Nonlinear Dynamics and Systems Theory, **1**(2) (2001) 121–131

Set Based Constant Reference Tracking for Continuous-Time Constrained Systems

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Received: June 15, 1999; Revised: August 5, 2000

Abstract: In the paper study the possibility of tracking constant reference signals for a linear time-invariant dynamic system in the presence of state constraints. Resort to the theory of invariant sets due to its good capability of handling this kind of problem. Attention is placed on the determination of suitable sets for the attainable steady state values and of suitable control laws which guarantee that every possible output steady state value belonging to this set can be reached from any initial state belonging to a proper set. Then, based on recent results on the possibility of associating to these sets explicit smooth control laws, an explicit controller is derived which allows the system to asymptotically track constant reference signals and guarantees that no constraints violation occurs. Finally, an example of the implementation of the proposed control law will be reported.

Keywords: *Asymptotic stability domains; Lyapunov method; Lyapunov functions; non-linear systems; sets; uniform asymptotic stability.*

Mathematics Subject Classification (2000): 34C35, 34D05, 34D20, 34D45, 34H05, 54H20, 93C10, 93C15, 93C50, 93C60, 93D05, 93D20, 93D30.

1 Introduction

In most recent literature concerning linear time-invariant continuous-time dynamic systems much emphasis has been put on the constrained stabilization problem [1, 2, 3, 4, 5] but little has been done to derive stabilizing regulators which guarantee perfect asymptotic tracking of constant reference signal in the presence of state and control constraints. This problem can for instance be solved by recasting it as an l^1 problem, though this results in high complexity regulators due to the nature of the problem which in general results, according to [6], in being a multiblock problem. Another way to proceed is that of exploiting invariant regions as done in [7, 8, 9]. In [8] the authors have proposed a discrete-time reference governor which behaves significantly well in the presence of state and control constraints and whose expression is given in implicit form and can be derived from that of the "maximal output admissible set" [4] of a proper dynamic system. The mentioned governor acts as a nonlinear first order filter which limits the reference signal whenever the state is almost to exit from the maximal output admissible set. In this work we focus our attention on continuous-time systems with state constraints only and, instead of limiting instant by instant the reference signal, we provide a polyhedral set of signals the output can track. Then, exploiting some recent results concerning the possibility of "smoothing" polyhedral Lyapunov functions [10], we show how it is possible to associate a control law in explicit form to this set.

2 Notation

For a vector $x \in I\!R^n$ we denote by $\|x\|_\infty = \max_i |x_i|$. We call C-set a convex and compact set having the origin as an interior point. Given a C-set $\mathcal{S}$ we denote by $\partial\mathcal{S}$ and $int\mathcal{S}$ the border and interior of $\mathcal{S}$, respectively, and we denote the scaled set $\lambda\mathcal{S}$, for $\lambda \geq 0$, as $\lambda\mathcal{S} = \{y\colon y = \lambda x\ x \in \mathcal{S}\}$. Given a continuous function $\Psi\colon I\!R^n \to I\!R$ and $k \in I\!R$ we define the (possibly empty) closed set $\bar{\mathcal{N}}[\Psi, k]$ as $\bar{\mathcal{N}}[\Psi, k] = \{x \in I\!R^n\colon \Psi(x) \leq k\}$. We say that $\Psi\colon I\!R^n \to I\!R$ is a Gauge function if, for every $x, y \in I\!R^n$ it fulfills the following properties: $\Psi(x) > 0$, if $x \neq 0$, $\Psi(\lambda x) = \lambda\Psi(x)$, for every $\lambda \geq 0$, and $\Psi(x + y) \leq \Psi(x) + \Psi(y)$. If Ψ is a Gauge function, the set $\bar{\mathcal{N}}[\Psi, k]$ is a C-set for all $k > 0$. Any C-set $\mathcal{S}$ induces a Gauge function (the so-called Minkowski functional of $\mathcal{S}$) which is defined as $\Psi_{\mathcal{S}}(x) \doteq \inf\{\mu \geq 0\colon x \in \mu\mathcal{S}\}$ or, equivalently, as $\Psi_{\mathcal{S}}(x) \doteq \inf\{\mu \geq 0\colon \frac{x}{\mu} \in \mathcal{S}\}$. A polyhedral C-set $\mathcal{P} \in I\!R^n$ can be written as $\mathcal{P} = \{x\colon \max_{i=1,s} F_i x \leq 1\}$, or in compact form as $\mathcal{P} = \{x\colon Fx \leq \bar{1}\}$, where $F \in I\!R^{s\times n}$ is a full column rank matrix, $\bar{1}$ is the s-dimensional column vector $[1\,1 \cdots 1]^T$ and the inequality sign has to be intended component-wise. We will say that an homogeneous function $\Psi(x)$ from $I\!R^n$ to $I\!R^+$ is a polyhedral function if it is the Minkowski functional of a polyhedral C-set. If $\mathcal{P} = \{x\colon Fx \leq \bar{1}\}$, then $\Psi_{\mathcal{P}}(x) = \max_i F_i x$.

