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Input-Output Decoupling with Stability for Bond Graph Models

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Abstract: In this paper, the geometric approach and the bond-graph methodology are combined to characterize the structure of square linear systems modeled by bond-graph. A new concept is defined to emphasize the symbolic expressions of the fixed modes of the decoupled model and to design decoupling state feedback laws.

Keywords: *Bond graphs; linear systems; non-interacting control; stability properties.*

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1 Introduction

The bond graph is an appreciated tool for physical systems modelling. Based on power flows representation, it enables the description of the system through energy storage and dissipative elements [10, 16]. In a control objective, the structure of the chosen model is also of greatest importance: closed loop requirements may depend on groups of elements of the open loop model. Refining these parts of the model would enable to meet the control goals more efficiently, provided that these refinements also improve the model accuracy. In an input-output decoupling objective, the aim of this work is to identify, on the bond graph model describing the system, the elements involved in major properties of the control solution.

Suitable tools for both structural analysis and synthesis of input-output decoupling control laws are defined by the geometric approach [1, 22]. In particular, many contributions have been brought about input-output decoupling by regular static state feedback, in which the structure of the open loop model is of greatest interest. This structure specially enables to know whether the model is decouplable [5 − 8, 11, 13]. If so, some poles of the decoupled model are also shown to be independent of the control law, so-called fixed modes [9, 12]. Suitable tools for the structural synthesis of such input-output decoupling control laws are defined by the geometric approach. Using particular state space subspaces [4], the designer may choose the number of degrees of freedom introduced by the control law. An unstable unassigned mode would lead to an unstable decoupled model, making this control strategy unrealistic.

In this paper, thanks to geometric concepts, structural analysis methods are emphasized for the input-output decoupling of linear square bond graph models by regular static state feedbacks. Graphical methods are first developed to determine, in terms of fixed modes, if a stable solution exists for the regular input-output decoupling problem. If so, the bond graph methodology is then used to compute state feedbacks insuring stability of the decoupled model.

2 Basic Concepts for Model Analysis

In this part, the basic concepts for model analysis are recalled with different approaches. These concepts are used in the main part of this paper for the characterization of feedback laws.

Consider square linear time-invariant systems $(\sum) = (A, B, C)$ described by equation

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t), \end{cases} \tag{1}$$

where $x(t) \in \mathcal{X} \approx \mathbb{R}^n$ is the state, $u(t) \in \mathcal{U} \approx \mathbb{R}^m$ is the control input, $y(t) \in \mathcal{Y} \approx \mathbb{R}^m$ is the output to be controlled. The same notation is used for maps and their matrix representations in particular bases $A\colon \mathcal{X} \to \mathcal{X}$, $B\colon \mathcal{U} \to \mathcal{X}$, $C\colon \mathcal{X} \to \mathcal{Y}$. $\mathcal{B}$ is the image of B and $\mathcal{K}$ the kernel of C. System (1) is supposed to be invertible.

2.1 Algebraic approach

The infinite structure allows us to express whether a model is decouplable by a regular static state feedback. The stability property of the decoupled model is deduced from the finite structure. It means that the controlled model can be made stable if the fixed modes are stable. The algebraic way for the study of these two structures is now recalled.

2.1.1 Infinite structure. The infinite structure is characterized by the row and global infinite zero orders.

Definition 2.1 Let n_i be the smallest integer verifying $c_i A^{k-1} B = 0$, $k < n_i$ and $c_i A^{n_i - 1} B \neq 0$, with c_i the i-th row of matrix C.

Definition 2.2 [5] n_i is called the i^{th} row infinite zero order, associated with the i^{th} output variable.

Property 2.1 *n_i is the number of derivations of the i^{th} output variable necessary to make appear explicitly at least one of the control input variables.*

Definition 2.3 [20] Let $T(s)$ be the transfer matrix of system (1). The Smith-McMillan form at infinity of $T(s)$ is the unique matrix $\Phi(s)$ defined by equation

$$T(s) = B_1(s).\Phi(s).B_2(s), \tag{2}$$

with

$$\Phi(s) = \operatorname{diag}\left\{ s^{-n'_1}, s^{-n'_2}, \ldots, s^{-n'_m} \right\},$$

and B_1 and B_2 are two non unique bicausal matrices.

Definition 2.4 [5] The set of non increasing integers $\{n'_1, n'_2, \dots, n'_m\}$ is the set of global infinite zero orders of $(\sum)$. It contains m numbers because the system is invertible.

2.1.2 Finite structure.

Definition 2.5 [17] Let $P(s)$,

$$P(s) = \begin{bmatrix} sI - A & B \\ -C & 0 \end{bmatrix}, \tag{3}$$

be the system matrix of $(\sum)$. The invariant zeros of $(\sum)$ are the zeros of the Smith form of $P(s)$. They also are the roots of $\det P(s)$ because the system is square.

2.2 Geometric approach

Some geometric results are now recalled that define invariant subspaces used in input-output decoupling.

Definition 2.6 [23] A subspace $\mathcal{W}$ is $(A, \mathcal{B})$ invariant subspace if it satisfies the inclusion $A\mathcal{W} \subset \mathcal{W} + \mathcal{B}$.

In terms of state feedback, $\mathcal{W}$ is an $(A, \mathcal{B})$ invariant subspace iff there exists a set $\mathcal{F}(A, B; W)$ of state feedback matrices F such as $(A + BF)\mathcal{W} \subset \mathcal{W}$. Let $\mathcal{L}(A, \mathcal{B}; \Psi)$ be the set of $(A, \mathcal{B})$ invariant subspaces included in the subspace Ψ. This subspace is closed for addition, it thus contains a supremal element.

Property 2.2 *The subspace $\mathcal{L}(A, \mathcal{B}; \Psi)$ contains a unique supremal element denoted as $\mathcal{V}^*(\Psi) = \sup \mathcal{L}(A, \mathcal{B}; \Psi)$.*

The subspace $\mathcal{V}^*(\Psi)$ is the limit of the algorithm (4)

$$\begin{cases} \mathcal{V}^0 = \mathcal{X}, \\ \mathcal{V}^\mu = \Psi \cap A^{-1}(\mathcal{B} + \mathcal{V}^{\mu-1}), \end{cases} \tag{4}$$

called "Controlled Invariant Subspace Algorithm" [23].

For control purposes, a particular set of subspaces is used: $(A, \mathcal{B})$ invariant subspaces included in the kernel of the output matrix, denoted $\mathcal{L}(A, \mathcal{B}; \mathcal{K})$. The supremal element is usually denoted as $\mathcal{V}^* = \sup \mathcal{L}(A, \mathcal{B}; \mathcal{K})$. It can be obtained by using equation (4), with $\Psi = \mathcal{K}$. For control purposes, the orthogonal complement of the subspace $\mathcal{V}^*$ is used in this paper. It is the limit of algorithm (5):

$$\begin{cases} \mathcal{V}^{0\perp} = 0, \\ (\mathcal{V}^{\mu-1})^\perp = \mathcal{K}^\perp + A^t(\mathcal{B}^\perp \cap (\mathcal{V}^{\mu-1})^\perp). \end{cases} \tag{5}$$

There is a fundamental property between $\mathcal{V}^*$ and the observability property of the controlled system. This property can be expressed as:

Property 2.3 *Subspace $\mathcal{V}^*$ is the greatest non observable subspace built with state feedback.*

Stable dynamics are associated with a second set of $(A, \mathcal{B})$ invariant subspaces: stabilizable subspaces.

