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Stability and Hopf Bifurcation in Differential Equations with One Delay

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Abstract: A class of parameter dependent differential equations with one delay is considered. A decomposition of the parameter space into domains where the corresponding characteristic equation has a constant number of zeros with positive real part is provided. The local stability analysis of the zero solution and the computation of all Hopf bifurcation points with respect to the delay is given.

Keywords: *Nonlinear delay differential equations; zeros of quasi-polynomials; local stability; Hopf bifurcation*

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1 Introduction

Local stability and bifurcation analysis of systems of nonlinear differential equations with one time delay of the following type

$$\dot{x}(t) = Ax(t) + Bx(t-\tau) + F(x(t), x(t-\tau)), \tag{1.1}$$

where $\tau \geq 0;\ A,\, B \in \mathbb{R}^{n,n},\ F \in C^k(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n),\ k \geq 1,\ F(0,0) = DF(0,0) = 0$, often leads to the consideration of quasi-polynomials $\Phi_{\tau,\lambda} \colon \mathbb{C} \to \mathbb{C};\ \tau \geq 0,\ \lambda \in \mathbb{C}$, given by

$$\Phi_{\tau,\lambda}(s) := (s+1)\exp(\tau s) - \lambda. \tag{1.2}$$

In this context it is of particular relevance to know how the zeros of $\Phi_{\tau,\lambda}$ are distributed in the complex plane, whether they lie in the left or right half plane, and finally, how they depend on the parameters τ and λ.

The objective of this work is to divide the τ-halfline and the λ-plane into domains where $\Phi_{\tau,\lambda}$ has a constant number of zeros with positive real part and to investigate the local stability of the zero solution and the Hopf bifurcation points of systems given by (1.1) with appropriate matrices A and B.

Systems of type (1.1) occur in several fields of science. For example, they model electro-optical circuits which display bistability and chaotic behavior (see [12, 17]), they describe dynamical processes in neural networks (see [1, 22]), they model protein synthesis (see [2]) and they arise in the study of white blood-cell production (see [21]). Interested readers may find further applications, for example, in [15, pp.1–8]; [13, pp.72–81], [18, pp.1–34], [19, pp.1–17].

The problem to estimate the zeros of (1.2) with positive real part, the stability analysis of equilibria and the computation of Hopf bifurcation points of (1.1) has attracted the interest of several authors. For instance, Hayes [16] discusses quasi-polynomial equations equivalent to $\Phi_{\tau,\lambda}(s) = 0$ with $\tau > 0$ and $\lambda \in \mathbb{R}$ (see also [5, pp.444–446], [6]). El'sgolts and Norkin [11, pp.134–136] give a partition of the (A, B)-plane consisting of regions where the corresponding characteristic quasi-polynomials of the linear approximation of (1.1) with $n = 1$ and $A, B \in \mathbb{R}$ has a constant number of zeros with positive real part (see also [9, pp.305–309], [19, pp.56, 57]). Braddock and Van den Driessche [7] estimate the domains in λ-plane, where corresponding quasi-polynomials of the form $\Phi(s) = (s + \mu) \exp(\tau s) - \lambda$ have no zeros with positive real part and discuss the local stability of the trivial solution $x(t) = 0$ of (1.1). Bélair [4] also investigates the local stability of the trivial solution of (1.1) with $A = -I_n$, and proves the existence of a Hopf bifurcation point in the one dimensional case $n = 1$ with $B < 0$. Godoy and dos Reis [14] explore (1.1) with $n = 2$, $A = -I_2$ and B having eigenvalues in $\mathbb{C} \setminus \{\mathbb{R} \cup i\mathbb{R}\}$, and provide a partition of the τ-halfline $(\tau \geq 0)$ in segments where the corresponding characteristic quasi-polynomials of the linear approximation of (1.1) have a constant number of zeros with positive real part (for the case that B has eigenvalues in $\mathbb{C} \setminus \mathbb{R}$, see [3]).

In this work we extend the results above in the following way. For given $\tau \geq 0$ $(\lambda \in \mathbb{C})$ we divide the λ-plane (τ-halfline) into regions (intervals) with constant number of zeros with positive real part of the corresponding quasi-polynomials $\Phi_{\tau,\lambda}(s)$ (Section 2). We investigate the local τ-dependent stability of the zero solution of (1.1) for a large class of matrices A and B (Section 3), and we compute all Hopf bifurcation points of (1.1) with τ as bifurcation parameter (Section 4).

2 Zeros of $\Phi_{\tau,\lambda}$ with Positive Real Part

Consider the quasi-polynomial equation

$$\Phi_{\tau,\lambda}(s) = (s + 1) \exp(\tau s) - \lambda = 0 \tag{2.1}$$

for given $\tau > 0$ and $\lambda \in \mathbb{C}$. The primary objective of this section is to divide the λ-plane into regions by a planar curve with following properties. Points λ lying on the curve represent quasi-polynomials $\Phi_{\tau,\lambda}$ having at least one pure imaginary root, and points in each region correspond to quasi-polynomials with the same number of zeros having positive real part, counted by their multiplicity. This method is well known as D-decomposition (just as D-subdivision or D-partition) (see [11, pp.132–138], [19, pp.55–60]). Then, as consequence of the D-decomposition of the λ-plane, we get a D-decomposition of the τ-halfline.

Let us first state a few elementary results on the roots of (2.1).