3 Problem Statement

In this work we consider a continuous-time reachable and observable square dynamic system (that is with an equal number of inputs and outputs) in its standard form, say described by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t), \end{aligned} \tag{1}$$

where $A \in I\!R^{n\times n}$, $B \in I\!R^{n\times m}$ and the output matrix $C \in I\!R^{m\times n}$. The main additional requirement for this system is that the state never exceeds prescribed bounds represented by the C-set $\mathcal{X}$, say

$$x(t) \in \mathcal{X} \quad \text{for every} \quad t \geq 0.$$

Since a necessary and sufficient condition for the constant tracking problem to have a solution is that the system has no transmission zeros at the origin, we will work under the following assumption.

Assumption 3.1 *The pencil matrix*

$$A_c = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$$

is invertible.

For this kind of system the constrained stabilization problem is quite a well established subject [2, 1, 3]. If we assume $\mathcal{X}$ to be a polyhedral C-set we know that a stabilizing control law exists if and only if there exists a contractive set for (1) contained in $\mathcal{X}$. If we add the requirement on the output infinity norm not to exceed a prescribed value μ then the above statement must be slightly modified in the sense that the solution requires the determination of a contractive set for (1) contained in $\mathcal{X}^* = \mathcal{X} \bigcap \{x\colon \|Cx\|_\infty \le \mu\}$ (this fact has been used in [11, 12] for the solution of l^1 problems with state feedback). In view of the reachability assumption it is easy to see that the afore-mentioned problem always has a solution (for instance a stabilizing linear regulator will do the job); nevertheless the interest in this kind of problem is usually mostly concerned with the criterion on the basis of which the stabilizing control law has to be chosen. One "natural" criterion is that of maximizing the domain of attraction to the origin included in the given set $\mathcal{X}$ as done in [2].

By exploiting this criterion we will consider the constrained tracking problem and we will take advantage of recent results [10] on the possibility of deriving suitable smooth controllers in explicit form for the solution of the constrained stabilization problem for tracking purposes. Before stating our problem it is worth recalling that, in view of Assumption 3.1 and of the constraints on the state, the set of admissible constant reference signals $\mathcal{Y}_R$ which the system will be able to track will be necessarily bounded. The problem we will focus our attention on can then be stated in the following way:

Problem 3.1 *Given the continuous-time dynamic system* (1) *and the state constraints set* $\mathcal{X}$ *find a state feedback control law* $u = \Phi(x)$ *and a set of reference signals* $\mathcal{Y}_R$ *such that for every constant reference signal* $\bar{y} \in \mathcal{Y}_R$ *the state evolution never exceeds the prescribed bounds for every* $t \ge 0$ *and such that* $\lim\limits_{t\to\infty} y(t) \to \bar{y}$.

