \mathrm{Cl}\, B_\mu(J)], \\ w_\mu(x;J), & \forall\, x \in [\mathrm{Cl}\, B_\mu(J) - J], \end{cases} \tag{19b}$$

$$w_2(x;J) \ge \begin{cases} \varsigma_2(x;J) + w_\mu(x_\tau;J), \\ \qquad x_\tau = \boldsymbol{x}(\tau;t,x), & \forall\,(t,x) \in R_i \times [N_m(R_i;J) - \mathrm{Cl}\, B_\mu(J)], \\ w_\mu(x;J), & \forall\, x \in [\mathrm{Cl}\, B_\mu(J) - J], \end{cases} \tag{19c}$$

where $w_\mu(.)$ is defined by (3). Now (3), (12), positive definiteness of $p(.)$ on $R_i \times S(t;J)$ with respect to J, invariance of J and (17)–(19) yield the following for $(t_0,x_0) = (t,x)$:

$$w_1(x;J) \le v(t,x;J), \quad \forall\,(t,x) \in R_i \times [N(t) - \mathrm{In}\, J], \tag{20a}$$

$$v(t,x;J) \le w_2(x;J), \quad \forall\,(t,x) \in R_i \times [N_m(R_i;J) - \mathrm{In}\, J], \tag{20b}$$

$$v(t,x;J) \le 0, \qquad \forall\,(t,x) \in R_i \times J, \tag{20c}$$

$$v(t,x;J) = 0, \qquad \forall\,(t,x) \in R_i \times \partial J. \tag{20d}$$

From $p(.) \in L(R_i,S;f;J)$, (2b), (16) and (20) it follows that a solution function $v(.)$ to (2) is decrescent, positive definite and continuous on $R_i \times N(t)$ with respect to J. Let be assumed that there exist two such solutions $v_1(.)$ and $v_2(.)$ of (2). Hence,

$$\begin{gathered} v_1(t_0,x_0;J) - v_2(t_0,x_0;J) = \int_{t_0}^{\infty} \big\{ p[\sigma,\boldsymbol{x}_1(\sigma;t_0,x_0);J] - p[\sigma,\boldsymbol{x}_2(\sigma;t_0,x_0);J] \big\}\, d\sigma, \\ \forall\,(t,x_0) \in R_0 \times N(t_0). \end{gathered} \tag{21}$$

Uniqueness of the motions $\boldsymbol{x}(.;t_0,x_0)$, $\forall\,(t_0,x_0) \in R_i \times N(t_0)$ (the weak smoothness property), $S(t_0;J) \supseteq N(t_0)$ and uniqueness of $p(t,x)$ for every $(t,x) \in R_i \times S(t;J)$ [due to positive definiteness of $p(.)$ on $S(t;J)$] imply

$$\int_{t_0}^{\infty} \big\{p[\sigma,\boldsymbol{x}_1(\sigma;t_0,x_0);J]-p[\sigma,\boldsymbol{x}_2(\sigma;t_0,x_0);J]\big\}\,d\sigma$$

$$=\int_{t_0}^{\infty} \big\{p[\sigma,\boldsymbol{x}(\sigma;t_0,x_0);J]-p[\sigma,\boldsymbol{x}(\sigma;t_0,x_0);J]\big\}\,d\sigma=0, \qquad \forall\,(t,x_0)\in R_0\times N(t_0).$$

This and (21) prove

$$v_1(t_0,x_0;J)\equiv v_2(t_0,x_0;J).$$

Hence, the function $v(.)$ is the unique solution to (2). This completes the proof of necessity of the condition 4-a-i). If $\partial N(t_0)\neq\emptyset$ then let $t_0\in R_i$ be arbitrary and x_k, $k=1,2,\dots$, be a sequence converging to u, $x_k\to u$ as $k\to\infty$, $x_k\in N(t_0)$, for all $k=1,2,\dots$, and $u\in\partial N(t_0)$. Let $\zeta\in R^+$ be arbitrarily chosen so that $N\supset \mathrm{Cl}\,P_\zeta(t;J)$ for all $t\in R_i$. Such ζ exists because $p(.)$ is positive definite on $S(t;J)$, $N\subseteq S(t;J)$, $\forall\,t\in R_i$, and defines $\mathrm{Cl}\,P_\zeta(t;J)$, and because N is a neighbourhood of J. Let τ_k, $\tau_k=\tau(t_0,x_k;\zeta;J)\in R_+$, be the first instant satisfying (22),

$$\boldsymbol{x}(t;t_0,x_0)\in \mathrm{Cl}\,P_\zeta(t;J), \quad \forall\,t\in[t_0+\tau_k,\infty). \tag{22}$$