Lemma 2.1

a) *$s \in \mathbb{C}$ is a zero of $\Phi_{\tau,\lambda}$ if and only if $\bar{s}$ is a zero of $\Phi_{\tau,\bar{\lambda}}$.*

b) *For $|\lambda| \le 1$ equation (2.1) has no solution with positive real part. For $|\lambda| > 1$ equation (2.1) has a finite number of solutions with positive real part. Furthermore, if such solutions exist, they belong to the open and bounded set*

$$S_\lambda := \Big\{ s \in \mathbb{C} \mid 0 < \operatorname{Re} s < |\lambda| - 1 \;\; \textit{and} \;\; |\operatorname{Im} s| < \sqrt{|\lambda|^2 - 1} \Big\}. \tag{2.2}$$

c) *Any root s of (2.1) with $\tau s \ne -(1+\tau)$ is simple.*

Proof a) Part a) is evident.

b) For all $s \in \mathbb{C}$ with $|s+1| \ge |\lambda|$ and $\operatorname{Re} s > 0$ it holds

$$|s+1| > |\lambda \exp(-\tau s)|. \tag{2.3}$$

This implies that equation (2.1) has no roots with $|s+1| \ge |\lambda|$ and $\operatorname{Re} s > 0$. So all roots of (2.1) with positive real part have to satisfy $|s+1| < |\lambda|$. We set $S_\lambda := \{s \in \mathbb{C}\colon |s+1| < |\lambda|, \operatorname{Re} s > 0\}$. Because S_λ is a bounded and connected subset of $\mathbb{C}$, the analytic function $\Phi_{\tau,\lambda}$ has only a finite number of zeros s with $\operatorname{Re} s > 0$ (see [8, p.78]). For $|\lambda| \le 1$ the set S_λ is empty and consequently (2.1) has no roots with positive real part.

For $|\lambda| > 1$ it follows

$$S_\lambda = \Big\{ s \in \mathbb{C} \mid 0 < \operatorname{Re} s < |\lambda| - 1 \;\; \text{and} \;\; |\operatorname{Im} s| < \sqrt{|\lambda|^2 - 1} \Big\}.$$

c) For $\tau = 0$ the only root $s = \lambda - 1$ is simple. If $\tau > 0$ the assertion follows from

$$\frac{d}{ds}\, \Phi_{\tau,\lambda}(s) = [\tau(s+1) + 1] \exp(\tau s) \ne 0 \tag{2.4}$$

for any $s \in \mathbb{C} \setminus \Big\{ -\frac{1+\tau}{\tau} \Big\}$.

2.1 D-decomposition of the $\boldsymbol{\lambda}$-plane

Let us now consider the planar curve mentioned above. Equation (2.1) has a pure imaginary root $s = i\omega$ if and only if

$$\lambda = (i\omega + 1) \exp(i\omega\tau) =: K_\tau(\omega). \tag{2.5}$$

In the following we summarize a few useful properties of the function K_τ (see Figure 2.1).

Lemma 2.2 *For $\tau > 0$ let $K_\tau\colon \mathbb{R} \to \mathbb{C}$ be the function defined by (2.5). Then:*

a) *K_τ describes a spiral in $\mathbb{C}$ with decreasing radius and argument for $\omega \in (-\infty, 0]$ and increasing radius and argument for $\omega \in [0, \infty)$. Moreover the curve described by K_τ is convex and lies symmetrically to the $\operatorname{Re}\lambda$-axis, i.e. $K_\tau(\omega) = \lambda \Leftrightarrow K_\tau(-\omega) = \bar{\lambda}$.*

b) *For $\omega, \tilde{\omega} \in \mathbb{R}$ and $\omega \ne \tilde{\omega}$ with $\lambda = K_\tau(\omega) = K_\tau(\tilde{\omega})$ it follows that $\omega = -\tilde{\omega}$ and $\lambda \in \mathbb{R}$.*

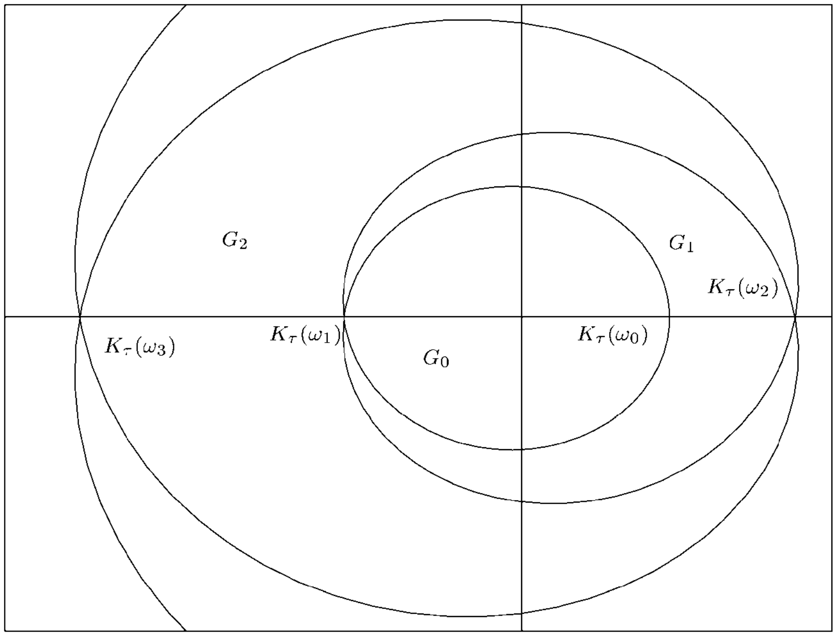

Figure 2.1. D-decomposition of the λ-plane.

Proof Part a) follows from (2.5), see also (2.6), (2.9), (2.10) below.

Now suppose that there exist $\omega, \tilde{\omega} \in \mathbb{R}$, $\omega \neq \tilde{\omega}$, with $K_\tau(\omega) = K_\tau(\tilde{\omega})$. Equation (2.5) yields $|K_\tau(\omega)|^2 = 1 + \omega^2 = 1 + \tilde{\omega}^2 = |K_\tau(\tilde{\omega})|^2$ and so $\omega = -\tilde{\omega}$. With a) we obtain $\lambda = K_\tau(\omega) = K_\tau(-\omega) = \bar{\lambda} \in \mathbb{R}$ and the proof is complete.