4 Tracking a Constant Reference Signal

In the previous section without going into much detail we have stated our problem and we have mentioned the set $\mathcal{Y}_R$ of admissible values the reference signal $\bar{y}$ can assume. To see how it is possible to derive such a set we have first recall some results concerning the use of invariant regions for the solution of this kind of problem. As a first step we recall that, given the continuous-time system (1), its discrete-time Euler Approximating System (EAS) is defined as follows:

$$\begin{aligned} x(k+1) &= (I + \tau A)x(k) + \tau B u(k), \\ y(x) &= Cx(k). \end{aligned} \tag{2}$$

For continuous and discrete-time systems it is possible to furnish the following definitions of domain of attraction [2].

Definition 4.1 A region $\mathcal{P} \subset \mathcal{X}$ is a *domain of attraction* (β-contractive region) for system (1) if there exists a constant $\beta > 0$ (often referred to as speed of convergence) such that for every initial condition $x(0) \in \mathcal{P}$ there exists a piecewise continuous control function $u(\cdot)\colon I\!R \to I\!R^m$ such that the evolution corresponding to $u(t)$ is such that:

$$\Psi_{\mathcal{P}}(x(t)) \leq \Psi_{\mathcal{P}}(x(0))e^{-\beta t}$$

for every $t \geq 0$ (we recall that $\Psi_{\mathcal{P}}$ is the Minkowski functional induced by $\mathcal{P}$ on $I\!R^n$).

Definition 4.2 A region $\mathcal{P} \subset \mathcal{X}$ is a *domain of attraction* (λ-contractive region) for system (2) if there exists a constant $\lambda < 1$ (often referred to as contractivity) such that for every initial condition $x(0) \in \mathcal{P}$ there exists a sequence $u(k) \in I\!R^m$ such that the corresponding evolution is such that:

$$\Psi_{\mathcal{P}}(x(k)) \leq \Psi_{\mathcal{P}}(x(0))\lambda^k$$

for every $t \geq 0$.

It can be proven that the existence of a β-contractive set $\mathcal{P}$ for system (1) is equivalent to the existence, for every $x \in \mathcal{P}$, of a value v such that:

$$D^+\Psi_{\mathcal{P}}(x, v) \doteq \limsup_{\tau \to 0^+} \frac{\Psi_{\mathcal{P}}(x + \tau(Ax + Bv)) - \Psi_{\mathcal{P}}(x)}{\tau} \leq -\beta\Psi_{\mathcal{P}}(x) \tag{3}$$

(the introduction of the generalized Lyapunov derivative allows to deal with non smooth functionals, see [10] for details). In the discrete-time case the above condition, for the existence of a λ-contractive set for (2), translates in the following one-step contractivity requirement:

$$\Psi_{\mathcal{P}}(x + \tau(Ax + Bv)) \leq \lambda\Psi_{\mathcal{P}}(x). \tag{4}$$

It is well known that the systems under consideration, for a given β, admit a maximal β-contractive set $\mathcal{S}_\beta$ contained in $\mathcal{X}$ and that this set is in general not polyhedral. From [2] it is known that it is possible to approximate arbitrarily well the largest contractive set $\mathcal{S}_\beta \subset \mathcal{X}$ by means of a polyhedral set $\mathcal{P} \subset \mathcal{X}$ which results in being a domain of attraction for system (1) with a speed of convergence $\bar{\beta}$ arbitrarily close to the prescribed one and the control $u = \phi(x)$ can be expressed in feedback form, where $\phi(x)$ is Lipschitz on $\mathcal{P}$. It is straightforward that the same applies (with the cited replacement of the set $\mathcal{X}$ with $\mathcal{X}^*$) when output bounds have to be considered.

This approximation is derived and can be effectively computed by exploiting the relation existing between a continuous-time system of the form (1) and its discrete-time EAS (2), according to the next result.