Existence of such τ_k is ensured by $x_k\in N(t_0)$, $N(t)\equiv D(t)$ and by the fact that $\cap[P_\zeta(t;J)\colon t\in R_i]$ is a neighbourhood of J due to decrescency of $p(.)$ on $R_i\times N(t)$ [42]. Continuity of $\boldsymbol{x}(t;t_0,x_0)$ in $(t;t_0,x_0)\in I_0\times R_i\times S(t_0;J)$ (the weak smoothness property), $S(t_0;J)\supseteq D(t_0;J)\equiv N(t_0)$, positive invariance of $D(t;J)$ [a) of Lemma A.1], the fact that $\cap[D(t;J)\colon t\in R_i]=D(R_i;J)$ is neighbourhood of J [(b) of Definition 4.1 through Definition 4.3] and $x_k\in N(t_0)$ imply

$$\tau_k\to\infty \quad \text{as} \quad k\to\infty. \tag{23}$$

Let $m\in\{1,2,\dots\}$ be such that $x_k\in\{N(t_0)-\mathrm{Cl}\,P_\zeta(t_0;J)\}$ for all $k=m,m+1,\dots$, and $x_k\to\partial N(t_0)$ as $k\to\infty$. Such x_k exists because $N(t_0)=D(t_0)$ is open [a-2) of Lemma A.1] and $N(t_0)\supset \mathrm{Cl}\,P_\zeta(t_0;J)$.

Let α be defined by

$$\alpha=\min\{p(t,x)\colon (t,x)\in R_i\times[S(t;J)-P_\zeta(t;J)]\}. \tag{24}$$

Since $p(.)\in L(R_i,S;f;J)$ then $\alpha\in R^+$. Hence, (12), (22), (24) and the definitions of α and τ_k yield $v(t_0,x_k;J)\geq\alpha\tau_k$, $\forall\,t_0\in R_i$, which together with (23) proves necessity of the condition 4-a-ii). The conditions under 4-b) follow from 4-a) due to (2), (4) and (6). This completes the proof of the necessity part.

Sufficiency. Let all the conditions of Theorem 6.1 hold. Since the function $v(.)$ is the solution to (2), [or, $u(.)$ is the solution to (4)], and it is positive definite and decrescent on $R_i\times N(t)$ with respect to J, $p(.)\in L(R_i,S;f;J)$, [or, $p(.)\in E(R_i,S;f;J)$], and $p(.)$ is decrescent positive definite on $R_i\times N(t)$ with respect to J, then J is uniformly asymptotically stable set on R_i, which is easy to verify by using Definition 4.1 through Definition 4.3 and by following Lyapunov [1], Lakshmikantham and

Leela [13], Yoshizawa [18], Zubov [19], Hahn [39], Grujić *et al.* [42], Miller and Michel [46], Krasovskii [49], Rouche *et al.* [50], Demidovich [51], Halanay [52], Hale [53], Kalman and Bertram [54]. Hence, J has both $D(t_0;J)$ at $t_0 \in R_i$ and $D(R_i;J)$ (Definitions 4.1–4.3) so that $D_a(t_0;J) \equiv D(t_0;J)$ and $D_a(R_i;J) = D(R_i;J)$ (Lemma A.2). In order to show that $N(t_0) \equiv D(t_0;J)$ and $N = D(R_i;J)$ we proceed as follows. The condition (ii) of the strong smoothness property guarantees $D(t_0;J) \subseteq S(t_0;J)$, $t_0 \in R_i$. Let $t_0 \in R_i$ be arbitrary and fixed. If $\partial N(t_0) = \emptyset$ then $N(t_0) = R^n$. Hence, $D(t_0;J) \subseteq N(t_0)$ is then only possible. If $D(t_0;J) \subset N(t_0)$ then $\partial D(t_0;J) \cap N(t_0) \neq \emptyset$ that implies $v(t_0,x;J) \to \infty$ as $x \to \partial D(t_0;J)$ (because the function $v(.)$ is the solution to (2), as shown in the proof of necessity), which contradicts the condition 4-a,i) because of $N(t_0) = R^n$. This implies $\partial D(t_0;J) \cap N(t_0) = \partial D(t_0;J) \cap R^n = \emptyset$. Since $D(t_0;J)$ is an open neighbourhood of J and J is compact connected set, then $D(t_0;J) = R^n$, i.e. $D(t_0;J) = N(t_0)$. Let it be now supposed that $\partial N(t_0) \neq \emptyset$, i.e. $N(t_0) \subset R^n$. If we assume now $\partial D(t_0;J) = \emptyset$, then $D(t_0;J) = R^n$ that implies $\partial N(t_0) \cap D(t_0;J) \neq \emptyset$. This and the condition 4-a,ii) show that there is a set $L \subseteq \partial N(t_0) \cap D(t_0;J)$ such that $v(t_0,x;J) \to \infty$ as $x \to L \subseteq \partial N(t_0) \cap D(t_0;J)$, which is impossible because the function $v(.)$ is the unique solution of (2), which is continuous on $R_i \times D(t;J)$, as shown in details in the necessity part. Assumed $\partial D(t_0;J) = \emptyset$ fails. Let $\partial D(t_0;J) \neq \emptyset$ be considered. If $\partial D(t_0;J) \cap \partial N(t_0) = \emptyset$ then either $D(t_0;J) = N(t_0)$ or $D(t_0;J) \subset N(t_0)$ or $N(t_0) \subset D(t_0;J)$ because both are open neighbourhoods of the set J and their boundaries are nonempty. The last two cases are not possible due to positive definiteness of the function $v(.)$ on $R_i \times N(t)$ and its construction via the equations (2) as shown above. If $\partial D(t_0;J) \cap \partial N(t_0) \neq \emptyset$ then either $\partial D(t_0;J) = \partial N(t_0)$, which implies $D(t_0;J) = N(t_0)$, or $\partial D(t_0;J) \cap N(t_0) \neq \emptyset$ and/or $D(t_0;J) \cap \partial N(t_0) \neq \emptyset$. If $\partial D(t_0;J) \cap N(t_0) \neq \emptyset$ then it means that the function $v(.)$ blows up (to ∞) on $N(t_0)$, which contradicts its continuity on $N(t_0)$ due to the condition 4-a,i). If $D(t_0;J) \cap \partial N(t_0) \neq \emptyset$ then it means that the function $v(.)$ blows up on $D(t_0;J)$ that is impossible due to (16) because $v(.)$ is generated by (2). Hence, $\partial D(t_0;J) = \partial N(t_0)$ that implies $D(t_0;J) = N(t_0)$, which holds as the overall result. Now, $N = \cap[N(t)\colon t \in R_i]$ (the condition 2) and the conditions b-3 of Definitions 4.1 and 4.2 together with b) of Definition 4.3 imply $D(R_i;J) = N$. Positive definiteness of $p(.)$ on $S(t;J)$, $p(.) \in L(R_i,S;f;J)$, the equation (2a), $N(t) \subseteq S(t;J)$ for all $t \in R_i$, the condition 4-a,i) and a) of Lemma A.1 imply