Every $\lambda \in \mathbb{C}$ can be written in polar coordinates, namely

$$\lambda = \rho e^{i\theta}, \tag{2.6}$$

where $\rho \geq 0$ is the radius and θ the argument of λ. Inserting (2.6) into (2.5) yields

$$(1 + i\omega) = \rho e^{i(\theta - \omega\tau)}. \tag{2.7}$$

From (2.7) we obtain following conditions for θ and ρ

$$\omega\tau - \theta \in \left(2k\pi - \frac{\pi}{2}, 2k\pi + \frac{\pi}{2}\right), \quad k \in \mathbb{Z}, \tag{2.8}$$

$$\sqrt{1 + \omega^2} = \rho = |\lambda|, \tag{2.9}$$

$$\omega = \tan(\theta - \omega\tau). \tag{2.10}$$

The next lemma deals with solutions of (2.10). We first set

$$I_k(\tau, \theta) := \left(\frac{1}{\tau}\left(2k\pi + \theta - \frac{\pi}{2}\right), \frac{1}{\tau}\left(2k\pi + \theta + \frac{\pi}{2}\right)\right), \quad k \in \mathbb{Z}, \ \tau > 0. \tag{2.11}$$

Lemma 2.3 *For any given $\tau > 0$, $\theta \in [0, 2\pi)$ and $k \in \mathbb{Z}$, equation (2.10) has a unique solution $\omega_k(\tau,\theta) \in I_k(\tau,\theta)$ with the following properties:*

$$\omega_k(\tau,\theta) \in \left(\frac{1}{\tau}\left(2k\pi + \theta - \frac{\pi}{2}\right), \frac{1}{\tau}(2k\pi + \theta)\right) \quad \textit{for} \quad k > 0, \tag{2.12}$$

$$\omega_0(\tau,\theta) \in \left(0, \frac{\theta}{\tau}\right) \quad \textit{for} \quad \theta \neq 0 \quad \textit{and} \quad \omega_0(\tau,0) = 0, \tag{2.13}$$

$$\omega_k(\tau,\theta) \in \left(\frac{1}{\tau}(2k\pi + \theta), \frac{1}{\tau}\left(2k\pi + \theta + \frac{\pi}{2}\right)\right) \quad \textit{for} \quad k < 0. \tag{2.14}$$

Proof $\tan(\theta - \omega\tau)$ is a decreasing function of $\omega \in I_k(\tau,\theta)$ with $\tan(\theta - \omega\tau) > 0$ for $\omega \in \left(\frac{1}{\tau}\left(2k\pi + \theta - \frac{\pi}{2}\right), \frac{1}{\tau}(2k\pi + \theta)\right)$, $\tan(\theta - \omega\tau) = 0$ for $\omega \in \frac{1}{\tau}(2k\pi + \theta)$ and $\tan(\theta - \omega\tau) < 0$ for $\omega \in \left(\frac{1}{\tau}(2k\pi + \theta), \frac{1}{\tau}\left(2k\pi + \theta + \frac{\pi}{2}\right)\right)$. This yields the assertions of the lemma.

For the construction of the regions with constant number of zeros of $\Phi_{\tau,\lambda}$ having positive real part, we need the intersection points of the curve K_τ with the $\operatorname{Re}\lambda$-axis. These intersection points are given by (2.10) with $\theta = 0$, if $k = 2l$ and $\theta = \pi$, if $k = 2l+1$, $l \in \mathbb{N}_0$. Because of symmetry properties of $K_\tau(\omega)$, see Lemma 2.2, we only consider the case $\omega \geq 0$. From Lemma 2.3 we obtain:

Lemma 2.4 *For $\tau > 0$ there is an increasing sequence of real numbers $0 = \omega_0^R < \omega_1^R < \dots$, where $\omega_k^R \in I_l(\tau,\theta)$ with $\theta = 0$ if $k = 2l$ and $\theta = \pi$ if $k = 2l+1$, $l \in \mathbb{N}_0$, such that*

a) $K_\tau(\omega_k^R) \in \mathbb{R}$ *and, if* $\omega \neq \omega_k^R$, $K_\tau(\omega) \notin \mathbb{R}$ *for any* $k \in \mathbb{N}_0$,
b) $(K_\tau(\omega_{2l}^R))_{l\in\mathbb{N}_0}$ *is an unbounded strictly increasing sequence with* $K_\tau(\omega_0^R) = 1$,
c) $\left(K_\tau(\omega_{2l+1}^R)\right)_{l\in\mathbb{N}_0}$ *is an unbounded strictly decreasing sequence with* $K_\tau(\omega_1^R) < -1$.

Using the sequence $(\omega_k^R)_{k\in\mathbb{N}_0}$ we now define segments of the curve described by K_τ lying in the upper and lower half of the λ-plane:

$$C_{\tau,k}^{\pm} := \{\lambda \in \mathbb{C}\colon \lambda = \operatorname{Re} K_\tau(\omega) \pm i|\operatorname{Im} K_\tau(\omega)|,\ \omega \in [\omega_k^R, \omega_{k+1}^R]\} \tag{2.15}$$

and $G_{\tau,k}$ as the region bounded by $C_{\tau,k}^+$ and $C_{\tau,k}^-$:

$$G_{\tau,k} := \{\mu \in \mathbb{C}\colon \operatorname{Re}\mu = \operatorname{Re}\lambda,\ -\operatorname{Im}\lambda < \operatorname{Im}\mu < \operatorname{Im}\lambda,\ \lambda \in C_{\tau,k}^+\} \tag{2.16}$$

for given $k \in \mathbb{N}_0$ and $\tau > 0$. Further we set

$$G_{\tau,-1} := \emptyset. \tag{2.17}$$

We summarize a few useful properties of the regions $G_{\tau,k}$ (see Figure 2.1) in the following.

Lemma 2.5 *Assume $\tau > 0$. For any $k \in \mathbb{N}_0$ the regions $G_{\tau,k}$ are bounded, connected and open subsets of the λ-plane, symmetric to the $\operatorname{Re}\lambda$-axis, satisfying*

a) $0 \in G_{\tau,k} \subset G_{\tau,k+1}$,
b) $G_{\tau,k+1} \setminus \overline{G_{\tau,k}} \neq \emptyset$,
c) $(\overline{G_{\tau,k+2}} \setminus G_{\tau,k+1}) \cap (\overline{G_{\tau,k+1}} \setminus G_{\tau,k}) = \partial G_{\tau,k+1} = C^+_{\tau,k+1} \cup C^-_{\tau,k+1}$,
d) $(\overline{G_{\tau,k+2}} \setminus G_{\tau,k+1}) \cap (\overline{G_{\tau,k}} \setminus G_{\tau,k-1}) = \partial G_{\tau,k+1} \cap \partial G_{\tau,k} = \{K_\tau(\omega^R_{k+1})\} \subset \mathbb{R}$,
e) $G_{\tau,0} \cap \{\lambda \in \mathbb{C}\colon\ \operatorname{Re}\lambda \geq 1\} = \emptyset$.

