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Optimal Design of Robust Control for Uncertain Systems: a Fuzzy Approach

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Abstract: The problem of designing controls for a linear dynamic system under input disturbance is considered. The input disturbance is bounded but the bound information is either deterministic or fuzzy. The control design is purely deterministic. However, the resulting system performance is interpreted differently, depending on the bound information. It may be deterministic or fuzzy (i.e. with a spectrum of outcome to various degrees). Finally, the optimal design problem of the control scheme, in which the cost is in quadratic form, is solved.

Keywords: *Uncertain systems; robust control; fuzzy approach.*

Mathematics Subject Classification (2000): 93C42, 92B12, 93B51, 93D05.

1 Introduction

Fuzzy theory was originally introduced to *describe information* (for example, the linguistic information) that is in lack of a sharp boundary with its environment (see [1]). However, it soon turned into the direction that mainly focuses on the use of *fuzzy reasoning* for control, estimation, decision making, etc. The application of fuzzy reasoning has enjoyed its advantage that it is model free. The designer's effort is mainly focused on tuning some parameters based on linguistic reasoning. It has been shown to be rather effective for a large amount of complex problems.

The current paper, on the other hand, proposes a rather different angle. It endeavors to explore applications of the original intention of fuzzy theory, namely, *information description*. In particular, we cast the framework within the context of control theory.

Granted that the probability theory is quite self-contained, criticism of its validity in describing the real world does exist. It is interesting to notice that Kalman [2], among others, despite his early devotion to the use of probability in mathematical system theory, is now critical on part of its foundation. Kalman contended that probability theory might not be all that suitable to describe the majority of randomness. In a sense, the link between a rather sophisticated mathematical tool and the physical world might be loose. We stress, however, that Kalman's recent comment on probability does not automatically assume him an advocate for fuzzy theory. His view on the latter has been unchanged (see [3] and [4]).

The *fuzzy* approach, as originally proposed by Zadeh [1] on the other hand, takes *the extent of occurrence* point of view. Historically, the merge between the probability theory and control/system theory, which can be traced back to the fifties, has been highly successful and received little criticism. In the state space framework, Kalman initiated the effort of looking into the estimation problem (see [5]) and control problem (see [6]) when a system is under stochastic noise. The effort has received tremendous attention. As it turns out, there is now a quite impressive arena on stochastic system and control theory (see, e.g., [7]) that can not be ignored by any practitioners.

In this work, we shall attempt to pursue a possible use of fuzzy description of uncertainty in robust control design. This may be viewed as an alternative proposal to combine the fuzzy theory and control theory. The objectives are two fold. First, we explore fuzzy descriptions of system performance should more information of the uncertainty (in the fuzzy sense) be provided. This adds more insight on the system performance. Furthermore, this also shows a way to view the system performance with human needs (which are often best described in a fuzzy sense). Second, we consider an optimal design of the robust control. The combined average system performance (over the fuzzy description) and control effort is to be extremized by an appropriate choice of a design parameter. This may be viewed as an analogous development to the LQG design in stochastic control.

2 Uncertain System and Robust Control

Consider the following uncertain system

$$\dot{x}(t) = Ax(t) + Bu(t) + Bv(x(t),t), \qquad x(t_0) = x_0, \tag{2.1}$$

where $t \in \mathbf{R}$ is the "time" (or more precisely, the independent variable), $x(t) \in \mathbf{R}^n$ is the state, $u(t) \in \mathbf{R}^m$ is the control, $v(x(t),t) \in \mathbf{R}^m$ is the (unknown) input disturbance, A, B are (known) constant matrices. The function $v(\cdot,t)$ is continuous. The function $v(x,\cdot)$ is Lebesgue measurable. The task is to choose the control u such that the state $x(t)$ of the controlled system of (2.1) enters a region around $x = 0$ after a finite time and remains there thereafter.

Assumption 2.1 The pair (A,B) is stabilizable.