$$v[t_0+\tau, \boldsymbol{x}(t_0+\tau,t_0,x_0);J] \le v(t_0,x_0;J) - \xi(\varsigma;p;v;N;R_i)\tau(t_0,x_0;\varsigma;J;p),$$

where $\zeta \in R^+$ is arbitrarily small,

$$\xi(\varsigma;p;v;N;R_i) = \min\{p(t,x;J)\colon (t,x) \in R_i \times [N - V_\psi(R_i;J)]\} \in R^+, \quad \psi = \varphi_1(\varsigma),$$
$$V_\psi(R_i;J) = \cap[V_\psi(t;J)\colon t \in R_i],$$
$$\varphi_1(\|x\|) \le v(t,x;J), \quad \forall\,(t,x) \in R_i \times N(t), \quad \varphi_1(.) \in K,$$
$$v(t,x;J) \le \varphi_2(\|x\|), \quad \forall\,(t,x) \in R_i \times N, \quad \varphi_2(.) \in K,$$

so that

$$\tau(t_0,x_0;\varsigma;J;p) \le [\varphi_2(\|x_0\|) - \varphi_1(\varsigma)]\xi^{-1}(\varsigma;p;v;N;R_i),$$
$$\sup[\tau_m(t_0,x_0;\varsigma;J;p)\colon t_0 \in R_i] \le [\varphi_2(\|x_0\|) - \varphi_1(\varsigma)]\xi^{-1}(\varsigma;p;v;N;R_i) < \infty, \quad \forall\,x_0 \in N,$$

and, therefore, the conditions under (b) of Definitions 4.1 through 4.3 are satisfied. This completes the proof of sufficiency of the conditions 1-4a). Sufficiency of the conditions 1-3,4b) is implied by sufficiency of 1-4a) and (6), which completes the proof.

If system (3.1) possesses the weak smoothness property then conditions of Theorem 6.1 change.

Theorem 6.2 *In order for a compact connected invariant set J of system (1) with the weak smoothness property to have the domain $D(R_i; J)$ of uniform asymptotic stability on R_i, for a set $N(t_0)$, $N(t_0) \subseteq S(t_0; J)$ for all $t_0 \in R_i$, to be the domain of its asymptotic stability at $t_0 \in R_i$: $N(t_0) \equiv D(t_0; J)$, and for a set N, $N \subseteq S(R_i, J)$, to be the domain of its uniform asymptotic stability on R_i, $N = D(R_i; J)$, it is both necessary and sufficient that the following holds:*

1) *the set $N(t)$ is open continuous neighbourhood of J for every $t \in R_i$,*
2) *the set N is a connected neighbourhood of J such that $N = \cap[N(t)\colon t \in R_i] = N_m(R_i; J)$,*
3) *$f(t, x) = 0$ for all $t \in R_i$ is possible only for $x \notin [N(t) - J]$,* *and*
4) *for any decrescent positive definite function $p(.)$ on $R_i \times R^n$ with respect to J, which obeys:*
   (a) *$p(.) \in L(R_i, R^n; f; J)$, the equations (2) have the unique solution function $v(.)$ with the following properties:*
      (i) *$v(.)$ is decrescent positive definite function on $R_i \times R^n$ with respect to J,*
      (ii) *if the boundary $\partial N(t)$ of $N(t)$ is nonempty then $x \to \partial N(t)$, $x \in N(t)$, implies $v(t, x; J) \to \infty$ for every $t \in R_i$,*

   or obeying:
   (b) *$p(.) \in E(R_i, R^n; f; J)$, the equations (4) have the unique solution function $u(.)$ with the following properties:*
      (i) *$u(.)$ is decrescent positive definite function on $R_i \times N(t)$ with respect to J,*
      (ii) *if the boundary $\partial N(t)$ of $N(t)$ is nonempty then $x \to \partial N(t)$, $x \in N(t)$, implies $u(t, x; J) \to 1$ for every $t \in R_i$.*

Proof The proof will be a modification and generalization of that of Theorem 2 in [38]. The modification is caused by non-differentiability of system motions, which was assumed in [38].