Proof By construction (see (2.16) and (2.15)) we obtain the boundness, connectivity and openness of $G_{\tau,k}$. Lemma 2.1a and 2.2 provide the symmetry.

For $x \in \left[\omega^R_k, \omega^R_{k+1}\right)$ and $y \in \left[\omega^R_{k+1}, \omega^R_{k+2}\right)$ $(k \in \mathbb{N}_0)$ we have $x < y$ and (2.9) implies $|K(x)| < |K(y)|$. The definition of $G_{\tau,k}$ and $C^{\pm}_{\tau,k}$, $k \in \mathbb{N}_0$, yield the assertions a), b), c) and d). $K_\tau\big(\omega^R_{k+1}\big) \in \mathbb{R}$ follows from Lemma 2.4a.

Since $\frac{dK_\tau}{d\omega}(0) = i(1+\tau)$, the curve K_τ is tangent to the straight line $\{\lambda \in \mathbb{C}\colon\ \operatorname{Re}\lambda = 1\}$ at $\lambda = 1$. The convexity (see Lemma 2.2) of K_τ and the definition (see (2.16)) of $G_{\tau,k}$ implies part e).

Proposition 2.1 *Let $\tau > 0$ and $k \in \mathbb{N}_0$. By passing from region $G_{\tau,k}$ into region $G_{\tau,k+1} \setminus \overline{G_{\tau,k}}$ along the positive $\operatorname{Im}\lambda$-axis exactly one root of (2.1) with positive real part appears.*

Proof Lemma 2.2 and 2.5 provide the existence of an unbounded strictly increasing sequence of positive real numbers $(\beta^I_k)_{k\in\mathbb{N}_0}$ such that

$$\partial G_{\tau,k} \cap \{i\,\beta \in \mathbb{C}\colon\ \beta > 0\} = \{i\beta^I_k\}$$

for $k \in \mathbb{N}_0$. Suppose $\lambda = i\,\beta^I_k$. First we consider the case $k = 2l$, $l \in \mathbb{N}_0$. For $\lambda = i\,\beta^I_k$ (2.1) has a root $s_{0,k} = i\omega^I_k$, with $2l\pi < \omega^I_k\tau < 2l\pi + \frac{\pi}{2}$. Notice that $\omega^I_k = \omega_l(\tau, \frac{\pi}{2})$ with ω_l as in Lemma 2.3. $s_{0,k}$ is the only root s of (2.1) for $\lambda = i\beta^I_k$ with $\operatorname{Re} s = 0$ (see Lemma 2.2 and 2.3).

Now consider the case $k = 2l+1$, $l \in \mathbb{N}_0$. For $\lambda = i\,\beta^I_k$ (2.1) has a root $s_{0,k} = -i\omega^I_k$, with $(2l+1)\pi < \omega^I_k\tau < (2l+1)\pi + \frac{\pi}{2}$. Notice that $\omega^I_k = \omega_l(\tau, \frac{3}{2}\pi)$ with ω_l as in Lemma 2.3. $s_{0,k}$ is the only root s of (2.1) for $\lambda = i\beta^I_k$ with $\operatorname{Re} s = 0$ (see Lemma 2.2 and 2.3).

In both cases there holds

$$\sin\tau(-1)^k\omega^I_k > 0. \tag{2.18}$$

Since $s_{0,k}$ is a simple root of (2.1) (see Lemma 2.1c) the implicit function theorem (see [10]) provides the existence of $\delta > 0$ and a unique differentiable function

$$s\colon\ \big(\beta^I_k - \delta,\, \beta^I_k + \delta\big) \to \mathbb{C},$$

where $s(\beta)$ solves equation (2.1) for $\lambda = i\,\beta$ and $s(\beta^I_k) = i(-1)^k\omega^I_k$. Moreover it holds

$$\frac{ds(\beta^I_k)}{d\beta} = \frac{\tau\beta^I_k + \sin\tau(-1)^k\omega^I_k + i\,\cos\tau(-1)^k\omega^I_k}{|\cos\tau(-1)^k\omega^I_k - i\big(\tau\beta^I_k + \sin\tau(-1)^k\omega^I_k\big)|^2}. \tag{2.19}$$

Using (2.18) this yields

$$\frac{d\,\operatorname{Re} s(\beta^I_k)}{d\beta} = \frac{\tau\beta^I_k + \sin\tau(-1)^k\omega^I_k}{|\cos\tau(-1)^k\omega^I_k - i\big(\tau\beta^I_k + \sin\tau(-1)^k\omega^I_k\big)|^2} > 0. \tag{2.20}$$

Therefore we can choose δ sufficiently small such that

$$\operatorname{Re} s(\beta) \begin{cases} < 0 & \text{for} \quad \beta_k^I - \delta < \beta < \beta_k^I, \\ = 0 & \text{for} \quad \beta = \beta_k^I, \\ > 0 & \text{for} \quad \beta_k^I < \beta < \beta_k^I + \delta. \end{cases}$$

On the other hand we know that $i(-1)^k \omega_k^I$ is the only solution with zero real part of (2.1) for $\lambda = i\beta_k^I$ (see Lemma 2.2) and that the real part of every solution of (2.1) is bounded above (see Lemma 2.1b). So the assertion of the proposition is proved.

Lemma 2.6 *Let $k \in \mathbb{N}_0$. For every $\lambda \in G_{\tau,k} \setminus \overline{G_{\tau,k-1}}$ the number of zeros with positive real parts (counted by their multiplicities) of (2.1) is constant.*

Proof First recall that all solutions with positive real part are in the open and bounded set S_λ (see Lemma 2.1b). Let $S := \bigcup\limits_{\lambda \in G_{\tau,k} \setminus \overline{G_{\tau,k-1}}} S_\lambda$. S is an open and bounded set. By definition it holds $|\Phi_{\tau,\lambda}(z)| > 0$ for all $z \in \partial S$. By Theorem 9.17.4 of [10, p.243], an application of Rouche's theorem, the number of zeros with positive real part is constant for all $\lambda \in G_{\tau,k} \setminus \overline{G_{\tau,k-1}}$.

We are now in a position to state the main result of this section.