Assumption 2.2 There is a known scalar $\overline{u} \geq 0$ such that

$$\max_{\substack{x\in\mathbf{R}^n\\ t\in\mathbf{R}}} \|v(x,t)\| \leq \overline{u}. \tag{2.2}$$

Choose constant $n \times n$ matrices $Q > 0$ and $R > 0$. Solve the following Riccati equation

$$A^T P + PA - PBR^{-1}B^T P + Q = 0 \tag{2.3}$$

for the solution $P > 0$, which is also an $n \times n$ matrix. Notice that the solution $P > 0$ exists and is unique if (A,B) is stabilizable. We propose the control u as follows:

$$u(t) = -\frac{1}{2}R^{-1}B^T Px(t) - \gamma B^T Px(t), \tag{2.4}$$

where $\gamma > 0$ is a scalar constant. The choice of γ will be made later.

Definition 2.1 Consider a dynamical system

$$\dot{\xi}(t) = f(\xi(t), t) \tag{2.5}$$

with $\xi(t_0) = \xi_0$. The solution of the system (suppose it exists) is *uniformly ultimately bounded* if for any $r > 0$ with $\|\xi_0\| \le r$, there are $\bar{d}(r) > 0$ and $\tilde{T}(\bar{d}(r), r) \ge 0$ such that

$$\|\xi(t)\| \le \bar{d}(r) \tag{2.6}$$

for all $t \ge t_0 + \tilde{T}(\bar{d}(r), r)$.

Theorem 2.1 [8] *Consider that the system (2.1) is subject to Assumptions 2.1 and 2.2. Suppose that the control (2.4) is applied. For each $\gamma > 0$, the resulting controlled system is uniformly ultimately bounded. Furthermore, the size of the ultimate boundedness region, i.e., $\bar{d}(r)$, can be made arbitrarily small by choosing a sufficiently large γ.*

There is a trade-off between the performance and the control effort. As a result, an optimal quest for the design may be interesting. It is also possible that, based on further understanding of the input disturbance, one is able to extract more information about its bound. We describe the information in the following.

Assumption 2.3 There is a scalar $\nu \ge 0$ such that

$$\max_{\substack{x \in \mathbf{R}^n \\ t \in \mathbf{R}}} \|v(x,t)\| \le \nu. \tag{2.7}$$

The membership value of ν in a region $U\colon = [\underline{u}, \overline{u}]$, $\overline{u} \ge \underline{u} \ge 0$, is prescribed by a fuzzy number N, whose membership function is $\mu_N\colon U \to [0,1]$.

The fuzzy description of the uncertainty bound, as shown in Assumption 2.3, enables us to pursue a fuzzy-based interpretation of the system performance. By Assumption 2.3, given that ν is in the fuzzy set N, the possibility that $\nu = u$, where $u \in [\underline{u}, \overline{u}]$, is given by $\mu_N(u)$.

For later purpose, we are also interested in the fuzzy number $N \cdot N$. This is discussed as follows. Let

$$v(x,t) = [\, v_1(x,t) \quad v_2(x,t) \quad \cdots \quad v_m(x,t)\,]^T. \tag{2.8}$$

It is possible that sometimes the designer only knows the fuzzy description of the bound of each component $v_i(x,t)$, $i = 1, 2, \cdots, m$. Suppose that $|v_i(x,t)| \le \nu_i$ for all x, t. The scalar ν_i belongs to a region $U_i\colon = [\underline{u}_i, \overline{u}_i]$, $\overline{u}_i \ge \underline{u}_i \ge 0$, which is the universe of discourse of a fuzzy number N_i. This fuzzy number is prescribed by a membership function $\mu_{N_i}\colon U_i \to [0,1]$.