Necessity. Let system (1) possess the weak smoothness property. Let the invariant set J have the uniform asymptotic stability domain $D(R_i; J)$ on R_i so that it has also the asymptotic stability domain $D(t_0; J)$ at every $t_0 \in R_i$. Let $N(t_0) = D(t_0; J) \subseteq S(t_0; J)$ for all $t_0 \in R_i$ so that $D(R_i; J) \subseteq S(J)$ and $N = D(R_i; J)$. Let a positive definite decrescent function $p(.)$ on $R_i \times R^n$ with respect to J be arbitrarily selected so that $p(.) \in L(R_i, R^n; f; J)$, {or, $p(.) \in E(R_i, R^n; f; J)$}. From now on we should repeat the proof of necessity of the conditions of Theorem 6.1 in order to complete this proof.

Sufficiency. Let system (1) possess the weak smoothness property and let the conditions 1)$-$4) hold. The set J is uniformly asymptotically stable on R_i, which can be easily verified by following Lyapunov [1], Lakshmikantham and Leela [13], Yoshizawa [18], Zubov [19], Hahn [39], Grujić *et al.* [42], Miller and Michel [46], Krasovskii [49], Rouche *et al.* [50], Demidovich [51], Halanay [52], Hale [53], Kalman and Bertram [54]. Hence, J has both the domain $D(R_i; J)$ of uniform asymptotic stability and the domain $D(t_0; J)$ of asymptotic stability at $t_0 \in R_i$ (Definition 4.3). Let $x_0 \in [R^n - N(t_0)]$ and $t_0 \in R_i$ be arbitrary. Continuity of $\boldsymbol{x}(t; t_0, x_0)$ in $t \in R_0$ (the weak smoothness property), continuity of $p(.)$ on $R_i \times R^n$ due to its positive definiteness on $R_i \times R^n$, the generation of $v(.)$ via (2) and the condition 4-a-ii), [4-b-ii)] imply $\boldsymbol{x}(t; t_0, x_0) \in [R^n - N(t)]$ for all $t \in I_0$. Therefore, $D(t_0; J) \subseteq \mathrm{Cl}\, N(t_0)$ and $D(R_i; J) \subseteq \mathrm{Cl}\, N$. Since $v(.)$ is generated via (2) then (as shown in the proof of the necessity part of Theorem 6.1) $v(t, x) \to \infty$ as $x \to \partial D(t; J)$, $x \in D(t; J)$, for every $t \in R_i$, which, together with the condition 4-a-1) proves $\partial D(t; J) \cap N(t) = \emptyset$ for every $t \in R_i$. This result, $D(t; J) \subseteq \mathrm{Cl}\, N(t)$, and the fact that both $N(t)$ and $D(t; J)$ are open neighbourhoods of J [condition 1) and a-2) of Lemma A.1] imply $N(t) \equiv D(t; J)$ and $N = D(R_i; J)$. By repeating the end of the proof of sufficiency of Theorem 6.1 we show that

$$\sup[\tau_m(t_0, x_0, \zeta; J)\colon t_0 \in R_i] < +\infty \quad \text{for every} \quad (x_0, \zeta) \in D_a(R_i; J) \times R^+,$$

which completes the proof.

Theorems 6.1 and 6.2 are based on the usage of $p(.) \in L(.)$, $\{p(.) \in E(.)\}$. The function $p(.)$ should obey the condition 3) of Definition 5.1, [3) of Definition 5.2], if we wish to determine exactly $D(t; J)$ and $D(R_i; J)$. Such a condition is not necessary for the test of only uniform asymptotic stability of J.

7 Uniform Asymptotic Stability of Invariant Sets

Uniform stability properties of time-varying systems are interesting for their independence of the initial moment t_0, which is a characteristic of stability properties of time-invariant systems.

Theorem 7.1 *In order for a compact connected invariant set J of system (1) possessing the weak smoothness property to be uniformly asymptotically stable on R_i it is both necessary and sufficient that there is an open connected neighbourhood A of J such that the following is valid:*

1) *$f(t, x) = 0$ for all $t \in R_i$ is possible only for $x \notin (A - J)$,*
2) *for any decrescent positive definite function $p(.)$ on $R_i \times A$ with respect to J, which obeys the conditions 1) and 2) of Definition 5.1, the equations (2) have a unique solution function $v(.)$ that is decrescent positive definite function on $R_i \times A$ with respect to J.*

Proof The proof will be a modification of that of Theorem 3 in [38]. The modification is due to non-differentiability of system motions, which was demanded in [38].