Definition 2.7 $\mathcal{W}$ is a stabilizable $(A, \mathcal{B})$ invariant subspace iff there exists a set of state feedback matrices $F \in \mathcal{F}(A, B; \mathcal{W})$ verifying equation

$$\sigma(\mathcal{W}|A + BF|\mathcal{W}) \subset C_-. \tag{6}$$

C_- is a set of negative real part eigenvalues. A stabilizable subspace $\mathcal{W}$ is thus an $(A, \mathcal{B})$ invariant subspace with which a state feedback matrix $F \in \mathcal{F}(A, B; \mathcal{W})$ is built such as $(A+BF)$ is stable on $\mathcal{W}$. Suppose $\mathcal{L}^-(A, \mathcal{B}; \mathcal{K})$ the set of stabilizable subspaces included in the kernel of the output matrix. This subspace verifies the following property:

Property 2.4 *The set $\mathcal{L}^-(A, \mathcal{B}; \mathcal{K})$ contains a unique supremal element denoted as $\mathcal{V}^*_{\text{stab}} = \sup \mathcal{L}^-(A, \mathcal{B}, \mathcal{K})$. It satisfies equation*

$$\mathcal{V}^*_{\text{stab}} \subset \mathcal{V}^*. \tag{7}$$

Among all output nulling trajectories, subspace $\mathcal{V}^*_{\text{stab}}$ only characterizes those which are stable. Guarantying in the same time decoupling and stability property of the decoupled system, it will be used for the control law synthesis. Among the set of output nulling trajectories, free dynamics and fixed dynamics are pointed out. They will be characterized for bond graph models.

Other subspaces are briefly used in this paper: $(\mathcal{K}, A)$-invariant subspaces.

Definition 2.8 $\mathcal{W}$ is a $(\mathcal{K}, A)$-invariant subspace iff it satisfies equation (8).

$$A(\mathcal{W} \cap \mathcal{K}) \subset \mathcal{W}. \tag{8}$$

Let $\mathcal{S}(\mathcal{K}, A; \mathcal{B})$ the set of $(\mathcal{K}, A)$-invariant subspaces containing subspace $\mathcal{B}$. This set contains a minimal element denoted as $S^* = \inf \mathcal{S}(\mathcal{K}, A; \mathcal{B})$. It is the limit of algorithm

$$\begin{cases} S^0 = 0, \\ S^\mu = \mathcal{B} + A(\mathcal{K} \cap S^{\mu-1}), \end{cases} \tag{9}$$

called "Conditional Invariant Subspace Algorithm".

As described by equations (4) and (9), subspaces $\mathcal{V}^*$ and S^* are obtained with dual algorithms. The following relation can be written:

Property 2.5 $\sup \mathcal{L}(A, \mathcal{B}; \mathcal{K}) = (\inf \mathcal{S}(\mathcal{B}^\perp, A^t; \mathcal{K}^\perp))^\perp$.

Property 2.6 *For invertible systems, subspaces $\mathcal{V}^*$ and S^* satisfy equation*

$$\mathcal{V}^* + S^* = \mathcal{X}. \tag{10}$$

Property 2.7 *For invertible systems, equation*

$$\dim \mathcal{V}^* = n - \sum_i n'_i \tag{11}$$

is satisfied.

According to equation (11), if $\mathcal{V}^*_i$ is the supremal subspace of subsystem $(\sum_i) = (A, B, c_i)$ included in $\ker c_i$, a basis for each subspace $\mathcal{V}^{*\perp}_i$ is the limit of algorithm (5) with $\mathcal{K} = \ker c_i$ and is given by equation

$$\mathcal{V}^{*\perp}_i = \text{vect}\left\{c_i^t, ..., (c_i A^{n_i - 1})^t\right\}. \tag{12}$$

2.3 Bond graph approach

Let us consider, in the following, bond graph models with complete integral causality assignment. The minimal state vector thus deduced is x whereas the state space equation is described by equation (1). The previous results can be applied on the state space representation. The object of this part is to recall some results about infinite zeros and invariant zeros of such models, directly with a graphical approach. Particularly, the equivalence between null invariant zeros of bond graph models with an integral causality assignment, denoted BGI, and infinite zeros of bond graph models with a derivative causality assignment, denoted BGD, is emphasized.

2.3.1 Infinite structure. Consider bond graph models with an integral causality assignment.

Definition 2.9 The length of a causal path is equal to the number of dynamical elements met when following the path.

Definition 2.10 When they contain at least one dynamical element, two causal paths are said to be different if they do not have any common dynamical element.

Property 2.8 [19] *n_i is equal to the length of the shortest causal path between the i^{th} output detector and all the input sources.*

Property 2.9 [19] *The number of global infinite zeros is equal to the number of different input-output causal paths. Their orders are computed as in equation*

$$\begin{cases} n'_m = L_1, \\ n'_{m-k+1} = L_k - L_{k-1}, \end{cases} \tag{13}$$

where L_k is the sum of the lengths of the k shortest input-output different causal paths.

If there are several choices of m different shortest input-output causal paths, the gains of the shortest different causal paths from at least two output detectors to all the input sources may be proportional. It means that, in this case, the control inputs do not appear independently in the output derivatives. Hence, the integers computed according to equations (13) do not define the global infinite zero orders of the model. For independence between control inputs and output derivatives to be performed, at least one output variable must be derived more times. The order of the i^{th} global infinite zero is thus greater than the length of the shortest causal path from the associated output detector to the input sources. For any invertible bond graph model with m inputs and m outputs, there exists at least one choice of m different input-output causal paths.

2.3.2 Finite structure. Graphical methods allow the characterization of the invariant zeros of $(\sum)$ straight from its bond graph model. Particularly, considering the bond graph model obtained by removing from the initial one each choice of m different input-output causal paths and expressing each characteristic polynomial, one determines the invariant zeros of the global square model [19]. Null invariant zeros can be derived straightforward with a causal approach. A new concept is now defined on the BGD: the i^{th} output infinite zero order n_{id}, associated with the i^{th} output variable. It will be pointed out that n_{id} is equal to the number of null invariant zeros of the i^{th} row subsystem.

Let us assign the derivative causality on the bond graph model of the system. As the derivative causality assignment can be performed, the state matrix A is invertible. A more general approach is proposed in [3]. Hence, the associated mathematical representation is given by equation

$$\begin{cases} x = A^{-1}\dot{x} - A^{-1}Bu, \\ y = CA^{-1}\dot{x} - CA^{-1}Bu. \end{cases} \tag{14}$$

Definition 2.11 Let n_{id} be the smallest integer verifying $c_iA^{-(k+1)}B = 0,\ k < n_{id}$ and $c_iA^{-(n_{id}+1)}B \neq 0$.

n_{id} is thus the number of integrations of the i^{th} output variable necessary to make appear explicitly at least one of the control input variables.

Property 2.10 *n_{id} is equal to the length of the shortest causal path between the i^{th} output detector and all the input sources on the BGD.*

Definition 2.12 n_{id} is called the order of the i^{th} row infinite zero associated with the i^{th} output variable on the BGD.