With the membership function $\mu_{N_i}(\cdot)$ prescribed, one obtains its α-cuts $[\underline{u}_{i_\alpha}, \overline{u}_{i_\alpha}]$. The square of the α-cuts, that is, $[\underline{u}_{i_\alpha}, \overline{u}_{i_\alpha}] \cdot [\underline{u}_{i_\alpha}, \overline{u}_{i_\alpha}]$, is obtained (see [9]). The sum of all these α-cuts, i.e., $\sum_{i=1}^{m} [\underline{u}_{i_\alpha}, \overline{u}_{i_\alpha}] \cdot [\underline{u}_{i_\alpha}, \overline{u}_{i_\alpha}]$, also can be obtained for each α (see [9]). Finally, one may use decomposition theorem to reach the membership function for the fuzzy number $N \cdot N$.

If the designer already knows the membership function $\mu_N(\cdot)$, then it is easy to obtain the membership function of the fuzzy number $N \cdot N$. All it takes is to take the square of the α-cuts of $\mu_N(\cdot)$, summarize them, and then invoke the decomposition theorem. We now state the following fuzzy-based system performance.

Theorem 2.2 *Consider that the system (2.1) is subject to Assumptions 2.1 and 2.3. Suppose that the control (2.4) is applied. For any* $u \in [\underline{u}, \overline{u}]$ *and any* $r > 0$ *with* $\|x_0\| \le r$*, the possibility that*

$$\|x(t)\| \le \hat{d}(u) \quad \textit{for all} \quad t \ge t_0 + \tilde{T}$$

is given by $\mu_{N \cdot N}(u)$*, where*

$$\hat{d}(u) = \underline{\hat{d}}(u) + \epsilon, \tag{2.9}$$

$$\underline{\hat{d}}(u) := \sqrt{\frac{u^2}{2\gamma\lambda_m(Q)}}. \tag{2.10}$$

Proof By [8], for any $\nu = u$,

$$\dot{V} \le -\lambda_m(Q)\|x\|^2 + \frac{u^2}{2\gamma}. \tag{2.11}$$

This means that $\dot{V}$ is negative definite for all $\|x\|$ such that

$$\|x\| > \sqrt{\frac{u^2}{2\gamma\lambda_m(Q)}} =: \underline{\hat{d}}(u). \tag{2.12}$$

From Assumption 2.3, the possibility that $\nu = u$ is $\mu_N(u)$. Thus the possibility that $\dot{V}$ is negative for all $\|x\| > \underline{\hat{d}}(u)$ is $\mu_{N \cdot N}(u)$. By Theorem 2.1, for any $t \ge t_0 + \tilde{T}$, $\|x(t)\| \le \bar{d}$. Since $\bar{d} > \underline{\hat{d}}(u)$, this in turn shows that the possibility of $\|x(t)\| \le \hat{d}(u)$ is given by $\mu_{N \cdot N}(u)$.

Remark 2.1 The theorem asserts that, given the uniform ultimate boundedness result in Theorem 2.1, and the additional information provided by Assumption 2.3, one can further prescribe a possibility distribution that the state enters another region, which is in general of smaller size. This is a totally new aspect of the system performance, as compared with the previous work in robust control. The special way of incorporating fuzzy logic theory with control system analysis is believed to be the first time.

The input disturbance bound ν is often obtained via observed data and analyzed by the engineer. The observed data is, by nature, always limited. The source of the disturbance is unlikely to be exactly repeated. Hence any interpretation via the *frequency of occurrence*, as the number of repetitions approaches to infinity, suffers from a lack of basis. An alternative interpretation of the bound for circumstances like this would have to be fuzzy in its nature. For examples, one may need to adopt the fuzzy (linguistic) terms such as "*close to*" or "*very close to*" a (crisp) value.

The system performance is also often judged by the engineer in terms of the need of human being: One may choose a (crisp) set point and intend to have the performance to be "*close to*" or "*very close to*" it, after a finite time. These again fall into the fuzzy category. A typical example of this nature is the "comfort" control in Heating, Ventilating, and Air Conditioning (HVAC) (see, e.g., [10]). On top of this, the engineer also has the discretion to impose a hard bound (through, e.g., the prescription of the size of uniform ultimate boundedness region) on the performance, which must be met with absolutely no exceptions. All these can be addressed by the current framework.