Necessity. Let system (1) possess the weak smoothness property. Let the invariant set J be uniformly asymptotically stable on R_i so that it has the domain $D(R_i; J)$ of uniform asymptotic stability (Definitions 4.1 through 4.3). Necessity of the condition 1) is proved in the same way as in the proof of Theorem 6.1. Since $D(R_i; J)$ and $S(R_i; J)$ are neighbourhoods of J then $D(R_i; J) \cap S(R_i; J) \neq \emptyset$. Let M be an open connected neighbourhood of J, which obeys $M \subseteq D(R_i; J) \cap S(R_i; J)$, and let $p(.)$ be an arbitrary decrescent positive definite function on $R_i \times M$ obeying the conditions 1) and 2) of

Definition 5.1. Hence, there exist positive definite functions $\psi_k(.)$ with respect to J, $\psi_k(.)\colon R^n \times 2^{R^n} \to R,\ k = 1,2$, which satisfy (25),

$$\psi_1(x;J) \leq p(t,x;J) \leq \psi_2(x;J), \quad \forall\,(t,x) \in R_i \times (M - \operatorname{In} J). \tag{25}$$

From the conditions 1) and 2) of Definition 5.1 it results that there is a solution $v(.)$ to (2), which is well defined and continuous on $\operatorname{Cl} B_\mu(J)$ and obeys (3). The set $L = M \cap B_\mu(J)$ is also open and connected neighbourhood of J and $L \subseteq D(R_i;J)$. Let $\varepsilon \in R^+$ be arbitrarily selected so that $B_\varepsilon(J) \subseteq L$. Hence, $B_\varepsilon(J) \subseteq D(R_i;J)$. Let $\rho \in R^+$ obeying $B_\rho(J) \subseteq D_s(\varepsilon;J),\ D_s(\varepsilon;J) = \cap\{D_s(t_0,\varepsilon;J)\colon t_0 \in R_i\}$ (Definitions 4.2 and 4.3), be arbitrarily selected. By following the proofs of (15) and (16) we prove that the function $v(.)$ has the next property since $B_\rho(J) \subseteq D_s(\varepsilon;J) \subseteq B_\varepsilon(J) \subseteq L \subseteq M$ [the second inclusion is implied by the definition of $D_s(\varepsilon;J)$],

$$|v(t,x;J)| < \infty, \quad \forall\,(t,x) \in R_i \times B_\rho(J), \quad v(t,x;J) \in C[R_i \times B_\rho(J)]. \tag{26}$$

By following the proof of (20) we show that there are $w_k(x;J) \in C[B_\rho(J)],\ w_k(x;J) = 0$ for every $x \in \partial J$ and $w_k(x;J) > 0$, for every $x \in [B_\rho(J) - J],\ k = 1,2$, such that

$$w_1(x;J) \leq v(t,x;J) \leq w_2(x;J), \quad \forall\,(t,x) \in R_i \times [B_\rho(J) - \operatorname{In} J]. \tag{27}$$

The results (26), (27), $w_k(x;J) \in C[B_\rho(J)]$, and $w_k(x;J) = 0$ for every $x \in \partial J$ and $w_k(x;J) > 0$, for every $x \in [B_\rho(J) - J],\ k = 1,2$, prove that the solution $v(.)$ to (2) is decrescent positive definite function on $R_i \times A$, for $A = B_\rho(J)$. Its uniqueness is proved in the same way as in the proof of the necessity part of Theorem 6.1. Hence, all the conditions are necessary for uniform asymptotic stability of J on R_i.

Sufficiency. Sufficiency of the conditions of Theorem 7.1 for uniform asymptotic stability of J on R_i of system (1) is easy to verify by following Lyapunov [1], Lakshmikantham and Leela [13], Yoshizawa [18], Zubov [19], Hahn [39], Grujić *et al.* [42], Miller and Michel [46], Krasovskii [49], Rouche *et al.* [50], Demidovich [51], Halanay [52], Hale [53], Kalman and Bertram [54], or by following the proof of sufficiency of the conditions of Theorem 6.2.

Comment 7.1 The theorems are valid for global uniform asymptotic stability of J if $S(t;J) \equiv R^n$ without demanding radial unboundedness of $v(.)$ due to (2) and the properties of $p(.)$.

8 Conclusion

The consistent Lyapunov methodology enables us to construct exactly a system Lyapunov function and to determine accurately the domain of asymptotic stability of invariant sets. This is achieved for non-differentiable time-varying non-linear systems. The results provide the conditions that are both necessary and sufficient, and which are not expressed in terms of existence of a system Lyapunov function. They permit *an arbitrary choice* of a non-differentiable decrescent positive definite function $p(.)$ from the functional family $L[R_i,S;f;J)]$, {or from $E[R_i,S;f;J)]$}. They are formulated in terms of properties of a solution function $v(.)$ to $D^+v(.) = -p(.)$ (2), {or in terms of properties of a solution function $u(.)$ to $D^+u(.) = -[1-u(.)]p(.)$, (4)}, respectively, which are obtained for a selected function $p(.)$. Definitions 5.1 and 5.2 determine the families $L[R_i, S; f; J)]$ and $E[R_i, S; f; J)]$ so that they are independent of a stability property of the system. If an obtained solution $v(.)$, $\{u(.)\}$, is also decrescent positive definite then (Theorem 7.1) the invariant set IS uniformly asymptotically stable. If $v(.)$, $\{u(.)\}$, does not possess any of these features then the invariant set IS NOT uniformly asymptotically stable. The solution to the problem of uniform asymptotic stability is obtained under a *single* application of Theorem 7.1. The same holds for the determination of both the domain of asymptotic stability of the invariant set J at any initial time $t_0 \in R_i$ and for its domain of uniform asymptotic stability (Theorems 6.1 and 6.2). These results generalize those of [38].