Extending the previous result to the whole system, let us now define for the BGD the new concept of global infinite zero orders, noted $\{n'_{1d}, ..., n'_{md}\}$.

Definition 2.13 Let $\{n'_{1d}, ..., n'_{md}\}$ the integer set verifying equation

$$\begin{cases} n'_{md} = L_{1d}, \\ n'_{(m-k+1)d} = L_{kd} - L_{(k-1)d}, \end{cases} \tag{15}$$

where L_{kd} is the sum of the lengths of the k shortest different input-output causal paths on the BGD. These integers are called global infinite zero orders of the BGD.

These integers are obtained directly on the BGD with the same approach as the set $\{n'_1, ..., n'_m\}$ on the BGI.

Property 2.11 *n_{id} is equal to the number of null invariant zeros of the i^{th} row subsystem.*

Theorem 2.1 *Let $\{n'_{1d}, ..., n'_{md}\}$ be the set of global infinite zero orders of the BGD. The number of null invariant zeros of the BGI is equal to $\sum_{k=1}^{m} n'_{kd}$.*

The proof of this theorem is proposed in appendix. Note that the BGD has frequently direct input-output causal paths. In that case, several choices of m different shortest input-output causal paths are often found. Then, when computing the integers from the bond graph model with derivative causality assignment, take care of the proportionality between the gains of these causal paths.

3 Regular Static State Feedback Decoupling with Stability

In this part, $(\sum)$ is supposed to be invertible, controllable and observable [18]. A static state feedback control law, described as equation

$$u = Fx + Gv, \tag{16}$$

is applied on equation (1). It is called regular when matrix G is square invertible.

3.1 Algebraic and geometric approaches

Let $\{n_i\}$ be the set of row infinite zero orders and $\{n'_i\}$ the set of global infinite zero orders. If $(\sum)$ is decouplable by a regular static state feedback, this control strategy is called rssf in the next [9].

3.1.1 Structural condition for decoupling with stability. Let Ω be the decoupling matrix defined as in equation

$$\Omega = \begin{bmatrix} c_1 A^{n_1-1} B \\ \vdots \\ c_m A^{n_m-1} B \end{bmatrix}. \tag{17}$$

Property 3.1 [5, 15] *n_i is invariant under rssf.*

Property 3.2 *$(\sum)$ is decouplable by rssf iff Ω is invertible.*

Theorem 3.1 [5] *$(\sum)$ is decouplable by rssf iff equivalent equations*

$$\{n_i\} = \{n'_i\} \tag{18}$$

and

$$\mathcal{V}^* = \bigcap_{i=1}^{m} \mathcal{V}_i^* \tag{19}$$

are satisfied.

When decoupling $(\sum)$ by state feedback, some poles of the decoupled model are unobservable and independent of the control law. They are called fixed modes. These fixed modes are defined straight from the open-loop model [9]: they are all or only some of the invariant zeros of the open-loop model. In order to achieve decoupling with stability, a second set of conditions must be satisfied. Let us denote $Z^+(c_i, A, B)$ the set of unstable invariant zeros of system (c_i, A, B).

Theorem 3.2 [12] *$(\sum)$ is decouplable with stability by rssf iff equations*

$$\begin{cases} \{n_i\} = \{n'_i\}, \\ Z^+(C, A, B) = \sum\limits_{i=1}^{m} Z^+(c_i, A, B) \end{cases} \tag{20}$$

are satisfied.

Theorem 3.3 [12] *$(\sum)$ is decouplable with stability by rssf iff equations*

$$\begin{cases} \mathcal{V}^* = \bigcap\limits_{i=1}^{m} \mathcal{V}_i^*, \\ \mathcal{V}^*_{\text{stab}} = \bigcap\limits_{i=1}^{m} \mathcal{V}^*_{i\,\text{stab}} \end{cases} \tag{21}$$

are satisfied.

3.1.2 Decoupling and disturbance rejection. In this section, symbolic expressions of regular static state feedback control laws $u(t) = Fx(t) + Gv(t)$ insuring input output decoupling are recalled. The geometric approach consists on identifying state subspaces with adequate properties for the control law goal [21].

These methods use geometric supports derived on the control law synthesis for disturbance rejection, and particularly on the concept of decoupling subspace [4]. In a first step, the methodology for disturbance rejection is recalled. Then, this concept is used in order to achieve input-output decoupling, by considering all the control inputs, except one, as a disturbance input for each output variable. Two sets of decoupling subspaces are used in this paper: $\mathcal{V}_i^* = \sup \mathcal{L}\,(A, B; \ker c_i)$ and $\mathcal{V}_{i\,\mathrm{stab}}^* = \sup \mathcal{L}^-\,(A, B; \ker c_i)$. The properties of the associated decoupled system are recalled.

Consider the SISO system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Ed(t), \\ y(t) = cx(t), \end{cases} \tag{22}$$

where $d(.) \in \mathcal{D} \approx \mathcal{R}^q$ is the disturbance.

The goal is to find a control law such that the transfer function matrix from $d(s)$ to $y(s)$ be zero. This goal is achievable if the following theorem is satisfied.

Theorem 3.4 [22] *The output variable $y(t)$ of system (22) can be decoupled from the disturbance vector $d(t)$ iff there exists a $(A, \mathcal{B})$ invariant subspace D satisfying equation*

$$\operatorname{Im} E \subset D \subset \mathcal{V}^* \subset \ker c. \tag{23}$$

D is called decoupling subspace of the disturbance $d(t)$.

The supremal decoupling subspace is $\mathcal{V}^* = \sup \mathcal{L}\,(A, \mathcal{B}; \ker c)$. The controlled system is described by equation

$$\begin{cases} \dot{x}(t) = (A + BF)x(t) + Bv(t) + Ed(t), \\ y(t) = cx(t), \end{cases} \tag{24}$$

where $v(t)$ is the new control input variable.

The state feedback matrix is calculated by considering the following property.

Proposition 3.1 [4] *Consider the SISO system (22) satisfying Theorem 3.4. Suppose that its infinite zero order is such that $n_0 \geq 1$. The feedback matrices F which render $(A + BF)$ invariant each decoupling subspace D are calculated following the equations*

$$\begin{cases} cA^{(n_0-1)}(A + BF) = h, \\ h.D = 0. \end{cases} \tag{25}$$

The column matrix h^t is a linear combination of subspace $D^\perp$ basis vectors. Parameters defining this linear combination are the degrees of freedom in the control law. The number of degrees of freedom is thus equal to $\dim D^\perp$.

According to Theorem 3.4, the supremal decoupling subspace is $\mathcal{V}^* = \sup \mathcal{L}(A, \mathcal{B}; \ker c)$. This solution minimizes the number of degrees of freedom for the control law. A better solution is given by subspace $\mathcal{V}_{\mathrm{stab}}^* \subset \mathcal{V}^*$ – Property 2.4.

Theorem 3.5 [22] *Suppose that system (22) is stabilizable. The output variable $y(t)$ can be decoupled from the disturbance $d(t)$ while guarantying stability iff equation*

$$\operatorname{Im} E \subset \mathcal{V}_{\mathrm{stab}}^* \subset \mathcal{V}^* \tag{26}$$

is satisfied.