Theorem 2.1 *Let $k \in \mathbb{N}_0$, $\tau > 0$. For any given $\lambda \in G_{\tau,k} \setminus \overline{G_{\tau,k-1}}$ the number of zeros with positive real parts (counted by their multiplicities) of (2.1) is exactly k.*

Proof The theorem is proved by induction on $k \in \mathbb{N}_0$. First notice that $0 \in G_{\tau,0}$ and that (2.1) with $\lambda = 0$ has no solution with positive real part. Consequently for all $\lambda \in G_{\tau,0}$ equation (2.1) has no solution with positive real part (see Lemma 2.6).

Suppose that (2.1) for $\lambda \in G_{\tau,k} \setminus \overline{G_{\tau,k-1}}$ has exactly $k \in \mathbb{N}_0$ solutions with positive real part. Proposition 2.1 yields that (2.1) has exactly $k+1$ solutions with positive real part for $\lambda \in G_{\tau,k+1} \setminus \overline{G_{\tau,k}}$. The theorem is proved.

2.2 D-decomposition of the τ-halfline.

Now we want to use the preceding results to give an D-decomposition of the τ-halfline. For any given $\tau > 0$ and $\theta \in [0, 2\pi)$ we define a sequence $\big(\lambda_k(\tau,\theta)\big)_{k \in \mathbb{N}_0}$ by

$$C_{\tau,k}^+ \cap \{\lambda \in \mathbb{C} \colon \lambda = \rho e^{i\theta}, \ \rho \ge 0\} = \{\lambda_k(\tau,\theta)\}, \quad k \in \mathbb{N}_0 \quad \text{if} \quad \theta \in [0,\pi],$$
$$C_{\tau,k}^- \cap \{\lambda \in \mathbb{C} \colon \lambda = \rho e^{i\theta}, \ \rho \ge 0\} = \{\lambda_k(\tau,\theta)\}, \quad k \in \mathbb{N}_0 \quad \text{if} \quad \theta \in (\pi,2\pi).$$

Lemma 2.7 *For $\tau > 0$ it holds*

a) *For any $k \in \mathbb{N}_0$ $(k \in \mathbb{N})$ and $\theta \in (0, 2\pi)$ $(\theta \in [0, 2\pi))$ $|\lambda_k(\tau,\theta)|$ is a decreasing function of $\tau > 0$. $\lambda_0(\tau, 0) = 1$ for all $\tau > 0$.*

b) $\lim\limits_{\tau \to 0+} |\lambda_k(\tau,\theta)| = \infty$ *provided $k > 0$ or $k = 0$ and $\theta \in \left[\frac{\pi}{2}, \frac{3}{2}\pi\right]$*

$\lim\limits_{\tau \to 0+} \lambda_0(\tau,\theta) = 1 + i \tan\theta$ *if $\theta \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{3}{2}\pi, 2\pi\right)$.*

c) $\lim\limits_{\tau \to \infty} \lambda_k(\tau,\theta) = e^{i\theta}$.

Proof Suppose $\theta \in (0, \pi)$. By construction of $C^+_{\tau,k}$, there is $\omega_k(\tau,\theta) \in (\omega^R_k, \omega^R_{k+1})$ such that $\lambda_k(\tau,\theta) = K_\tau(\omega_k(\tau,\theta))$ if $k = 2l$ and $\overline{\lambda_k(\tau,\theta)} = K_\tau(\omega_k(\tau,\theta))$ if $k = 2l+1$, $l \in \mathbb{N}_0$. Now consider $\omega_k(\tau,\theta)$ as function of $\tau > 0$. By differentiating (2.10) with respect to τ we obtain:

$$\frac{d\omega_k(\tau,\theta)}{d\tau} = -\frac{\omega_k(\tau,\theta)(1+\omega_k^2(\tau,\theta))}{1+\tau(1+\omega_k^2(\tau,\theta))} < 0. \tag{2.21}$$

Consequently $\omega_k(\tau,\theta)$ is a decreasing function of $\tau > 0$, and thus, by (2.9), $|\lambda_k(\tau,\theta)|$ is also a decreasing function of $\tau > 0$. This proves part a) with $\theta \in (0,\pi)$. Part a) with $\theta \in (\pi, 2\pi)$ follows by symmetry (see Lemma 2.2). For $k = 0$ and $\theta = 0$ there holds $\lambda_0(\tau, 0) = 1$. The cases $(\theta = 0,\ k \in \mathbb{N})$ and $(\theta = \pi,\ k \in \mathbb{N}_0)$ can be proved in a similar way.

Equations (2.8) and (2.9) provide b) and part c) follows from (2.8), (2.9) and (2.12).

Using the lemma above we obtain

Lemma 2.8 *For any* $\tau_1,\ \tau_2 > 0$ *with* $\tau_1 < \tau_2$ *there holds*

a) $G_{\tau_2,k} \subsetneqq G_{\tau_1,k}$ *for any* $k \in \mathbb{N}_0$*;*
b) $\partial G_{\tau_1,0} \cap \partial G_{\tau_2,0} = \{1\}$ *and* $\partial G_{\tau_1,k} \cap \partial G_{\tau_2,k} = \emptyset$*, for* $k \in \mathbb{N}$.

To complete the discussion about the τ-dependence of the regions $G_{\tau,k}$ we consider the limiting cases $\tau = 0$ and $\tau \to \infty$.

Lemma 2.9 *Let* $\tau = 0$*. Equation (2.1) has exactly one solution, namely* $s = \lambda - 1$.

For $\tau \in (0, \infty)$ we set $z = \tau s$. From (2.1) for $\tau \to \infty$ we obtain

$$\Phi_\lambda(z) := \exp(z) - \lambda = 0. \tag{2.22}$$

It is easy to prove, that

Lemma 2.10 *For* $|\lambda| < 1$ Φ_λ *has only zeros with negative real part, and for* $|\lambda| > 1$ Φ_λ *has only zeros with positive real part.* z *is a zero of* Φ_λ *with* $\operatorname{Re} z = 0$ *if and only if* $|\lambda| = 1$.