3 Optimal Design of γ

The previous section shows a system performance which can be guaranteed by a deterministic control design. By the analysis, the size of the uniform ultimate boundedness region decreases as γ increases. As γ approaches to infinity, the size approaches to 0. This rather strong performance is accompanied by a (possibly) large control effort, which is reflected by γ. From the practical design point of view, the designer may be also interested in seeking an optimal choice of γ for a compromise among various conflicting criteria. This is associated with the minimization of a performance index.

We first explore more on the deterministic performance of the uncertain system. By the Rayleigh's principle,

$$\lambda_m(P)\|x\|^2 \le x^T P x = V \le \lambda_M(P)\|x\|^2 \tag{3.1}$$

and hence

$$-\|x\|^2 \le -\frac{1}{\lambda_M(P)}\, V. \tag{3.2}$$

With this into (2.11), we have

$$\dot V(t) \le -\frac{\lambda_m(Q)}{\lambda_M(P)}\, V(t) + \frac{\nu^2}{2\gamma}, \tag{3.3}$$

where $V_0 = V(t_0) = x_0^T P x_0$. This is a *differential inequality.* The following is needed for our analysis of (3.3).

Definition 3.1 [11] If $w(\psi, t)$ is a scalar function of the scalars ψ, t in some open connected set $\mathcal{D}$, we say a function $\psi(t),\ t_0 \le t \le \bar t,\ \bar t > t_0$ is a *solution of the differential inequality*

$$\dot\psi(t) \le w(\psi(t), t) \tag{3.4}$$

on $[t_0, \bar t)$ if $\psi(t)$ is continuous on $[t_0, \bar t)$ and its derivative on $[t_0, \bar t)$ satisfies (3.4).

Theorem 3.1 [11] *Let $w(\phi, t)$ be continuous on an open connected set $\mathcal{D} \in \mathbf{R}^2$ and such that the initial value problem for the scalar equation*

$$\dot\phi(t) = w(\phi(t), t), \qquad \phi(t_0) = \phi_0 \tag{3.5}$$

has a unique solution. If $\phi(t)$ is a solution of (3.5) on $t_0 \le t \le \bar t$ and $\psi(t)$ is a solution of (3.4) on $t_0 \le t < \bar t$ with $\psi(t_0) \le \phi(t_0)$, then $\psi(t) \le \phi(t)$ for $t_0 \le t \le \bar t$.

Instead of exploring the solution of the differential inequality, which is often non-unique and not available, the theorem suggests that it may be feasible to study the upper bound of the solution. The reasoning is, however, based on that the solution of (3.5) is unique.

Theorem 3.2 [12] *Consider the differential inequality (3.4) and the differential equation (3.5). Suppose that for some constant $L > 0$, the function $w(\cdot)$ satisfies the Lipschitz condition*

$$|w(v_1, t) - w(v_2, t)| \le L|v_1 - v_2| \tag{3.6}$$

for all points (v_1, t), $(v_2, t) \in \mathcal{D}$. Then any function $\psi(t)$ that satisfies the differential inequality (3.4) for $t_0 \le t < \bar{t}$ satisfies also the inequality

$$\psi(t) \le \phi(t) \tag{3.7}$$

for $t_0 \le t \le \bar{t}$.

We consider the differential equation

$$\dot{r}(t) = -\frac{\lambda_m(Q)}{\lambda_M(P)}\, r(t) + \frac{\nu^2}{2\gamma}, \qquad r(t_0) = V_0. \tag{3.8}$$

The right-hand side satisfies the global Lipschitz condition with

$$L = \frac{\lambda_m(Q)}{\lambda_M(P)}. \tag{3.9}$$

We proceed with solving the differential equation (3.8). This results in

$$r(t) = \left(V_0 - \frac{\lambda_M(P)}{\lambda_m(Q)}\,\frac{\nu^2}{2\gamma}\right) \exp\left[-\frac{\lambda_m(Q)}{\lambda_M(P)}\,(t - t_0)\right] + \frac{\lambda_M(P)}{\lambda_m(Q)}\,\frac{\nu^2}{2\gamma}. \tag{3.10}$$