Theorem 4.1 [13] *Suppose system* (1) *admits a β-contractive C-set $\mathcal{P} \subset \mathcal{X}$. Then for all $0 < \beta' < \beta$ there exists $\tau > 0$ such that $\mathcal{P}$ is λ'-contractive for the discrete-time system* (2) *with $0 < \lambda' = 1 - \tau\beta'$. Conversely, if system* (2) *admits a λ-contractive C-set $\mathcal{P}$ then $\mathcal{P}$ is β-contractive for system* (1) *with $\beta = \frac{(1-\lambda)}{\tau}$.*

Given the above definitions it is hence possible to define the set $\mathcal{Y}_R$ of admissible constant reference signals which the system will be able to track. Suppose a β-contractive set $\mathcal{P}$ has been found and consider the following equation:

$$A_c \begin{bmatrix} \bar{x} \\ \bar{u} \end{bmatrix} = \begin{bmatrix} 0 \\ \bar{y} \end{bmatrix}. \tag{5}$$

Since A_c is invertible the solution to the above set of equations can be written as

$$\bar{x} = K_{\bar{y}\bar{x}}\bar{y}, \tag{6}$$

$$\bar{u} = K_{\bar{y}\bar{u}}\bar{y}. \tag{7}$$

From (6) we see that all the admissible equilibrium states belong to the subspace $K_{\bar{y}\bar{x}}\bar{y}$ so that the admissible constant reference signals which do not lead to state constraints violation are given by

$$\mathcal{Y}_R = \{\bar{y} \colon K_{\bar{y}\bar{x}}\bar{y} \in \mathcal{P}\}, \tag{8}$$

while from (7) we know that to track an arbitrary constant signal $\bar{y} \in \mathcal{Y}_R$ the control value will have to converge to the value $\bar{u} = \bar{u}(\bar{y}) = K_{\bar{y}\bar{u}}\bar{y}$.

The next step for the solution of Problem 3.1 is that of determining a suitable control law such as to guarantee that the state constraints are never violated and the output converges to the given constant reference value $\bar{y}$. In view of Assumption 3.1 this amounts to requiring that $\lim\limits_{t\to\infty} x(t) = K_{\bar{y}\bar{x}}\bar{y}$.

To this aim consider a reference value $\bar{y} \in \alpha\mathcal{Y}_R$, $\alpha < 1$ (the need for the introduction of the parameter α will be clear in the sequel; the introduction of α basically amounts to discarding trackable signals corresponding to states belonging to the border of $\mathcal{P}$) and consider the following functional, which is the Gauge functional associated to the set $\mathcal{P}$ and centered in $\bar{x}(\bar{y})$

$$\Psi^{\bar{y}}(x) \doteq \inf\{\mu \ge 0 \colon \bar{x}(\bar{y}) + \frac{1}{\mu}\,(x - \bar{x}(\bar{y})) \in \mathcal{P}\}.$$

The following lemma allows us to compute explicitly $\Psi^{\bar{y}}(x)$ whenever $\mathcal{P}$ is a polyhedral C-set.

Lemma 4.1 *If $\mathcal{P} = \{x \colon Fx \le \bar{1}\}$, then for every $\bar{y} \in \alpha\mathcal{Y}_R$, $\alpha < 1$, and $x \in \mathcal{P}$*

$$\Psi^{\bar{y}}(x) = \max_i \frac{F_i(x - \bar{x}(\bar{y}))}{1 - F_i\bar{x}(\bar{y})}. \tag{9}$$

Moreover $\Psi^{\bar{y}}(x) = 1$ whenever $x \in \partial\mathcal{P}$.