Remark 2.1 For any $\tau > 0$ there holds

$$G_{\infty,0} \subsetneqq G_{\tau,0} \subsetneqq G_{0,0}, \tag{2.23}$$

where

$$G_{\infty,0} := \{\lambda \in \mathbb{C}\colon |\lambda| < 1\} \quad \text{and} \quad G_{0,0} := \{\lambda \in \mathbb{C}\colon \operatorname{Re}\lambda < 1\}. \tag{2.24}$$

In order to be able to state the main results on the D-decomposition of the τ-halfline we define positive real numbers $\tau_k(\lambda)$ for $\lambda \in \mathbb{C}$, $|\lambda| > 1$, such that $\lambda \in \partial G_{\tau,k}$ if and only if $\tau = \tau_k(\lambda)$ for $k \in \mathbb{N}_0$ $(k \in \mathbb{N})$ if $\operatorname{Re}\lambda < 1$ $(\operatorname{Re}\lambda \geq 1)$. For $\operatorname{Re}\lambda \geq 1$ we set $\tau_0(\lambda) := 0$. Moreover let $\tau_{-1}(\lambda) := 0$.

As a consequence of Lemma 2.7 we obtain

Proposition 2.2 *If $\lambda \in \mathbb{C} \setminus \mathbb{R}$ with $|\lambda| > 1$ then $(\tau_k(\lambda))_{k\in\mathbb{N}_0}$ is an unbounded and strictly increasing sequence.*

If $\lambda \in \mathbb{R}$ with $\lambda > 1$ then $(\tau_k(\lambda))_{k\in\mathbb{N}_0}$ is an unbounded and increasing sequence with $\tau_{2k-1}(\lambda) = \tau_{2k}(\lambda) < \tau_{2k+1}(\lambda)$.

If $\lambda \in \mathbb{R}$ with $\lambda < -1$ then $(\tau_k(\lambda))_{k\in\mathbb{N}}$ is an unbounded and increasing sequence with $\tau_{2k}(\lambda) = \tau_{2k+1}(\lambda) < \tau_{2k+2}(\lambda)$.

Remark 2.2 One can compute $\tau_k(\lambda)$ explicitly. Because of the symmetry properties of $K_\tau(\omega)$, see Lemma 2.2, it is sufficient to consider $\mathbb{C} \ni \lambda = |\lambda|e^{i\theta}$ with $\operatorname{Im}\lambda \geq 0$, i.e. $\theta \in [0, \pi]$. It holds

$$\tau_{2k}(\lambda) = \frac{2k\pi + \theta - \arctan\left(\sqrt{|\lambda|^2 - 1}\right)}{\sqrt{|\lambda|^2 - 1}}$$

for $k \in \mathbb{N}_0$ $(k \in \mathbb{N})$ if $\operatorname{Re}\lambda \leq 1$ $(\operatorname{Re}\lambda > 1)$ and

$$\tau_{2k+1}(\lambda) = \frac{2(k+1)\pi - \theta - \arctan\left(\sqrt{|\lambda|^2 - 1}\right)}{\sqrt{|\lambda|^2 - 1}}$$

for $k \in \mathbb{N}_0$. Note that $\arctan\left(\sqrt{\lambda^2 - 1}\right) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Theorem 2.2

a) *Let $\lambda \in \mathbb{C} \setminus \mathbb{R}$ with $|\lambda| > 1$. For $\operatorname{Re}\lambda < 1$ $(\operatorname{Re}\lambda \geq 1)$ and $\tau \in (\tau_{k-1}(\lambda), \tau_k(\lambda)]$, $k \in \mathbb{N}_0$ $(k \in \mathbb{N})$ the number of zeros with positive real part of (2.1) counted by their multiplicities is exactly k.*

b) *Let $\lambda \in \mathbb{R}$. For $\lambda > 1$ and $\tau \in (\tau_{2k}(\lambda), \tau_{2k+2}(\lambda)]$, $k \in \mathbb{N}_0$, equation (2.1) has exactly $2k+1$ solutions with positive real part. For $\lambda < -1$ and $\tau \in (\tau_{2k-1}(\lambda), \tau_{2k+1}(\lambda)]$, $k \in \mathbb{N}_0$, equation (2.1) has exactly $2k$ solutions with positive real part.*

Proof Theorem 2.1, Lemma 2.8 and the definition of $\tau_k(\lambda)$ yield the assertions.

3 Stability of Delay Differential Equations

We consider the following system of delay differential equations:

$$\dot{x}(t) = Ax(t) + Bx(t-\tau) + F(x(t), x(t-\tau)), \tag{3.1}$$

where $\tau > 0$; $A, B \in \mathbb{R}^{n,n}$, $F \in C^k(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$, $k \geq 1$ and $F(0,0) = DF(0,0) = 0$. It follows $\bar{x} = 0$ is an equilibrium point of (3.1).

3.1 Characteristic equation

The linear part of the system (3.1) is given by

$$\dot{x}(t) = Ax(t) + Bx(t-\tau), \tag{3.2}$$

where $\tau \geq 0$; $A, B \in \mathbb{R}^{n,n}$. The corresponding characteristic equation satisfies:

$$\det(sI - A - B\exp(-\tau s)) = 0. \tag{3.3}$$

We are interested in special matrices A and B, for which it is possible to study the properties of the solutions of the characteristic equation above by help of (2.1).

Definition 3.1 We say the matrices $A, B \in \mathbb{R}^{n,n}$ satisfy *condition (C)* if there is a regular (unitary) matrix $M \in \mathbb{C}^{n,n}$ such that $A = M(D_A + T_A)M^{-1}$ and $B = M(D_B + T_B)M^{-1}$, where $D_A = \operatorname{diag}(-p_1, \ldots, -p_n) \in \mathbb{R}^{n,n}$, with $p_i > 0$, $i \in \{1, \ldots, n\}$, $D_B = \operatorname{diag}(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^{n,n}$, and T_A, T_B are upper triangular with all diagonal entries equal to zero.

Example 3.1 If $A = \operatorname{diag}(-p, \ldots, -p) \in \mathbb{R}^{n,n}$, with $p > 0$, and $B \in \mathbb{R}^{n,n}$ is a general matrix, or if $A \in \mathbb{R}^{n,n}$ is a matrix with n real negative eigenvalues and $B = \operatorname{diag}(\lambda, \ldots, \lambda) \in \mathbb{R}^{n,n}$, with $\lambda \in \mathbb{R}$, then A and B satisfy the condition (C) (see [4, 7]).