Therefore

$$V(t) \le r(t) \tag{3.11}$$

or

$$V(t) \le \left(V_0 - \frac{\lambda_M(P)}{\lambda_m(Q)}\,\frac{\nu^2}{2\gamma}\right) \exp\left[-\frac{\lambda_m(Q)}{\lambda_M(P)}\,(t - t_0)\right] + \frac{\lambda_M(P)}{\lambda_m(Q)}\,\frac{\nu^2}{2\gamma} \tag{3.12}$$

for all $t \ge t_0$. By the same argument, we also have, for any t_s and any $\tau \ge t_s$,

$$V(\tau) \le \left(V_s - \frac{\lambda_M(P)}{\lambda_m(Q)}\,\frac{\nu^2}{2\gamma}\right) \exp\left[-\frac{\lambda_m(Q)}{\lambda_M(P)}\,(\tau - t_s)\right] + \frac{\lambda_M(P)}{\lambda_m(Q)}\,\frac{\nu^2}{2\gamma}, \tag{3.13}$$

where $V_s = V(t_s) = x^T(t_s)Px(t_s)$. The time t_s is when the control scheme (2.4) starts to be executed. It does not need to be t_0.

By the Rayleigh's principle $V(\tau) \ge \lambda_m(P)\|x(\tau)\|^2$, the right-hand side of (3.13) provides an upper bound of $\lambda_m(P)\|x(\tau)\|^2$. This in turn leads to an upper bound of $\|x(\tau)\|^2$. For each $\tau \ge t_s$, let

$$\eta(\nu, \gamma, \tau, t_s) := \left(V_s - \frac{\lambda_M(P)}{\lambda_m(Q)}\,\frac{\nu^2}{2\gamma}\right) \exp\left[-\frac{\lambda_m(Q)}{\lambda_M(P)}\,(\tau - t_s)\right], \tag{3.14}$$

$$\eta_\infty(\nu, \gamma) := \frac{\lambda_M(P)}{\lambda_m(Q)}\,\frac{\nu^2}{2\gamma}. \tag{3.15}$$

Notice that for each ν, γ, t_s, $\eta(\nu, \gamma, \tau, t_s) \to 0$ as $\tau \to \infty$.

One may relate $\eta(\nu, \gamma, \tau, t_s)$ to the transient portion and $\eta_\infty(\nu, \gamma)$ the steady state portion of the system performance. Since there is no knowledge of the input disturbance $v(x, t)$ except its possible bound, it is only realistic to refer to $\eta(\nu, \gamma, \tau, t_s)$ and $\eta_\infty(\nu, \gamma)$ while analyzing the system performance. We also notice that both $\eta(\nu, \gamma, \tau, t_s)$ and $\eta_\infty(\nu, \gamma)$ are dependent on ν. The value of ν is not known except that it lies within a set U (i.e., the universe of discourse) to the degree that is defined by $\mu_N(\cdot)$.

Definition 3.2 For any function $f\colon [\underline{u},\overline{u}] \to \mathbf{R}$, the D-operation $D[f(\nu)]$ is defined as follows:

$$D[f(\nu)] = \frac{\int_{\underline{u}}^{\overline{u}} f(\nu)\mu_N(\nu)d\nu}{\int_{\underline{u}}^{\overline{u}} \mu_N(\nu)d\nu}. \tag{3.16}$$

Remark 3.1 In a sense, the D-operation $D[f(\nu)]$ takes an average value of $f(\nu)$ over $\mu_N(\nu)$. In the special case that $f(\nu)=\nu$, this is reduced to the well-known center-of-gravity defuzzification method (see, e.g., [13]). If N is crisp (i.e., $\mu_N(\nu)=1$ for all ν), then $D[f(\nu)] = f(\nu)$. This is reduced to the classical case.