Proof It follows from simple algebra by first noting that, since $\bar{x}(\bar{y}) \in int\mathcal{P}$, the quantity $1 - F_i\bar{x}(\bar{y})$ is strictly greater than zero for every i. Hence

$$\begin{aligned}\Psi^{\bar{y}}(x) &\doteq \inf\left\{\mu \ge 0 \colon \bar{x} + \frac{1}{\mu}\,(x - \bar{x}) \in \mathcal{P}\right\}\\ &= \inf\left\{\mu \ge 0 \colon F_i\left(\bar{x} + \frac{1}{\mu}\,(x - \bar{x})\right) \le 1 \ \ \forall\, i\right\}\\ &= \inf\left\{\mu \ge 0 \colon \frac{F_i(x - \bar{x})}{1 - F_i\bar{x}} \le \mu \ \ \forall\, i\right\}.\end{aligned}$$

The next lemma shows that the functional $\Psi^{\bar{y}}(x)$ just introduced, whenever $\mathcal{P}$ is a domain of attraction, can be regarded as a Lyapunov function for the dynamic of the error $e(t) = x(t) - \bar{x}(\bar{y})$ when the reference signal is a constant. For the sake of clarity and given the above-mentioned possibility of approximating the largest β-contractive set for system (1) by means of a polyhedral set, without lack of generality we will limit our attention to the case of polyhedral C-sets, although the next lemma can be proven true for any contractive C-set.

Lemma 4.2 *Let $\mathcal{P} = \{x \colon Fx \le \bar{1}\}$ be a β-contractive polyhedral C-set for system (1) and let $\mathcal{Y}_R$ be defined as in (8). Then for every constant value $\bar{y} \in \alpha\mathcal{Y}_R$, $\alpha < 1$, there exists $0 < \beta_1 < \beta$ and a state feedback control function $u = \phi_1(x, \bar{x})$ such that for every $x(0) \in \mathcal{P}$ the corresponding state evolution is such that*

$$\Psi^{\bar{y}}(x(t)) \le e^{-\beta_1 t}\Psi^{\bar{y}}(x(0)) \tag{10}$$

for every $t \ge 0$.

Proof Consider a constant reference value $\bar{y}$ and let $\bar{x}$ and $\bar{u}$ be the corresponding state and control values. Setting $e(t) = x(t) - \bar{x}$ and $v(t) = u(t) - \bar{u}$ leads to the following description of the error dynamics:

$$\dot{e} = \dot{x} - \dot{\bar{x}} = Ax + Bu - (A\bar{x} + B\bar{u}) = Ae + Bv. \tag{11}$$

Since $\Psi^{\bar{y}}(x) = \max_i \frac{F_i(x-\bar{x})}{1-F_i\bar{x}} = \max_i \frac{F_i}{1-F_i\bar{x}}\, e = \Psi_1(e)$, showing that (10) holds amounts to prove that $\mathcal{P}_1 = \left\{e \colon \frac{F_i}{1-F_i\bar{x}}\, e \le 1,\ \ i = 1, \ldots, s\right\}$ is a β_1-contractive domain for system (11). The latter, in view of Theorem 4.1, can be proven by determining τ and λ_1 such that $\mathcal{P}_1$ is λ_1 contractive for the discrete-time EAS of (11), say for every $e \in \mathcal{P}_1$ there exists v such that

$$\max_i \frac{F_i}{1 - F_i\bar{x}}\,(e + \tau(Ae + Bv)) \le \lambda_1 \max_j \frac{F_j}{1 - F_j\bar{x}}\, e. \tag{12}$$

Let us first consider $e \in \partial\mathcal{P}_1$ (hence $x \in \partial\mathcal{P}$). Expanding $v = u - \bar{u}$ in (12), the above requires, for every i, that

$$\begin{aligned} &\frac{F_i}{1 - F_i\bar{x}}\,(x - \bar{x} + \tau(Ax - A\bar{x} + Bu - B\bar{u})) \\ &\quad = \frac{F_i(x + \tau(Ax + Bu)) - 1}{1 - F_i\bar{x}} + 1 \le \lambda_1. \end{aligned} \tag{13}$$