The consistent Lyapunov methodology for the non-linear systems is inverse to Lyapunov's original methodology for the non-linear systems. The former is consistent due to its consistency with Lyapunov's methodology for time-invariant linear systems and generalizes it in the framework of both linear and non-linear systems, while the latter is not.

The consistent Lyapunov methodology provides the complete solution to the uniform asymptotic stability problem after its *single* application, which is not guaranteed by Lyapunov's original methodology for non-linear systems. No repetition of the procedure is needed in the former case if the test result is negative.

The consistent Lyapunov methodology can be further developed to other classes of dynamical systems such as discrete-time systems, stochastic systems and those governed by functional-differential or partial differential equations.

Appendix

Lemma A.1 *Let system (1) possess the weak smoothness property and let a compact connected invariant set J be uniformly attractive on R_i with the instantaneous domain $D_a(t; J)$ of attraction obeying $D_a(t; J) \subseteq S(t; J)$ for all $t \in R_i$ and with the domain $D_a(R_i; J)$ of uniform attraction on R_i.*

a) *If $R_i \subset R$ then*
   1) *$(t_0, x_0) \in R_i \times D_a(t_0; J)$ implies $x(t; t_0, x_0) \in D_a(t_0; J)$ for all $t \in R_i$, which means that $D_a(t; J)$ is invariant on R_i,*
   2) *$D_a(t; J)$ is open continuous neighbourhood of J at any $t \in R_i$: $D_a(t; J) \equiv \operatorname{In} D_a(t; J)$, $D_a(t; J) \in C(R_i)$,*
   3) *$D_a(R_i; J)$ is connected neighbourhood of J. If $D_a(t; J) = D_a(R_i; J)$ for all $t \in R_i$ then $D_a(R_i; J)$ is also invariant on R_i.*

b) *If $R_i = R$ then*
   1) *$D_a(t; J)$ is invariant, that is that $(t_0, x_0) \in R_i \times D_a(t_0; J)$ implies $x(t; t_0, x_0) \in D_a(t_0; J)$ for all $t \in R$,*
   2) *$D_a(t; J)$ is open continuous neighbourhood of J at any $t \in R$: $D_a(t; J) \equiv \operatorname{In} D_a(t; J)$, $D_a(t; J) \in C(R)$,*
   3) *$D_a(J)$ is connected neighbourhood of J. If $D_a(t; J) = D_a(J)$ for all $t \in R$ then $D_a(J)$ is also invariant.*

Proof We will follow the proof of Lemma A.1 of [38] in order to show its validity also for non-differentiable time-varying non-linear systems.

Let system (1) possess the weak smoothness property and let a compact connected invariant set J be uniformly attractive on R_i with the instantaneous domain $D_a(t; J)$ of attraction obeying $D_a(t;J) \subseteq S(t;J)$ for all $t \in R_i$ and with the domain $D_a(R_i;J)$ of uniform attraction on R_i. Hence, $D_a(R_i;J) = \cap[D_a(t_0;J)\colon t_0 \in R_i]$ (Definition 4.1).