Lemma 3.1 *For any crisp constant $a \in \mathbf{R}$,*

$$D[af(\nu)] = aD[f(\nu)]. \tag{3.17}$$

We now propose the following performance index: For any t_s, let

$$\begin{aligned} J(\gamma,t_s) &:= D\Bigg[\int_{t_s}^{\infty} \eta^2(\nu,\gamma,\tau,t_s)\,d\tau\Bigg] + \alpha D[\eta_\infty^2(\nu,\gamma)] + \beta\gamma^2 \\ &=: J_1(\gamma,t_s) + J_2(\gamma) + J_3(\gamma), \end{aligned} \tag{3.18}$$

$\alpha,\beta > 0$. The performance index consists of three parts. The first part $J_1(\gamma,t_s)$ may be interpreted as the average (via the D-operation) of the overall transient performance (via the integration) from time t_s. The second part $J_2(\gamma)$ may be interpreted as the average (via the D-operation) of the steady state performance. The third part $J_3(\gamma)$ is due to the control cost. Both α and β are weighting factors. The weighting of J_1 is normalized to be unity.

Remark 3.2 A standard LQG (i.e., linear-quadratic-Gaussian) problem in stochastic control is to minimize a performance index which is the average (via the expectation value operation in probability) of the overall state and control accumulation. The current optimal design of γ may be viewed as a parallel problem, though not equivalent, in the fuzzy setting. However, one can not be too careful in distinguishing the difference. For example, the Gaussian probability distribution implies that the uncertainty is unbounded (although a higher bound is predicted by a lower probability). In the current consideration, the uncertainty bound is always finite.

Let $\kappa := \lambda_M(P)/\lambda_m(Q)$. One can show that

$$\begin{aligned} \int_{t_s}^{\infty} \eta^2(\nu,\gamma,\tau,t_s)\,d\tau &= \left(V_s - \frac{\lambda_M(P)}{\lambda_m(Q)}\,\frac{\nu^2}{2\gamma}\right)^2 \int_{t_s}^{\infty} \exp\left[-2\frac{\lambda_m(Q)}{\lambda_M(P)}(\tau-t_s)\right] d\tau \\ &= \left(V_s - \kappa\frac{\nu^2}{2\gamma}\right)^2 \frac{\kappa}{2}. \end{aligned} \tag{3.19}$$

Taking the D-operation,

$$\begin{aligned} D\Big[\int_{t_s}^{\infty} \eta^2(\nu,\gamma,t,t_s)\,dt\Big] &= D\Big[\Big(V_s-\kappa\frac{\nu^2}{2\gamma}\Big)^2\frac{\kappa}{2}\Big] \\ &= \Big(V_s-\frac{V_s\kappa}{\gamma}\,D[\nu^2]+\frac{\kappa^2}{4\gamma^2}\,D[\nu^4]\Big)\frac{\kappa}{2}. \end{aligned} \tag{3.20}$$

Next, we analyze the cost $J_2(\gamma)$:

$$D[\eta_\infty^2(\nu,\gamma)] = D\Big[\Big(\frac{\lambda_M(P)}{\lambda_m(Q)}\Big)^2\Big(\frac{\nu^2}{2\gamma}\Big)^2\Big] = \frac{\kappa^2}{4\gamma^2}\,D[\nu^4]. \tag{3.21}$$

With (3.20) and (3.21) into (3.18),

$$\begin{aligned} J(\gamma,t_s) &= \Big(V_s-\frac{V_s\kappa}{\gamma}\,D[\nu^2]+\frac{\kappa^2}{4\gamma^2}\,D[\nu^4]\Big)\frac{\kappa}{2}+\alpha\frac{\kappa^2}{4\gamma^2}\,D[\nu^4]+\beta\gamma^2 \\ &=: \kappa_1-\frac{\kappa_2}{\gamma}+\frac{\kappa_3}{\gamma^2}+\alpha\frac{\kappa_4}{\gamma^2}+\beta\gamma^2, \end{aligned} \tag{3.22}$$

where $\kappa_1 := \frac{\kappa}{\gamma}V_s$, $\kappa_2 := \frac{\kappa^2}{2}V_sD[\nu^2]$, $\kappa_3 := \frac{\kappa^4}{4}D[\nu^4]$, $\kappa_4 := \frac{\kappa^2}{4}D[\nu^4]$.