From Theorem 4.1 for every $\beta' < \beta$ there exists τ such that $\mathcal{P}$ is $\lambda' = 1 - \tau\beta'$-contractive for the EAS of (1), say for every $x \in \partial\mathcal{P}$, there exists $\tilde{u}$ such that for every i

$$F_i(x + \tau(Ax + B\tilde{u})) \le 1 - \tau\beta'.$$

Hence, setting $u = \tilde{u}$ in (13), results in

$$\frac{F_i(x + \tau(Ax + B\tilde{u})) - 1}{1 - F_i\bar{x}} + 1 \le -\frac{\tau\beta'}{1 - F_i\bar{x}} + 1 \le \lambda_1,$$

for some $\lambda_1 < 1$ in view of the fact that $1 - F_i\bar{x} > 0$ for every i. The extension to the case of e (respectively $x = e + \bar{x}$) in the interior of $\mathcal{P}_1$ (resp. $\mathcal{P}$) is straightforward due to the homogeneity of $\Psi_1(e)$. In fact for every x in the interior of $\mathcal{P}$ the error e can be written as $e = x - \bar{x} = \gamma(x_1 - \bar{x}) = \gamma e_1$, with $e_1 \in \partial\mathcal{P}_1$, for a proper scaling factor $\gamma < 1$. The one step contractivity requirement (12) can then be rewritten as

$$\max_i \frac{F_i}{1 - F_i\bar{x}}\left(\gamma e_1 + \tau(\gamma A e_1 + Bv)\right) \le \lambda_1\gamma. \tag{14}$$

Setting $v = \gamma v_1$ in (14) and dividing both terms by γ we get (13).

Now, since $\mathcal{P}_1$ is β_1-contractive for (11), it is possible to associate to $\mathcal{P}_1$ a Lipschitz continuous state feedback control law $\phi(e) = \phi(x - \bar{x})$. Going back from (11) to the original system (1) it is readily seen that $\phi_1(x, \bar{x}) = \bar{u} + \phi(x - \bar{x})$ is the desired control law.

The lemma just presented allows us partially to solve Problem 3.1 as it just states that whenever the initial condition lies in the set $\mathcal{P}$ and the reference signal is a constant value belonging to the interior of $\mathcal{Y}_R$ we can provide a Lipschitz continuous state feedback control function which guarantees that the corresponding state evolution belongs to $\mathcal{P}$ and asymptotically converges to the given steady state value. This might appear as an expected consequence of the existence of a contractive region (w.r.t. the origin) for system (1). Nevertheless, as we will see next, this way of proceeding allows us to determine an explicit feedback control law. Before going on with the next theorem we need to recall a result which is a restricted version of what has been presented in [14] concerning the possibility of deriving explicit continuous state feedback control law for the class of systems under consideration. This is obtained by smoothing the polyhedral function $\Psi_{\mathcal{P}}(x)$ so as to get, for a given positive integer $q > 0$, the Gauge function

$$\Psi_q(x) = \left(\sum_{i=1}^{s} \sigma_{2q}(F_i x) \right)^{\frac{1}{2q}} \tag{15}$$

with

$$\sigma_r(x) = \begin{cases} x^r & \text{if} \quad x \geq 0 \\ 0 & \text{otherwise} \end{cases}.$$

Introducing the function gradient

$$\nabla \Psi_q(x) = \left[\frac{\partial \Psi_q(x)}{\partial x_1}, \, \dots, \, \frac{\partial \Psi_q(x)}{\partial x_n} \right] = \Psi_q(x)^{(1-2q)} G_q(x) F,$$

where

$$G_q(x) = \left[\sigma_{2q-1}(F_1 x) \dots \sigma_{2q-1}(F_s x) \right],$$

the following result holds:

Theorem 4.2 [14] *Let $\mathcal{P} = \{x \colon Fx \leq \bar{1}\}$ be a β-contractive polyhedral C-set for system* (1). *Then for every $0 < \beta_1 < \beta$ there exists a positive integer q such that the set $\mathcal{P}_q = \{x \colon \Psi_q(x) \leq 1\}$ is β_1-contractive for system* (1). *Moreover it is possible to associate to $\Psi_q(x)$ the explicit smooth*[1] *state feedback control law*

$$u = \Phi(x) = -\mu_0 \Psi_q(x)^{2(1-q)} B^T F^T G_q(x), \tag{16}$$

where μ_0 is a finitely computable nonnegative constant.