In that case, subspace $\mathcal{V}_{\mathrm{stab}}^*$ is used for the calculus of matrix F. Matrix F satisfies the following property.

Property 3.3 [4] *Each state feedback matrix* F *such* $(A+BF)\mathcal{V}^*_{\rm stab} \subset \mathcal{V}^*_{\rm stab}$ *satisfies equation*

$$\sigma(\mathcal{V}^*_{\rm stab}|A + BF|\mathcal{V}^*_{\rm stab}) \subset C_-. \tag{27}$$

These results are right for multivariable systems. Consider now system $(\sum)$ described by equation (1). The decoupled system using a rssf $u(t) = Fx(t) + Gv(t)$ is described by equation

$$\begin{cases} \dot{x}(t) = (A + BF)x(t) + BGv(t), \\ y(t) = Cx(t), \end{cases} \tag{28}$$

with $v(t)$ the new input control vector.

Denote $\overline{v}_i(t)$ the vector $v(t)$ without its ith variable. Given that system (1) is decoupled, each output variable $y_i(t)$ is decoupled from the disturbance vector $\overline{v}_i(t)$, for $i = 1, \dots, m$ at the same time. The following property can than be written.

Property 3.4 [4] *Each* $(A + BF)$ *invariant subspace* D_i *satisfying equation*

$$\operatorname{Im} E_i \subset D_i \subset \mathcal{V}^*_i \subset \ker c_i, \qquad i = 1, \dots, m \tag{29}$$

is associated with each output variable $y_i(t)$ *of the decoupled system (28).*

E_i is the ith column of matrix E. The supremal decoupling subspace is $\mathcal{V}^*_i = \sup \mathcal{L}\,(A, \mathcal{B}; \ker c_i)$. For each of the m SISO subsystems, the control law $u(t) = Fx(t) + Gv(t)$ is such that the disturbance $\overline{v}_i(t)$ is included in a decoupling subspace D_i – equation (29). This subspace is $(A+BF)$-invariant. Properties 2.1 and 3.4 allow the definition of the decoupling control law $u(t) = Fx(t) + Gv(t)$.

Property 3.5 [4] *Consider a system which can be decoupled by a rssf. Let* Ω *be the decoupling matrix,* $\{n_i\}$ *its row infinite structure and* $\{D_i\}$ *a set of subspaces solution for the decoupling problem. A decoupled system is obtained with matrices* F *and* G *and the control law* $u(t) = Fx(t) + Gv(t)$ *following equations*

$$\begin{cases} h_i.D_i = 0, \qquad i = 1, \dots, m, \\ \Omega F = [h_i - c_i A^{n_i}]_{i=1,\dots,m}\,, \\ \Omega G = \operatorname{diag}\,[g_i]_{i=1,\dots,m}. \end{cases} \tag{30}$$

A formal expression of matrices F and G using Maple is derived from the set of decoupling subspaces. g_i, $i = 1, \dots, m$, are freely assignable parameters to choose static gains of the closed loop system. Each row matrix h_i is a linear combination of subspace $D_i^\perp$ basis vector. The number of degrees of freedom is thus function of the choice of the decoupling subspace. Two sets of decoupling subspaces are used in this paper: $\{\mathcal{V}^*_1, \dots, \mathcal{V}^*_m\}$ and $\{\mathcal{V}^*_{1\,\rm stab}, \dots, \mathcal{V}^*_{m\,\rm stab}\}$. These subspaces characterize the properties of the decoupled system.

Property 3.6 [22] *Consider a square controllable system decoupled by a rssf. Choosing* $\{\mathcal{V}^*_1, \dots, \mathcal{V}^*_m\}$ *as the set of decoupling subspaces, the unassignable modes of the decoupled systems are all the invariant zeros of the open loop system. If decoupling with stability is possible, choosing* $\{\mathcal{V}^*_{1\,\rm stab}, \dots, \mathcal{V}^*_{m\,\rm stab}\}$ *as the set of decoupling subspaces, unassignable modes of the decoupled system are strictly stable invariant zeros of the open loop system.*

A given rssf is thus associated with a given set of decoupling subspaces that introduces degrees of freedom used to assign some closed loop modes. The decoupling subspaces enable the choice of the number of the decoupled model unassignable modes.

The control law giving the maximum number of unassignable modes is obtained when taking as decoupling subspaces the greatest ones. The associated unassignable modes are all the invariant zeros of $(\sum)$: one unstable invariant zero makes unstable the decoupled model. However, if decoupling with stability is possible, a stable decoupled model may be designed. In this case, for bond graph models, the set of fixed modes is only composed of all the strictly stable invariant zeros [12].

A graphical necessary and sufficient condition is derived in the next section for the existence of at least a control law insuring stability of the decoupled model. This control law is associated with decoupling subspaces $\{\mathcal{V}^*_{1\,\mathrm{stab}}, \ldots, \mathcal{V}^*_{m\,\mathrm{stab}}\}$. The formal expressions of these decoupling subspaces are then expressed from the bond graph model of $(\sum)$.

Remark 3.1 The state feedback control law creates an unobservable subspace contained in $\mathcal{V}^*$. Given that the system is square, the controllable part in $\mathcal{V}^*$ is empty. All modes which are unobservable are also non controllable modes for the control law. They are non assignable modes in Property 3.6.

3.2 Bond graph approach

If the bond graph model has no invariant zero and if it is rssf, then it is rssf with stability because there are no fixed mode in that case. If some of the invariant zeros are strictly unstable, the problem does not have any solution for bond graph models because these invariant zeros are fixed modes. If none of these invariant zeros are strictly unstable, the only unstable invariant zeros are the null ones: the following study is dealing with this case.

A method allows us to determine if there exists a stable solution for the input-output decoupling problem. This method is based on the study of infinite zeros structures, whose main concepts are now recalled [2].

Then, it is shown how the bond graph formalism allows us to determine the invariant subspaces symbolic expression and then the control law symbolic expression, directly with a graphical approach.

An example is then proposed.

3.2.1 Structural approach analysis. Combining the previous results, Theorem 2.1 and Theorem 3.2 enable to derive a graphical necessary and sufficient condition for $(\sum)$ to be decouplable by rssf with stability.

Theorem 3.6 *Assume that $(\sum)$ does not have any strictly unstable invariant zero. A stable solution for the input-output decoupling of $(\sum)$ thus exists iff the infinite zeros structures of BGI and BGD verify equations*

$$\begin{cases} \{n_i\} = \{n'_i\}\,, \\ \{n_{id}\} = \{n'_{id}\}\,. \end{cases} \tag{31}$$

According to Property 2.11 and Theorem 3.2, the proof is immediate.

Hence, the bond graph model of $(\sum)$ enables to know graphically if a decoupling rssf exists insuring closed loop stability. In the next part, regular decoupling with stability is supposed to be possible.

3.2.2 Control law. As expressed by Property 3.5, the decoupling control law associated with a set of decoupling subspaces is computed thanks to the symbolic expressions of their orthogonal complements. No simple algorithm allows the calculation of these subspaces with a symbolic expression. The bond graph methodology gives a different way. Causal path length concepts on the bond graph models with integral and derivative causality assignment are now used to determine the expressions of the two sets of useful subspaces $\{(\mathcal{V}_i^*)^\perp, \dots, (\mathcal{V}_m^*)^\perp\}$ and $\{(\mathcal{V}_{1\text{stab}}^*)^\perp, \dots, (\mathcal{V}_{m\text{stab}}^*)^\perp\}$.