The optimal design problem is then the following constrained optimization problem: For any t_s,

$$\min_{\gamma} J(\gamma,t_s) \quad \text{subject to} \quad \gamma>0. \tag{3.23}$$

For any t_s, taking the first order derivative of J with respect to γ:

$$\frac{\partial J}{\partial\gamma} = \frac{\kappa_2}{\gamma^2}-2\,\frac{\kappa_3}{\gamma^3}-2\alpha\,\frac{\kappa_4}{\gamma^3}+2\beta\gamma = \frac{1}{\gamma^3}\left(\kappa_2\gamma-2\kappa_3-2\alpha\kappa_4+2\beta\gamma^4\right). \tag{3.24}$$

That

$$\frac{\partial J}{\partial\gamma}=0 \tag{3.25}$$

leads to

$$\kappa_2\gamma-2\kappa_3-2\alpha\kappa_4+2\beta\gamma^4=0 \tag{3.26}$$

or

$$\kappa_2\gamma+2\beta\gamma^4=2(\kappa_3+\alpha\kappa_4). \tag{3.27}$$

Equation (3.27) is a scalar quartic equation. For simplicity, in the rest of discussion, we shall rule out the trivial possibility of $\underline{u}=\overline{u}=0$, which results in $D[\nu^2]=0$ and $D[\nu^4]=0$. In other words, we only consider $D[\nu^2]>0$ and $D[\nu^4]>0$ and hence $\kappa_3>0$ and $\kappa_4>0$ (notice that $\kappa>0$). This in turn means that the solutions (there are two) γ to (3.27) are not identical to zero.

To observe the constraint $\gamma>0$, we now restrict ourselves to only the positive solution of (3.27). For the $\gamma>0$ that solves (3.27),

$$\begin{aligned} \frac{\partial^2 J}{\partial\gamma^2} &= -\frac{3}{\gamma^4}\left(\kappa_2\gamma^2\kappa_3-2\alpha\kappa_4+2\beta\gamma^4\right)+\frac{1}{\gamma^3}\left(\kappa_2+8\beta\gamma^3\right) \\ &= \frac{1}{\gamma^3}\left(\kappa_2+8\beta\gamma^3\right)>0. \end{aligned} \tag{3.28}$$

The positive solution of the scalar quartic equation (3.27), which depends on V_s, solves the constrained minimization problem (3.23). By the continuity of the left-hand side of (3.27) on γ, the solution $\gamma > 0$ to (3.27) always exists. In addition, since the left-hand side of (3.27) is strictly increasing in γ, the solution $\gamma > 0$ to (3.27) is unique. We summarize the main result as follows.

Theorem 3.3 *Consider that the system (2.1) is subject to Assumptions 2.1 and 2.3. Suppose that the control (2.4) is applied. For given V_s, the unique solution $\gamma > 0$ of (3.27) minimizes the performance index (3.18).*

The solutions of the quartic equation (3.27) depend on the *cubic resolvent* (see [14])

$$z^3 + (-4r)z - q^2 = 0, \tag{3.29}$$

where

$$r = -\frac{1}{\beta}\left(\kappa_3 + \alpha\kappa_4\right), \qquad q = \frac{\kappa_2}{2\beta}.$$