In Lemma 4.2 it has been shown that the polyhedral function (9) is a Lyapunov function for the error whenever the reference signal belongs to the interior of $\mathcal{Y}_R$, but nothing has been said about the effective determination of a stabilizing control law (in

[1] We mean smooth for every $x \neq 0$.

the sense that we have proved its existence though not furnishing any expression for it), due to the lack of differentiability of (9).

The next theorem will provide us with the requested expression for the controller. To this aim we first "smooth", similarly to what we have done in (15), the expression given by (9) and centered in $\bar{x}(\bar{y})$ by taking $q < \infty$ sufficiently large so as to get the function

$$\Psi_q^{\bar{y}}(x) = \left(\sum_{i=1}^{s} \sigma_{2q} \left(\frac{F_i(x - \bar{x})}{1 - F_i \bar{x}} \right) \right)^{\frac{1}{2q}}. \tag{17}$$

Simple algebra shows that the gradient $\nabla \Psi_q^{\bar{y}}(x)$ of (17) is:

$$\nabla \Psi_q^{\bar{y}}(x) = \left(\Psi_q^{\bar{y}}(x) \right)^{(1-2q)} G_q^{\bar{y}}(x) F_{\bar{y}},$$

where

$$G_q^{\bar{y}}(x) = \left[\sigma_{2q-1} \left(\frac{F_1(x - \bar{x})}{1 - F_1 \bar{x}} \right) \ \ldots \ \sigma_{2q-1} \left(\frac{F_s(x - \bar{x})}{1 - F_s \bar{x}} \right) \right]$$

and

$$F_{\bar{y}} = \begin{bmatrix} \frac{F_1}{1 - F_1 \bar{x}} \\ \cdots \\ \frac{F_s}{1 - F_s \bar{x}} \end{bmatrix}.$$

These expressions allow us to introduce the next theorem.

Theorem 4.3 *Let $\mathcal{P} = \{x \colon Fx \leq \bar{1}\}$ be a β-contractive polyhedral C-set contained in $\mathcal{X}$ for system* (1). *Then for every reference signal $\bar{y} \in \alpha \mathcal{Y}_R$, $\alpha < 1$, there exists $0 < \beta_1 < \beta$ and an integer q such that the control law*

$$\Phi(x, \bar{y}) = \bar{u}(\bar{y}) - \rho_0 \Psi_q^{\bar{y}}(x)^{2(1-q)} B^T F_{\bar{y}}^T G_q^{\bar{y}}(x), \tag{18}$$

where ρ_0 is a finitely computable nonnegative constant, is such that for every initial condition $x(0) \in \mathcal{P}$ the output of the corresponding evolution $y(t)$ asymptotically converges to $\bar{y}$ with speed equal to β_1 while assuring that $x(t) \in \mathcal{X}$ for every $t \geq 0$.

Proof From Lemma 4.2 we have that $\mathcal{P}_1 = \{e \colon \Psi_1(e) \leq 1\}$, where $\Psi_1(e) = \max_i \frac{F_i}{1 - F_i \bar{x}} e$, is a β_1-contractive set for system (11). The proof hence follows immediately by first recalling Theorem 4.2, which assures the existence of an explicit control law of the form (16) (which will result in being a function of $e = x - \bar{x}$), and by subsequently going back to the original system to obtain (18).