Consider the bond graph model with integral causality assignment. Let DE_i- resp. DE_{id}- be the i^{th} dynamical element in integral – resp. derivative – causality, associated with the i^{th} state vector component $x_i(t)$ on the bond graph model with integral – resp. derivative – causality assignment. Let be $G_k(DE_i, D_j)$ the constant term, without Laplace operator s, of the gain of a causal path of length k between the i^{th} dynamical element in integral causality DE_i and the j^{th} output detector D_j. Let $g(DE_i)$ be equal to $1/I$ for an I-element and $1/C$ for a C-element.

Property 3.7 $c_j A^k I^i = \sum G_k(DE_i, D_j).g(DE_i)$.

I^i is the identity matrix i^{th} column. From Property 3.7, the formal expressions of the subspaces $\{(\mathcal{V}_i^*)^\perp, \dots, (\mathcal{V}_m^*)^\perp\}$ can be obtained with a graphical manner. Consider now the bond graph model with derivative causality assignment. For $n_{id} \geq 1$, let $\mathcal{V}_{id}^*$ be such that:

$$(\mathcal{V}_{id}^*)^\perp = \text{span}\,\{(c_i A^{-1})^t, ..., (c_i A^{-n_{id}})^t\}. \tag{32}$$

Let $G_{kd}(DE_{id}, D_j)$ be the constant term of the gain of a causal path of length k between the ith dynamical element in derivative causality DE_{id} and the jth output detector D_j.

Property 3.8 $c_j A^{-k} I^i = \sum G_{k-1}(DE_{id}, D_j)$.

From the same graphical way as in Property 3.7, the formal expressions of the subspaces $\{(\mathcal{V}_{1d}^*)^\perp, \dots, (\mathcal{V}_{md}^*)^\perp\}$ can be obtained. The Property 3.9 is deduced from equation (12) and (32).

Property 3.9 $\dim(\mathcal{V}_i^*)^\perp = n_i$ *and* $\dim(\mathcal{V}_{id}^*)^\perp = n_{id}$.

Finally, the symbolic expression of the subspaces $(\mathcal{V}_{i\,\text{stab}}^*)^\perp$ can be derived.

Property 3.10 $(\mathcal{V}_{i\,\text{stab}}^*)^\perp = (\mathcal{V}_i^*)^\perp \oplus (\mathcal{V}_{id}^*)^\perp,\ i = 1, ..., m$.

$\{(\mathcal{V}_{i\,\text{stab}}^*), \dots, (\mathcal{V}_{m\,\text{stab}}^*)\}$ is the set of greatest decoupling subspaces insuring closed loop stability if stable decoupling is possible. The proof of Property 3.10, rather technical, is detailed in the appendix. The bond graph model of $(\sum)$ thus allows us to derive graphically the symbolic expressions of the subspaces needed for the synthesis of input-output (stable) decoupling rssf. An example using the previous analysis and computation methods is now presented.

4 Example

Let us define, Figure 4.1, the bond graph model $BG1$ containing two input sources $\{E_1, E_2\}$, two output detectors $\{D_1, D_2\}$ associated with outputs variable to be controlled

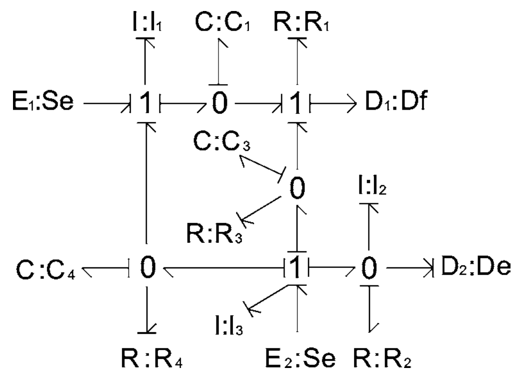

Figure 4.1. Bond-graph model $BG1$.

and six dynamical elements, each with integral causality assignment. This model is invertible, controllable, observable and decouplable by rssf [2].

Assigning a derivative causality on the whole set of dynamical elements leads to the bond graph model $BG2$ described Figure 4.2. The derivative causality can be assigned to each dynamical element. It means that the state matrix is invertible.

Remove from $BG1$ the two shortest different input-output causal paths $D_1 \to R_1 \to C_1 \to I_1 \to E_1$ and $D_2 \to R_2 \to I_3 \to E_2$. The remaining bond graph model contains three dynamical elements: the global model has thus three invariant zeros [19]. Some of these invariant zeros may be null. Studying the global infinite zero structure of $BG2$ allows us to determine graphically their number. The shortest causal path from the output detector D_1 to the input sources does not meet any dynamical element. Thus $n'_{2d} = 0$. Furthermore, there are causal paths of length 1 from the output detector D_2 to the input sources. Due to the R-element R_4, these causal paths are independent of those of length 0 defining n'_{2d}. Hence $n'_{1d} = 1$. Theorem 2.1 so allows to state that the global model has one null invariant zero.

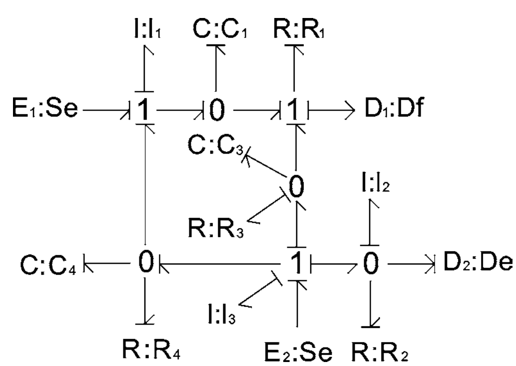

Figure 4.2. Bond-graph model $BG2$.

On $BG1$, finally removing each couple of different input-output causal paths, one computes the two remaining invariant zeros: $s = -1/R_4C_4$ and $s = -1/R_4C_4$ [19]. The three invariant zeros of $BG1$ define the unassignable modes when $(\mathcal{V}_1^*)^\perp$ and $(\mathcal{V}_2^*)^\perp$ are chosen as decoupling subspaces. None of the invariant zeros are strictly unstable.

Computing the row infinite zero orders of $BG2$ thus allows us to know if a stable solution exists for the input-output decoupling of $BG1$. The shortest causal paths from each output detector to the input sources are $D_1 \to R_1 \to E_1$ and $D_2 \to I_2 \to R_4 \to E_2$. Thus $n_{1d} = 0$ and $n_{2d} = 1$. Hence, according to Theorem 3.6, a stable decoupled model may be designed. The suitable rssf leads to a set of fixed modes composed of the only strictly stable invariant zeros.