Using the multiplicativity of the determinant function we prove

Lemma 3.1 *Let $A, B \in \mathbb{R}^{n,n}$ satisfy condition (C). Then:*

$$\det(sI - A - B\exp(-\tau s)) = \exp(-\tau s) \prod_{i=1}^{n} \big[(s + p_i)\exp(\tau s) - \lambda_i\big].$$

Remark 3.1

a) Setting $\acute{s} = \frac{s}{p}$, $\acute{\tau} = p\tau$ and $\acute{\lambda} = \frac{\lambda}{p}$ into $(s + p)\exp(s\tau) - \lambda = 0$ we obtain

$$\Phi_{\acute{\tau},\acute{\lambda}}(\acute{s}) := (\acute{s} + 1)\exp(\acute{s}\acute{\tau}) - \acute{\lambda} = 0. \tag{3.4}$$

b) From a) and Lemma 3.1. It follows: If $A, B \in \mathbb{R}^{n,n}$ satisfy (C), equation (3.3) can be reduced to n simpler equations of type (3.4) with $\acute{\tau} = p_i\tau$ and $\acute{\lambda} = \frac{\lambda_i}{p_i}$, $i \in \{1, \ldots, n\}$.

3.2 τ-dependent stability

In the following we study the τ-dependent stability properties of the trivial equilibrium $\bar{x} = 0$ of system (3.1).

Theorem 3.1 *Suppose the matrices $A, B \in \mathbb{R}^{n,n}$ satisfy (C). Then*

a) *If $|\lambda_i| \le p_i$ and $\lambda_i \ne p_i$ for all $i \in \{1, \ldots, n\}$, then $\bar{x} = 0$ is asymptotically stable for any $\tau \ge 0$.*

b) *If there is $l \in \{1, \ldots, n\}$ such that $\operatorname{Re}\lambda_l \ge p_l$ $(\operatorname{Re}\lambda_l > p_l)$ and $\lambda_l \ne p_l$, then $\bar{x} = 0$ is unstable for any $\tau > 0$ $(\tau \ge 0)$.*

c) *Suppose $\operatorname{Re}\lambda_i < p_i$ for all $i \in \{1, \ldots, n\}$. Further we suppose there exist $l \in \{1, \ldots, n\}$ such that $|\lambda_l| > p_l$. Then there is $0 < \tau^s$, such that $\bar{x} = 0$ is asymptotically stable for $0 \le \tau < \tau^s$ and unstable for $\tau > \tau^s$.*

Proof The case $\tau = 0$ is covered by Lemma 2.9. In the sequel we suppose $\tau > 0$. $|\lambda_i| \le p_i$ and $\lambda_i \ne p_i$ for all $i \in \{1, \ldots, n\}$ yields $\frac{\lambda_i}{p_i} \in G_{\infty,0} \subset G_{\tau p_i,0}$ (see Lemma 2.8a) for arbitrary $\tau > 0$. It follows that for any $\tau > 0$ the characteristic equation (3.3) has only roots with negative real part (see Theorem 2.1). Standard results on stability in first approximation (see [11, pp.160, 161]) prove part a.

If there is $l \in \{1, \ldots, n\}$ such that $\operatorname{Re}\lambda_l \ge p_l$ and $\lambda_l \ne p_l$, then there holds $\frac{\lambda_l}{p_l} \notin \overline{G_{\tau p_l,0}}$ (see Lemma 2.5d) for arbitrary $\tau > 0$. This implies (see Theorem 2.1) that for any τ the characteristic equation (3.3) has at least one root with positive real part, and thus part b is proved.

Now let $l \in \{1,\dots,n\}$ be such that $|\lambda_l| > p_l$ and $\operatorname{Re}\lambda_l < p_l$. By Lemmas 2.7 and 2.8 there exist a $\tau_l^s > 0$ such that

$$\begin{aligned}
&\frac{\lambda_l}{p_l} \in G_{\tau p_l,0}, && \tau < \tau_l^s,\\
&\frac{\lambda_l}{p_l} \in \partial G_{\tau p_l,0}, && \tau = \tau_l^s,\\
&\frac{\lambda_l}{p_l} \notin \overline{G_{\tau p_l,0}}, && \tau > \tau_l^s.
\end{aligned}$$

We set

$$\tau^s = \min\{\tau_l^s\colon l \in \{1,\dots,n\} \text{ with } |\lambda_l| > p_l > \operatorname{Re}\lambda_l\}.$$

Consequently the characteristic equation (3.3) has only roots with negative real part if $\tau < \tau^s$ and at least one root with positive real part if $\tau > \tau^s$.

Remark 3.2 τ^s in Theorem 3.1c is defined by

$$\tau^s = \min\left\{\frac{\theta_l - \arctan(\frac{1}{p_l}\sqrt{|\lambda_l|^2 - p_l^2})}{\sqrt{|\lambda_l|^2 - p_l^2}}\colon\ l \in \{1,\dots,n\} \text{ with } |\lambda_l| > p_l > \operatorname{Re}\lambda_l\right\},$$

where $\theta_l \in [0, 2\pi)$ such that $\lambda_l = |\lambda_l|\, e^{i\theta_l}$ and $\arctan\left(\frac{1}{p_l}\sqrt{|\lambda_l|^2 - p_l^2}\right) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

For the sake of completeness we consider the limiting case $\tau \to \infty$. For $\tau \in (0,\infty)$ we set $t' = \frac{t}{\tau}$ and $y(t') = x(t'\tau)$. Then (3.1) becomes

$$\frac{1}{\tau}\,\dot y(t') = Ay(t') + By(t'-1) + F(y(t'), y(t'-1)).$$

For $\tau \to \infty$ we obtain

$$Ay(t') + By(t'-1) + F(y(t'), y(t'-1)) = 0. \tag{3.5}$$

Theorem 3.2 *Suppose the matrices A, $B \in \mathbb{R}^{n,n}$ satisfy (C). Then*

a) *If $|\lambda_i| < p_i$ for all $i \in \{1,\dots,n\}$, then $\bar x = 0$ as solution of equation (3.5) is asymptotically stable.*
b) *If there is $l \in \{1,\dots,n\}$ such that $|\lambda_l| > p_l$, then $\bar x = 0$ is unstable.*