a) Let t_0 and $t^* \in R_i$, $t_0 \neq t^*$. Let $x^* = \boldsymbol{x}(t^*;t_0,x_0)$ for any $x_0 \in D_a(t_0;J)$. Then, $\boldsymbol{x}(t;t_0,x_0) \to J$ as $t \to \infty$. Since $\boldsymbol{x}(t;t^*,x^*) \equiv \boldsymbol{x}[t;t^*,\boldsymbol{x}(t^*;t_0,x_0)] \equiv \boldsymbol{x}(t;t_0,x_0)$, which is true due to (i) of the weak smoothness property and $D_a(t_0;J) \subseteq S(t_0;J)$, then $\boldsymbol{x}(t;t^*,x^*) \to J$ as $t \to \infty$. Hence, $x^* = \boldsymbol{x}(t^*;t_0,x_0) \in D_a(t^*;J)$ that proves the statement under a-1). Let $\zeta \in R^+$ be such that $B_{2\zeta}(J) \subset D_a(R_i;J)$. It exists due to Definition 4.1b. Let be assumed that $D_a(t;J)$ is not open for all $t \in R_i$. Let there exist $t_0' \in R_i$ and $x_0' \in \partial D_a(t_0';J) \cap D_a(t_0';J)$. Let $\varepsilon \in (0,\zeta/2)$. Then, (i) of the weak smoothness property and $D_a(t_0;J) \subseteq S(t_0;J)$, $t_0 \in R_i$, imply existence of $\theta \in R^+$, $\theta = \theta(t_0',x_0',\varepsilon)$, such that $\|x_0 - x_0'\| < \theta$ ensures $\|\boldsymbol{x}(t_0'+2\sigma';t_0',x_0) - \boldsymbol{x}(t_0'+2\sigma';t_0',x_0')\| < \varepsilon$, where $\sigma' = \tau(t_0',x_0',\zeta)$ (Definition 4.1a). Since $\varepsilon < \zeta/2$ and $\rho[\boldsymbol{x}(t_0'+2\sigma';t_0',x_0');J] < \zeta$ then $\boldsymbol{x}(t_0'+2\sigma';t_0',x_0) \in B_{2\zeta}(J) \subset D_a(R_i;J)$. Hence, $x_0 \in D_a(t_0';J)$. Any x_0 obeying $\|x_0 - x_0'\| < \theta$ may be selected in a θ-neighbourhood of x_0' out of $D_a(t_0';J)$, which is contradicted by the obtained $x_0 \in D_a(t_0';J)$. The former is true and the latter is wrong showing that there are not $t_0' \in R_i$ and $x_0' \in \partial D_a(t_0';J) \cap D_a(t_0';J)$. If $x_0' \in \partial D_a(t_0';J)$ then $x_0' \notin D_a(t_0';J)$. The set $D_a(t_0;J)$ is open for all $t_0 \in R_i$ and it is neighbourhood of J due to Definition 4.1. Therefore, $D_a(t;J) \equiv \operatorname{In} D_a(t;J)$ and it is a neighbourhood of J on R_i. Altogether, $D_a(t;J)$ is open neighbourhood of J on R_i. In order to prove $D_a(t;J) \in C(R_i)$ we will use a contradiction. Let there exist $t_0^* \in R_i$ such that $D_a(t;J)$ is discontinuous at t_0^*. As a consequence, there are $\varepsilon^* \in R^+$ and a sequence $K^* \subseteq \{1,2,\ldots,n,\ldots\}$ such that $t_k \to t_0^*$, $k \to \infty$, $k \in K^*$, and that there is $z^* \in D_a(t_0^*;J)$ for which $\rho[z^*, D_a(t_k;J)] \geq \varepsilon^*$, $\forall\, k \in K^*$, and/or there is $w^* \in D_a(t_k;J)$, $\forall\, k \in K^*$, which obeys $\rho[w^*, D_a(t_0^*;J)] \geq \varepsilon^*$. Let $\xi \in R^+$ obey both $\xi < \varepsilon/2$ and $B_\xi(J) \subseteq D_a(t;J)$ for all $t \in R_i$, which is possible due to uniform attraction of J on R_i [b-2) of Definition 4.1]. Let $m \in K^*$ be such that $t_m > t_0^* + \tau(t_0^*,z^*,\xi/2)$, $t_m \in R_i$. This guarantees (Definition 4.1): $\boldsymbol{x}(t_m;t_0^*,z^*) \in B_{\xi/2}(J)$. Let $\delta = \delta(t_0^*;z^*;m;\xi/2) \in R^+$, $\delta < \xi/2$, and $\psi = \psi(t_0^*;z^*;m;\xi/2) \in R^+$ obey that

$$|t_j - t_0^*| < \psi \quad \text{and} \quad \|x_0 - z^*\| < \delta,\ \ j \in K^* \quad \text{imply} \quad \|\boldsymbol{x}(t_m;t_j,x_0) - \boldsymbol{x}(t_m;t_0^*,z^*)\| < \xi/2,$$

which is possible due to continuity of the system motions [the weak smoothness property: (i-b) and (ii)]. Hence, $\boldsymbol{x}(t_m;t_0^*,z^*) \in B_{\xi/2}(J)$ implies $\boldsymbol{x}(t_m;t_j,x_0) \in B_\xi(J)$. This further yields $\boldsymbol{x}(t_m;t_j,x_0) \in D_a(t_m;J)$ and $x_0 \in D_a(t_j;J)$. Besides, $\|x_0 - z^*\| < \delta < \xi/2 < \varepsilon^*/4$ and $x_0 \in D_a(t_j;J)$ imply $\rho[z^*, D_a(t_j;J)] < \varepsilon^*$ that contradicts $\rho[z^*, D_a(t_k;J)] \geq \varepsilon^*$, $\forall\, k \in K^*$, and disproves existence of time $t_0^* \in R_i$ and $z^* \in D_a(t_0^*;J)$ for which $\rho[z^*, D_a(t_k;J)] \geq \varepsilon^*$, $\forall\, k \in K^*$. In the analogous way we show that there are not w^* and t_0^* as defined above. This proves continuity of $D_a(t;J)$ on R_i. The statement under a-2) is correct. Furthermore, $D_a(R_i;J)$ is neighbourhood of $x = 0$ by definition (Definition 4.1). Its connectedness is proved by contradiction. Let us assume that it is not connected. Then, there are disjoint sets D_{ak}, $k = 1,2,\ldots,N$, such that $D_a(R_i;J) = \cup[D_{ak}\colon k = 1,2,\ldots,N]$. One of D_{ak} is not a neighbourhood of J. Let it be D_{a1} and let D_{am}, $D_{am} \subset D_a(R_i;J)$, $m \in \{2,3,\ldots,N\}$, be connected neighbourhood of J that is possible because J is a compact connected set. Then $x_0 \in D_{a1}$ implies $\boldsymbol{x}(t;t_0,x_0) \to J$ as $t \to \infty$, $\forall\, t_0 \in R_i$. There is $t_1 \in R_0$ such that $\boldsymbol{x}(t_1;t_0,x_0) \notin D_a$ because of continuity of $\boldsymbol{x}(t;t_0,x_0)$ in $t \in R_0$, $\forall\, t_0 \in R_i$, and because D_{a1} is disjoint subset of $D_a(R_i;J)$, which is not neighbourhood of J. However, this is impossible due to $\boldsymbol{x}[t;t_1,x(t_1;t_0,x_0)] \equiv \boldsymbol{x}(t;t_0,x_0) \to J$ as $t \to \infty$. Hence, the assumption on disconnectedness of $D_a(R_i;J)$