The decoupling subspaces associated with the two previous decoupling strategies are the subspaces $(\mathcal{V}_i^*)$ and $(\mathcal{V}_{i\,\mathrm{stab}}^*)$, $i = 1, 2$. The expressions of their orthogonal complements are determined according to equations (12), (32) and Property 3.10. Symbolic computations with MAPLE enable the derivation from Property 3.5 of the two associated rssf and the two closed loop transfer matrices, where (a_1, b_1, b_2, b_3) depend of the bond graph parameters. For the first decoupling control law, the closed loop transfer matrix is given by equation

$$T(s) = \begin{bmatrix} g_1/(s^2 + p_1^1 s + a_1 p_0^1) & 0 \\ 0 & g_2/(s + b_1 p_0^2) \end{bmatrix}. \tag{33}$$

As expected, it is a third order matrix: the three remaining modes have been made unassignable. They are the invariant zeros of $BG1$. For the second decoupling control law, the closed loop transfer matrix is given by equation

$$T(s) = \begin{bmatrix} g_1/(s^2 + p_1^1 s + a_1 p_0^1) & 0 \\ 0 & g_2 s/(s^2 + b_2 p_0^2 s + b_3 p_0^2 + p_1^2) \end{bmatrix}. \tag{34}$$

It is a 4^{th} order matrix. As determined by the previous analysis, $s = -1/R_4C_4$ and $s = -1/R_4C_4$ are the fixed modes. According to Property 3.5, p_k^i are degrees of freedom available to tune closed loop dynamics and g_i are parameters used to assign closed loop static gains, $k = 0, 1$, $i = 1, 2$. Note that the model obtained by removing the R-element R_4 would still be decouplable; but closed loop stability could not be performed. Indeed, proportionality between the gains of the shortest different input-output causal paths would change the set of global infinite zero orders: $n'_{2d} = 0$ and $n'_{1d} = 2$. The row infinite zero orders staying unchanged, Theorem 3.6 states that no decoupling rssf exists, achieving closed loop stability.

5 Conclusion

In this paper, structural analysis methods are developed for the input-output decoupling of linear square bond graph models by regular static state feedback.

The poles of the decoupled model are first studied. Due to the non-interaction constraints, some of these poles are fixed: these modes are some of the invariant zeros of the open loop model. Input-output causal path concepts on both bond graph models with integral and derivative causality assignment are used to characterize the symbolic expressions of these invariant zeros. A graphical interpretation of a necessary and sufficient condition is also derived for the input-output decoupling problem with stability to be solvable.

The bond graph methodology is then used to compute a decoupling state feedback insuring stability of the decoupled model, when it is possible. An example is finally presented to detail these analysis and computation methods.

In this paper, it is recalled that the input-output decoupling problem is often achieved with algebraic and geometrical approaches. The bond graph approach is principally based on graphical manipulations and at each step of the procedures information on the model, thus on the physical process, can be analyzed.

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Appendix

A Row invariant zeros

Proof of Property 2.11.

□ Suppose $G_i(s)$ the transfer matrix of the subsystem $\sum(c_i, A, B)$ – equations (35)

$$\begin{cases} G_i(s) = N_i(s)/D(s), \\ N_i(s) = [N_{i1}(s) \dots N_{im}(s)]\,. \end{cases} \tag{35}$$

The state matrix is invertible, thus for bond graph models the subsystem $\sum(c_i, A, B)$ is structurally controllable and observable. The invariant zeros of the subsystem $\sum(c_i, A, B)$ are therefore its null transmission zeros and the null zeros of matrix $G_i(s)$.

The null zeros of matrix $G_i(s)$ are all the zeros of the polynomial matrix $N_i(s)$. These null zeros are zeros of matrix $N_i(s)$ invariant polynomials. As $N_i(s)$ is a row matrix, it has only one invariant polynomial, denoted $\lambda_1^i(s)$. $\lambda_1^i(s)$ is the gcd of the polynomials $\{N_{i1}(s), \dots, N_{im}(s)\}$. The row subsystem $\sum(c_i, A, B)$ null invariant zeros are the common null roots of the transfer matrix numerators. The transfer matrix $G_i(s)$ is given by equation

$$G_i(s) = c_i\,(sI - A)^{-1}\,B. \tag{36}$$

An equivalent expression is equation

$$G_i(s) = c_i\left(sA^{-1} - I\right)^{-1} A^{-1}B. \tag{37}$$

Around $s = 0$, equation (38) can be written

$$\left[\left(sA^{-1} - I\right)^{-1}\right]_{s\to 0} = -\left[I + sA^{-1} + s^2A^{-2} + \dots\right]. \tag{38}$$

A representation of $G_i(s)$ deduced from equations (37) and (38) is given by equation

$$[G_i(s)]_{s\to 0} = -\left[\left(c_iA^{-1}B\right) + \left(c_iA^{-2}B\right)s + \left(c_iA^{-3}B\right)s^2 + \dots\right]. \tag{39}$$

With the new Definition 2.11 of the integer n_{id}, the expression of the matrix $[G_i(s)]_{s\to 0}$ is given by equation

$$[G_i(s)]_{s\to 0} = -\left[(c_iA^{-(n_{id}+1)}B)s^{n_{id}} + (c_iA^{-(n_{id}+2)}B)s^{(n_{id}+1)} + \dots\right]. \tag{40}$$

The expression of matrix $G_i(s)$ around $s = 0$ is given by equation

$$[G_i(s)]_{s\to 0} = -s^{n_{id}}\left[(c_iA^{-(n_{id}+1)}B) + (c_iA^{-(n_{id}+2)}B)s + \dots\right]. \tag{41}$$

n_{id} is equal to the number of common null roots of each transfer matrix $G_i(s)$ numerator. The number of row subsystem $\sum(c_i, A, B)$ null invariant zeros is thus equal to n_{id}. □

B Global invariant zeros

□ Consider an invertible square system $\sum(C, A, B)$. Suppose $P(s)$ its system matrix and $G(s)$ its transfer matrix. These two matrices satisfy equation

$$\det[P(s)] = \det[sI - A] \cdot \det[G(s)]. \tag{42}$$

System $\sum(C, A, B)$ invariant zeros are the roots of $\det[P(s)]$. Given that the state matrix is invertible, the number of null invariant zeros of $\sum(C, A, B)$ is equal to the number of $\det[G(s)]$ null roots.

Consider the bond graph model with a derivative causality assignment (BGD), and its transfer matrix $G_d(s)$ deduced from equations (14). Around $s = 0$, this matrix satisfies equation

$$[G_d(s)]_{s\to 0} = \left[\left(-CA^{-1}B\right) + \left(-CA^{-2}B\right)s + \ldots\right]. \tag{43}$$

Suppose θ'_k the constant coefficient matrix of input output causal path gains of length k in the BGD. Equation (43) can be rewritten as equation

$$[G_d(s)]_{s\to 0} = \sum_{k=0}^{\infty} \theta'_k s^k \tag{44}$$

or equivalently as equation

$$[G_d(1/s)]_{s\to\infty} = \sum_{k=0}^{\infty} \theta'_k / s^k. \tag{45}$$

Suppose a bond graph model with direct transmission between the input sources and the output detectors. The output equation is given by

$$y(t) = Cx(t) + Du(t). \tag{46}$$

On the BGI, around $s \to \infty$ the transfer matrix $G(s)$ is written as equation

$$[G(s)]_{s\to\infty} = \left[D + \frac{CB}{s} + \frac{CAB}{s^2} + \ldots\right]. \tag{47}$$