Proof Since A is a regular matrix, equation (3.5) can be rewritten as

$$y(t) = Cy(t-1) + g(y(t'-1)),$$

where $C = -A^{-1}B$ and $g \in C^1(U,V)$; $U, V \subset \mathbb{R}^n$ neighborhoods of $\bar x = 0$, an appropriate function with $g(0) = Dg(0) = 0$. The eigenvalues μ_i of C satisfy $\mu_i = \frac{\lambda_i}{p_i}$, $i \in \{1,\dots,n\}$. From Lemma 2.10 and Remark 3.1: $|\mu_i| < 1$ if $|\lambda_i| < p_i$ for all $i \in \{1,\dots,n\}$ and if there is $l \in \{1,\dots,n\}$ such that $|\lambda_l| > p_l$ then $|\mu_l| > 1$. The Theorem about Stability by First Approximation for Difference Equations (see [20, p.104]) completes the proof.

4 Hopf Bifurcation

In this section we derive sufficient conditions for the occurrence of Hopf bifurcation points in (3.1) with bifurcation parameter τ.

Theorem 4.1 *Suppose that*

A1: *Matrices A and B satisfy condition (C).*

A2: *There are* $i_0, i_1 \in \{1,\dots,n\}$ *with* $i_0 \neq i_1$ *if* $\lambda_{i_0} \in \mathbb{C}\setminus\mathbb{R}$ *and* $i_0 = i_1$ *if* $\lambda_{i_0} \in \mathbb{R}$, *such that* $\frac{\overline{\lambda_{i_1}}}{p_{i_1}} = \frac{\lambda_{i_0}}{p_{i_0}}$ *and* $p_{i_0} < |\lambda_{i_0}|$. *Assume* $\operatorname{Im}\lambda_{i_0} \geq 0$. *Set*

$$\tau_{2k}^H := \frac{2k\pi + \theta_{i_0} - \arctan\left(\frac{1}{p_{i_0}}\sqrt{|\lambda_{i_0}|^2 - p_{i_0}^2}\right)}{\sqrt{|\lambda_{i_0}|^2 - p_{i_0}^2}}$$

for $k \in \mathbb{N}_0$ *if* $\operatorname{Re}\lambda_{i_0} < p_{i_0}$ *and* $k \in \mathbb{N}$ *if* $\operatorname{Re}\lambda_{i_0} \geq p_{i_0}$, *and*

$$\tau_{2k+1}^H := \frac{2(k+1)\pi - \theta_{i_0} - \arctan\left(\frac{1}{p_{i_0}}\sqrt{|\lambda_{i_0}|^2 - p_{i_0}^2}\right)}{\sqrt{|\lambda_{i_0}|^2 - p_{i_0}^2}}$$

for $k \in \mathbb{N}_0$, *where* $\theta_{i_0} \in [0,\pi)$ *such that* $\lambda_{i_0} = |\lambda_{i_0}|\, e^{i\theta_{i_0}}$.

A3: *For any* $i \in \{1,\dots,n\}\setminus\{i_0,i_1\}$, *for which there exist* $l \in \mathbb{N}_0$ *such that* $\frac{\lambda_i}{p_i} \in \partial G_{\tau_k^H p_i, l}$, *it follows* $\frac{1}{p_i}\sqrt{|\lambda_i|^2 - p_i^2} \neq \mathbb{N}$.

Then a Hopf bifurcation takes place at $\tau = \tau_k^H$ *for* $k \in \mathbb{N}_0$ *if* $\operatorname{Re}\lambda_{i_0} < p_{i_0}$ *respectively* $k \in \mathbb{N}$ *if* $\operatorname{Re}\lambda_{i_0} \geq p_{i_0}$.

Proof The Theorem is proved by verifying the hypotheses (H1) and (H2) of the Hopf Bifurcation Theorem (see [15, pp.331–333]). If $\tau = \tau_k^H$, equations (2.8) – (2.10) and (2.16) yield $\frac{\lambda_{i_0}}{p_{i_0}} \in \partial G_{\tau_k^H p_{i_0}, k}$. Lemma 3.1 and Remark 3.1 provide that $s_0 = ip_{i_0}\omega(\tau_k^H p_{i_0}, \theta_{i_0})$ is a purely imaginary root of the characteristic equation (3.3), where $\omega(\tau_k^H p_{i_0}, \theta_{i_0})$ is the unique solution of equation (2.10) in $I_k(\tau_k^H p_{i_0}, \theta_{i_0})$ (see Lemma 2.3). From Lemma 2.1c we obtain $i\omega(\tau_k^H p_{i_0}, \theta_{i_0})$ is a simple root of (2.1) for $\lambda = \frac{\lambda_{i_0}}{p_{i_0}}$ and $\tau = \tau_k^H p_{i_0}$, and consequently s_0 is a simple root of the characteristic equation (3.3) for $\tau = \tau_k^H p_{i_0}$. Further we get by (A3) that there are no other roots $s \neq s_0, \overline{s_0}$ of the characteristic equation (3.3) for $\tau = \tau_k^H$ which satisfy $s = ms_0$ with $m \in \mathbb{Z}$. This verifies hypothesis (H1) in [15, pp.331–333].

Since s_0 is a simple root the implicit function theorem (see [10]) provides the existence of $\delta > 0$ and a differentiable function $s\colon (\tau_k^H - \delta, \tau_k^H + \delta) \to \mathbb{C}$ with $s(\tau_k^H) = s_0$ and $s(\tau)$ solves (3.3). Moreover one can compute

$$\frac{d\operatorname{Re}s}{d\tau}(\tau_k^H) = p_{i_0}\frac{\omega^2(\tau_k^H p_{i_0}, \theta_{i_0})}{(1 + \tau_k^H p_{i_0})^2 + (\tau_k^H p_{i_0})^2\,\omega^2(\tau_k^H p_{i_0}, \theta_{i_0})} > 0.$$

Thus, hypothesis (H2) in [15, pp.331–333] is satisfied.

Remark 4.1 If $n = 1$ and $n = 2$ with $\lambda_{i_0} \in \mathbb{C}\setminus\mathbb{R}$, respectively, condition (A3) in Theorem 4.1 is always satisfied.

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