is incorrect, which completes the proof of all the statements under a) by noting that invariance of $D_a(R_i; J)$ on R_i results directly from 1) in case $D_a(t; J) = D_a(R_i; J)$ for all $t \in R_i$.

b) The assertions under b) directly follow from those under a) in case $R_i = R$.

Lemma A.2

a) *If a compact connected invariant set* J *of system (1) possessing the weak smoothness property is asymptotically stable at* $t_0 \in R_i$ *and its domain of attraction* $D_a(t_0; J)$ *at* $t_0 \in R_i$ *obeys* $D_a(t_0; J) \subseteq S(t_0; J)$ *then its domains* $D_a(t_0; J)$, $D_s(t_0; J)$ *and* $D(t_0; J)$ *are interrelated by* $D_a(t_0; J) \subseteq D_s(t_0; J)$ *and* $D(t_0; J) = D_a(t_0; J)$ *for all* $t_0 \in R_i$.

b) *If a compact connected invariant set* J *of system (1) possessing the weak smoothness property is uniformly asymptotically stable on* R_i *and its domain* $D_a(R_i; J)$ *of uniform attraction on* R_i *satisfies* $D_a(R_i; J) \subseteq S(R_i; J)$ *then its domains* $D_a(R_i; J)$, $D_s(R_i; J)$ *and* $D(R_i; J)$ *are interrelated by* $D_a(R_i; J) \subseteq D_s(R_i; J)$ *and* $D(R_i; J) = D_a(R_i; J)$.

Proof We will follow the proof of Lemma A.2 of [38] in order to verify its validity also for non-differentiable time-varying non-linear systems.

Let system (1) possess the weak smoothness property and J be its compact connected invariant set.

a) Let the set J be asymptotically stable at t_0 and $D_a(t_0; J) \subseteq S(t_0; J)$. Let $x_0 \in D_a(t_0; J)$ be arbitrary. Time-invariance of J and continuity of $\boldsymbol{x}(t; t_0, x_0)$ in $(t, t_0, x_0) \in R_0 \times R_i \times S(t_0; J)$, $D_a(t_0; J) \subseteq S(t_0; J)$ and $x_0 \in D_a(t_0; J)$ imply $\max\{\rho[\boldsymbol{x}(t; t_0, x_0), J]\colon t \in R_0\} < \infty$. Let $\varepsilon = 2\max\{\rho[\boldsymbol{x}(t; t_0, x0), J]\colon t \in R_0\}$. Hence, $x_0 \in D_s(t_0, \varepsilon; J)$ due to (a-1) of Definition 4.2, which implies $x_0 \in D_s(t_0; J)$ in view of (a-3) of Definition 4.2. Altogether, $x_0 \in D_a(t_0; J)$ yields $x_0 \in D_s(t_0; J)$ that proves $D_a(t_0; J) \subseteq D_s(t_0; J)$ for all $t_0 \in R_i$. This result and (a) of Definition 4.3 complete the proof of the statement under (a).

b) Let the set J be uniformly asymptotically stable on R_i and $D_a(R_i; J) \subseteq S(R_i; J)$. Let $x_0 \in D_a(R_i; J)$ be arbitrary. Hence, $\max\{\rho[\boldsymbol{x}(t; t_0, x_0), J]\colon (t; t_0) \in R_0 \times R_i\} < \infty$ due to time invariance of J and continuity of $\boldsymbol{x}(t; t_0, x_0)$ in $(t, t_0, x_0) \in R_0 \times R_i \times D_a(R_i; J)$. Let $\varepsilon = 2\max\{\rho[\boldsymbol{x}(t; t_0, x_0), J]\colon (t; t_0) \in R_0 \times R_i\} \in R^+$ so that obviously $x_0 \in D_s(\varepsilon, R_i; J) = \cap[D_s(t_0, \varepsilon; J)\colon t_0 \in R_i] \neq \emptyset$. Therefore $x_0 \in D_s(R_i; J)$ (Definition 4.3). The result that $x_0 \in D_a(R_i; J)$ implies $x_0 \in D_s(R_i; J)$ proves $D_a(R_i; J) \subseteq D_s(R_i; J)$ and $D(R_i; J) = D_a(R_i; J)$ (due to Definition 4.3). This completes the proof.

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