Suppose θ_k the constant coefficient matrix of input output causal path gains of length k in the BGI. Equation (47) can be rewritten as equation

$$[G(s)]_{s\to\infty} = \sum_{k=0}^{\infty} \theta_k / s^k. \tag{48}$$

With equations (44), (45) and (48), matrices $[G_d(s)]_{s\to 0}$ and $[G(s)]_{s\to\infty}$ can be written with the same formalism. It allows to conclude that the set of integers $\{n'_{1d}, \ldots, n'_{md}\}$

are obtained from matrix $G_d(1/s)$ Smith McMillan form at infinity. This matrix, denoted as $\Phi'(s)$ satisfies equations

$$\begin{cases} G_d(1/s) = J_1'(s) \cdot \Phi'(s) \cdot J_2'(s), \\ \det\left[\lim_{s\to\infty}\{J_k'(s)\}\right] \neq 0 \quad \text{with} \quad k = 1,2, \\ \Phi'(s) = \operatorname{diag}\left\{s^{-n'_{1d}}, \dots, s^{-n'_{md}}\right\}. \end{cases} \tag{49}$$

From equations (49) follows equation

$$\det\left[G_d(1/s)\right]_{s\to\infty} \approx K_1' \cdot K_2' \cdot 1/\left(s^{\left(\sum_{i=1}^m n'_{id}\right)}\right), \tag{50}$$

with K_1' and K_2' non zero constants, or equivalently equation

$$\det\left[G_d(s)\right]_{s\to 0} \approx K_1' \cdot K_2' \cdot s^{\left(\sum_{i=1}^m n'_{id}\right)}. \tag{51}$$

According that matrices $G(s)$ and $G_d(s)$ are equal, can be written equation

$$\det[G(s)]_{s\to 0} \approx K_1' \cdot K_2' \cdot s^{\left(\sum_{i=1}^m n'_{id}\right)}. \tag{52}$$

From equation (52), it comes that the number of null invariant zeros in BGI is equal to the sum of the infinite zero orders of the BGD for square models. The property remains valid for non square models, that is with $m > p$. □

C Stabilizing decoupling subspace

The proof is divided in three parts. At first, it is shown that the two subspaces $\mathcal{V}_i^{*\perp}$ and $\mathcal{V}_{id}^{*\perp}$ are such as $\mathcal{V}_i^{*\perp} \oplus \mathcal{V}_{id}^{*\perp} = \mathcal{V}_{is}^{*\perp}$ – **step 1**, then that $\mathcal{V}_{is}^*$ is a $(A, \mathcal{B})$ invariant subspace included in the subspace $\mathcal{V}_i^*$ – **step 2**. It is then shown that $\mathcal{V}_{is}^*$ is equal to $\mathcal{V}_{i\,\mathrm{stab}}^*$ – **step 3**.

Step 1: $\mathcal{V}_{is}^{*\perp} = \mathcal{V}_i^{*\perp} \oplus \mathcal{V}_{id}^{*\perp}$.

Consider the $\mathcal{V}_i^{*\perp}$ subspace basis defined by equation

$$\mathcal{V}_i^{*\perp} = \operatorname{vect}\left\{(c_i)^t, \dots, \left(c_i A^{n_i - 1}\right)^t\right\}, \qquad n_i \geq 1. \tag{53}$$

$\mathcal{V}_{id}^{*\perp}$ subspace basis is defined by equation

$$\mathcal{V}_{id}^{*\perp} = \operatorname{vect}\left\{\left(c_i A^{-1}\right)^t, \dots, \left(c_i A^{-n_{id}}\right)^t\right\}, \qquad n_{id} \geq 1 \tag{54}$$

if $n_{id} \geq 1$, else $\mathcal{V}_{id}^{*\perp} = 0$. n_{id} is the smallest integer satisfying equations

$$\begin{cases} c_i A^{-(k+1)} B = 0, \quad k < n_{id}, \\ c_i A^{-(n_{id}+1)} B \neq 0. \end{cases} \tag{55}$$

Consider S_i^* the smallest (c_i, A) invariant subspace containing $\mathcal{B}$, defined by the following algorithm

$$\begin{cases} S_i^0 = 0, \\ S_i^\mu = \mathcal{B} + A((\ker c_i) \cap S_i^{\mu-1}). \end{cases} \tag{56}$$

$S_i^{*\perp}$ and $\mathcal{V}_{id}^{*\perp}$ are related as equation

$$\mathcal{V}_{id}^{*\perp} \subset S_i^{*\perp}. \tag{57}$$

Indeed, suppose the product $(\mathcal{V}_{id}^{*\perp})^t \cdot S_i^*$, according to equations (54), (55) and (56), the first basis vector of subspace $\mathcal{V}_{id}^{*\perp}$ satisfies equation

$$c_i A^{-1} \cdot S_i^* = 0. \tag{58}$$

The same equation can be written for each $\mathcal{V}_{id}^{*\perp}$ basis vector. The last one satisfies equation

$$c_i A^{-n_{id}} \cdot S_i^* = 0. \tag{59}$$

It is thus possible to deduce equation

$$(\mathcal{V}_{id}^{*\perp})^t \cdot S_i^* = 0, \tag{60}$$

which implies equation (57).

Consider now the subspace $\mathcal{V}_{is}^{*\perp}$ satisfying equation

$$\mathcal{V}_{is}^{*\perp} = \mathcal{V}_i^{*\perp} + \mathcal{V}_{id}^{*\perp}. \tag{61}$$

$\sum(C, A, B)$ is right invertible and thus row subsystems $\sum(c_i, A, B)$ are in the same way right invertible. Equation (62) can be written

$$\mathcal{V}_i^* + S_i^* = \mathcal{X}. \tag{62}$$

Equation (63) can be deduced

$$\mathcal{V}_i^{*\perp} \cap S_i^{*\perp} = 0. \tag{63}$$

From equations (63) and (57) it comes:

$$\mathcal{V}_i^{*\perp} \cap \mathcal{V}_{id}^{*\perp} = 0. \tag{64}$$

According to equations (61) and (64), subspaces $\mathcal{V}_i^{*\perp}$ and $\mathcal{V}_{id}^{*\perp}$ satisfy equation

$$\mathcal{V}_{is}^{*\perp} = \mathcal{V}_i^{*\perp} \oplus \mathcal{V}_{id}^{*\perp}. \tag{65}$$

Step 2: $\mathcal{V}_{is}^*$ is a $(A, \mathcal{B})$ invariant subspace included in $\mathcal{V}_i^*$.

$\mathcal{V}_{is}^*$ is a $(A, \mathcal{B})$ invariant subspace iff it satisfies equation

$$A\mathcal{V}_{is}^* \subset \mathcal{V}_{is}^* + \mathcal{B}. \tag{66}$$

It is sufficient to prove that for each vector $x \in \mathcal{V}_{is}^*$, equation

$$\left\{\mathcal{V}_{is}^{*\perp} \cap \mathcal{B}^{\perp}\right\}^t \cdot Ax = 0 \tag{67}$$

is satisfied.