Let $p_1 := -4r$, $p_2 := -q^2$. The discriminant D of the cubic resolvent is given by

$$D = \left(\frac{p_1}{3}\right)^3 + \left(\frac{p_2}{2}\right)^2. \tag{3.30}$$

Since $r < 0$, $D > 0$. The solutions of the cubic resolvent are given by

$$z_1 = u + v, \tag{3.31}$$

$$z_2 = -\frac{(u+v)}{2} + (u-v)i\sqrt{\frac{3}{2}}, \tag{3.32}$$

$$z_3 = -\frac{(u+v)}{2} - (u-v)i\sqrt{\frac{3}{2}}, \tag{3.33}$$

where

$$u = \left(-\frac{p_2}{2} + \sqrt{D}\right)^{\frac{1}{3}}, \tag{3.34}$$

$$v = \left(-\frac{p_2}{2} - \sqrt{D}\right)^{\frac{1}{3}}. \tag{3.35}$$

The cubic resolvent possesses one real solution and two complex conjugate solutions. This in turn implies that the quartic solution has two real solutions and one pair of complex conjugate solutions. The maximum real solution, which is positive, of the quartic equation is given by

$$\gamma = \frac{1}{2}\left(\sqrt{z_1} + \sqrt{z_2} + \sqrt{z_3}\right). \tag{3.36}$$

With z_1, z_2, and z_3 into (3.36), a lengthy but straightforward algebra shows that the positive solution of the quartic equation is given by

$$\gamma = \frac{1}{2}\left(\sqrt{u+v} + \sqrt{7u^2 + 7v^2 - 10uv}\,\cos\frac{\theta}{2}\right), \tag{3.37}$$

where

$$\theta = \tan^{-1} \frac{\sqrt{\frac{3}{2}}(u-v)}{-\frac{1}{2}(u+v)}. \tag{3.38}$$

Remark 3.3 The calculation of γ in (3.37) requires V_s which depends on $x(t_s)$. In implementations, this can be obtained via on-line feedback of the state. Notice that t_s is the starting time of the execution of the control. It does not need to be identical to the initial time t_0. The control starts to activate as soon as it receives the feedback signal $x(t_s)$. The control scheme, which minimizes the performance index (3.18), also only depends on t_s, not t_0. Certainly, the controlled system with $x(t_s)$ the initial state is uniformly ultimately bounded.

By using (3.27), the cost J in (3.22) can be rewritten as

$$\begin{aligned} J &= \kappa_1 - \frac{\kappa_2}{\gamma} + \frac{\kappa_3}{\gamma^2} + \alpha\frac{\kappa_4}{\gamma^2} + \beta\gamma^2 \\ &= \kappa_1 - \frac{1}{\gamma^2}\left(\kappa_2\gamma + 2\beta\gamma^4\right) + \kappa_3\gamma^2 + \alpha\,\frac{\kappa_4}{\gamma^2} + 3\beta\gamma^2 \\ &= \kappa_1 - \frac{1}{\gamma^2}\left(\kappa_3 + \alpha\kappa_4 + 3\beta\gamma^4\right). \end{aligned} \tag{3.39}$$

With (3.37), the minimum cost is given by

$$J_{\min} = \kappa_1 - \frac{4}{(\sqrt{z_1}+\sqrt{z_2}+\sqrt{z_3})^2}\left(\kappa_3 + \alpha\kappa_4 + \frac{3}{8}\,\beta\big(\sqrt{z_1}+\sqrt{z_2}+\sqrt{z_3}\big)^4\right). \tag{3.40}$$

Remark 3.4 Combining the previous results, the robust control scheme (2.4) using the optimal design of $\gamma > 0$ renders the closed-loop system uniformly ultimately bounded (with the initial state $x(t_s)$). In addition, there is a possibility distribution associated with the size of the region that the state will enter.

4 Conclusions

The incorporation of uncertainty, which is described in a fuzzy sense, into a robust control framework is introduced. This is believed to the first attempt for such a merge. As to the prescription of the desirable performance, it is often the designer's discretion. Since in practice it is in fact more realistic to prescribe the performance in a fuzzy sense (such as "close to", "very close to"), the current framework fits in well with both the need (the performance) and the given (uncertainty).

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