5 Example

Consider the following two dimensional system

$$\dot{x}(t) = \begin{bmatrix} -0.3 & 1 \\ -1 & -0.3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} -5 \\ 5 \end{bmatrix},$$

$$y(t) = [\,2 \quad 0\,] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

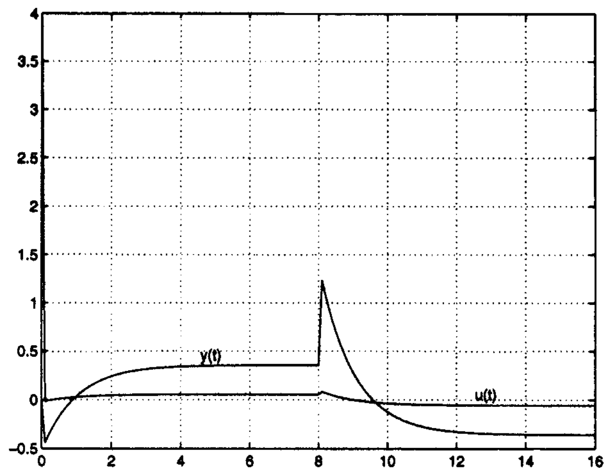

Figure 5.1. State space evolution.

with state constraints given by the set $\mathcal{X} = \{x\colon \|x\|_\infty \le 1\}$. A polyhedral 2-contractive set contained in $\mathcal{X}$ is $\mathcal{P} = \{x\colon \max_i F_i x \le 1\}$, where F is the following matrix

$$F = \begin{bmatrix} 0 & 1 \\ 0 & -1 \\ 1 & 0 \\ -1 & 0 \\ 1.391 & 1.540 \\ -1.391 & -1.540 \end{bmatrix}.$$

The resulting sets of admissible constant input and output values are $\mathcal{U}_R = [-0.073, 0.073]$ and $\mathcal{Y}_R = [-0.470, 0.470]$. We chose as a tracking value $\bar{y} = 0.358$ corresponding to $\alpha = 0.761$ and exploiting the results presented in Theorem 4.3 we determined the integer $q = 12$ such that the proposed control law (18) with $\rho_0 = 21.586$ guarantees asymptotic tracking of $\bar{y}$ for every $x_0 \in \mathcal{P}$ with speed of convergence $\beta_1 = 0.3$. Figure 5.1 depicts the state space evolution obtained starting from zero initial value and tracking value-equal to $\bar{y}$ for the first 8 seconds and $-\bar{y}$ for $t > 8$ together with different level surfaces of the Lyapunov functions $\Psi_{12}^{\bar{y}}$ and $\Psi_{12}^{-\bar{y}}$ (dotted) associated to the two tracking states $\bar{x}(\bar{y})$ and $-\bar{x}(\bar{y})$ which belong to the first and third quadrant and are indicated with a circled cross in the same figure.

Finally Figure 5.2 shows the evolution of the output as well as that of the control.

6 Conclusions

This work has dealt with perfect asymptotic tracking for state constrained dynamic systems. An alternative approach to the one proposed by Gilbert et al. [8], which is based on the concept of "maximal output admissible set" and recent results [10, 14], has been presented. This novel approach allows us to synthesize an explicit nonlinear state

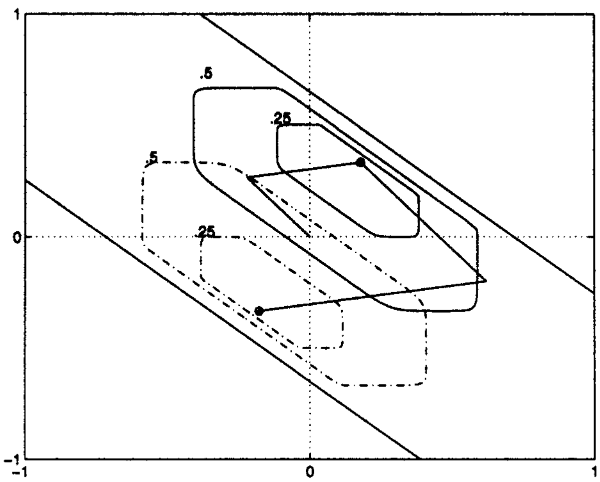

Figure 5.2. Control and output simulated plots.

feedback control law which guarantees perfect asymptotic tracking while maximizing the set of trackable signals which do not lead to state constraint violation.

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