Consider a subspace of $\left\{\mathcal{V}_{is}^{*\perp} \cap \mathcal{B}^{\perp}\right\}$. According to equation (65), a subspace basis is the union of subspace basis $\left\{\mathcal{V}_{i}^{*\perp} \cap \mathcal{B}^{\perp}\right\}$ and $\left\{\mathcal{V}_{id}^{*\perp} \cap \mathcal{B}^{\perp}\right\}$. Basis vectors of subspace $\mathcal{V}_{i}^{*\perp}$ belonging to subspace $\mathcal{B}^{\perp}$ are identified thanks to the integer n_i which satisfies equation

$$\begin{cases} c_i A^{(k-1)} B = 0, & k < n_i, \\ c_i A^{(n_i-1)} B \neq 0. \end{cases} \tag{68}$$

This equation can be rewritten as equation

$$\begin{cases} (c_i A^{(k-1)})^t \in \mathcal{B}^{\perp}, & k < n_i, \\ (c_i A^{(n_i-1)})^t \notin \mathcal{B}^{\perp}. \end{cases} \tag{69}$$

According to equations (53) and (69), subspace $\left\{\mathcal{V}_{i}^{*\perp} \cap \mathcal{B}^{\perp}\right\}$ can be describes by the following equation:

$$\left\{\mathcal{V}_{i}^{*\perp} \cap \mathcal{B}^{\perp}\right\} = \operatorname{vect}\left\{(c_i)^t, \dots, \left(c_i A^{n_i-2}\right)^t\right\}. \tag{70}$$

With the same manner, basis vectors of subspace $\mathcal{V}_{id}^{*\perp}$ belonging to subspace $\mathcal{B}^{\perp}$ are identified thanks to the integer n_{id} which satisfies equation

$$\begin{cases} (c_i A^{-(k+1)})^t \in \mathcal{B}^{\perp}, & k < n_{id}, \\ (c_i A^{-(n_{id}+1)})^t \notin \mathcal{B}^{\perp}. \end{cases} \tag{71}$$

According to equations (54) and (71), a basis of subspace $\left\{\mathcal{V}_{id}^{*\perp} \cap \mathcal{B}^{\perp}\right\}$ is described by equation

$$\left\{\mathcal{V}_{id}^{*\perp} \cap \mathcal{B}^{\perp}\right\} = \operatorname{vect}\left\{(c_i A^{-1})^t, \dots, (c_i A^{-n_{id}})^t\right\}. \tag{72}$$

Equations (70) and (72) allow to write a basis for subspace $\left\{\mathcal{V}_{is}^{*\perp} \cap \mathcal{B}^{\perp}\right\}$:

$$\left\{\mathcal{V}_{is}^{*\perp} \cap \mathcal{B}^{\perp}\right\} = \operatorname{vect}\left\{(c_i)^t, \dots, (c_i A^{n_i-2})^t \,\middle|\, (c_i A^{-1})^t, \dots, (c_i A^{-n_{id}})^t\right\}. \tag{73}$$

Let us prove that each vector $x \in \mathcal{V}_{is}^*$ satisfies equation (67). This equation can be rewritten as equation

$$\mathcal{V}_{is}^* = \mathcal{V}_{i}^* \cap \mathcal{V}_{id}^*. \tag{74}$$

It means that each vector x belonging to subspace $\mathcal{V}_{is}^*$ also belongs to $\mathcal{V}_{i}^*$. It satisfies equation

$$(\mathcal{V}_{i}^{*\perp})^t \cdot x = 0. \tag{75}$$

From equations (53) and (75), it comes that each vector $x \in \mathcal{V}_{is}^*$ satisfies equations

$$\begin{cases} c_i x = 0, \\ c_i A x = 0, \\ \vdots \\ c_i A^{(n_i-1)} x = 0. \end{cases} \tag{76}$$

According to equation (74), if x belong to subspace $\mathcal{V}_{is}^*$ it also belongs to subspace $\mathcal{V}_{id}^*$, and satisfies equation

$$(\mathcal{V}_{id}^{*\perp})^t \cdot x = 0. \tag{77}$$

Thus, each vector $x \in \mathcal{V}_{is}^*$ satisfies equations

$$\begin{cases} c_i A^{-1} x = 0, \\ c_i A^{-2} x = 0, \\ \vdots \\ c_i A^{-n_{id}} x = 0. \end{cases} \tag{78}$$

For each vector basis z belonging to subspace $\left\{\mathcal{V}_{is}^{*\perp} \cap \mathcal{B}^{\perp}\right\}$, expression $z^t Ax$ is calculated with $x \in \mathcal{V}_{is}^*$. For each vector $v_k = (c_i A^k)^t$, $k = 0, \ldots, (n_i - 2)$, from equation (76) it comes:

$$v_k^t Ax = 0, \qquad x \in \mathcal{V}_{is}^*. \tag{79}$$

For each vector $w_k = (c_i A^{-k})^t$, $k = 1, \ldots, n_{id}$, from equation (76) and (78) it comes equation

$$w_k^t Ax = 0. \tag{80}$$

Thus, each basis vector z belonging to subspace $\left\{\mathcal{V}_{is}^{*\perp} \cap \mathcal{B}^{\perp}\right\}$ satisfies equation

$$z^t Ax = 0, \qquad x \in \mathcal{V}_{is}^*. \tag{81}$$

For each vector $x \in \mathcal{V}_{is}^*$, it comes equation

$$\left\{\mathcal{V}_{is}^{*\perp} \cap \mathcal{B}^{\perp}\right\}^t \cdot Ax = 0. \tag{82}$$

Thus, $\mathcal{V}_{is}^*$ is a $(A, \mathcal{B})$ invariant subspace and from equation (74) it can be concluded that this subspace is included in subspace $\mathcal{V}_i^*$.

Step 3: $\mathcal{V}_{is}^* = \mathcal{V}_{i\,\mathrm{stab}}^*$.

Step 1 and Step 2 allow to prove that $\mathcal{V}_{is}^*$ satisfies the following properties:

$$\begin{cases} \mathcal{V}_{is}^{*\perp} = \mathcal{V}_i^{*\perp} \oplus \mathcal{V}_{id}^{*\perp}, \\ \mathcal{V}_{is}^* \text{ is a } (A, \mathcal{B}) \text{ invariant subspace,} \\ \dim(\mathcal{V}_{is}^{*\perp}) = n_i + n_{id}. \end{cases} \tag{83}$$

If the row subsystem $\sum(c_i, A, B)$ does not contain any strictly instable invariant zero, it is possible to write equation

$$\dim(\mathcal{V}_{is}^{*\perp}) = n_i + C^+(c_i, A, B). \tag{84}$$

Then, subspace $\mathcal{V}_{is}^*$ satisfies equation

$$\dim(\mathcal{V}_{is}^*) = \dim(\mathcal{V}_i^*) - C^+(c_i, A, B). \tag{85}$$

Equations (83) and (85) allow to conclude that subspace $\mathcal{V}_{is}^*$ satisfies the following properties:

$$\begin{cases} \mathcal{V}_{is}^* \text{ is a } (A, \mathcal{B}) \text{ invariant subspace,} \\ \mathcal{V}_{is}^* \subset \mathcal{V}_i^*, \\ \dim(\mathcal{V}_{is}^*) = \dim(\mathcal{V}_i^*) - C^+(c_i, A, B). \end{cases} \tag{86}$$

From equation (86) it follows the conclusion: if the row subsystem $\sum(c_i, A, B)$ does not contain any strictly instable invariant zero, subspace $\mathcal{V}_{is}^*$ is the greatest internally stable $(A, \mathcal{B})$ invariant subspace included in $(\ker c_i)$. It is thus equal to subspace $\mathcal{V}_{i\,\mathrm{stab}}^*$.
□