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Hausdorff Dimension Estimates by Use of a Tubular Carathéodory Structure and Their Application to Stability Theory

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Abstract: The paper is concerned with upper bounds for the Hausdorff dimension of flow invariant compact sets on Riemannian manifolds and the application of such bounds to global stability investigations of equilibrium points. The proof of the main theorem uses a special Carathéodory dimension structure in order to get contraction conditions for the considered Carathéodory measures which majorize the Hausdorff measures. The Hausdorff dimension bounds in the general case are formulated in terms of the eigenvalues of the symmetric part of the operator which generates the associated system in normal variations with respect to the direction of the vector field. For sets with an equivariant tangent bundle splitting dimension bounds are derived in terms of uniform Lyapunov exponents. A generalization of the well-known theorems of Hartman-Olech and Borg is given.

Keywords: *Hausdorff dimension; Carathéodory dimension structure; outer measures via tube covers; system in normal variations; global stability; uniform Lyapunov exponents; equivariant tangent bundle splitting; Riemannian manifolds.*

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1 Introduction

The first method of Lyapunov ([9, 36, 47, 49]) traditionally includes all the approaches for the stability investigation of a given solution of an ODE (or an other dynamical

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system) which consider the perturbed solutions by means of various types of linearized or variational equations. In particular this method can be used to construct explicitly (i.e. in the form of a series of known functions and exponential terms including the Lyapunov characteristic exponents) integral manifolds of stationary solutions in order to determine the stability character of these solutions.

As a rule in the given variational equation new coordinates are introduced in order to separate the normal components of the vector fields which act transversally to the flow lines. The main idea of reparametrization and the use of flow information in the transversal to an orbit direction goes back to ([20, 48]). Using these techniques the well-known theorems of Hartman-Olech and Borg ([4, 19, 20]) on global asymptotic stability are derived. For ODE's in $\mathbb{R}^n$ these results were extended and generalized in [29, 32] for other types of stability behavior (stability in the sense of Poincaré and Zhukovskij) including into the consideration Lyapunov functions. Variational systems written in normal coordinates are also used in stability theory to show orbital stability of solutions of a differential equation ([20, 31, 32]). For bounded semi-orbits these methods are extended in [31] to vector fields on Riemannian manifolds. In particular, in this paper sufficient conditions for orbital stability and instability are deduced by estimating the singular values of the fundamental operator of the linearized vector field.

Note that for simple mechanical systems in Lagrange form the physical paths can be interpreted as geodesics on a Riemannian manifold ([17, 23, 24]). A prototype of such systems with instability behavior in the sense of Zhukovskij are geodesic flows on the unit tangent bundle of a manifold with negative curvature ([10, 17, 23, 24, 42]). These systems are characterized by a uniform splitting of the tangent bundle into invariant subbundles (with respect to the linearization) having equal contracting or expanding rates in all points of the bundle. They belong to a special type of (strong) hyperbolic systems. Unfortunately most of the interesting equations are only quasi-hyperbolic ([7, 13, 42, 43]).

Stability investigations of flows are closely connected with global properties of invariant sets or attractors such as dimension (topological, Hausdorff, box-counting etc.) and the topological shape of these sets (connectness, point-like type etc.) ([14, 18]).

The first general results for upper Hausdorff dimension estimates of flow invariant sets in $\mathbb{R}^n$ in terms of singular values of the linearization are given by [6]. This approach was extended in [25, 39] to map-invariant sets on Riemannian manifolds and in [26, 28, 29] by including Lyapunov functions into the contraction conditions for outer Hausdorff measures. In [8, 46] the Douady-Oesterlé results were extended to estimates for evolution systems in general Hilbert spaces. Hausdorff dimension estimates of general flow invariant sets using the eigenvalues of the symmetric part of the operator part of the (standard) equation in variation are deduced in [45] for the $\mathbb{R}^n$ and in [39] for manifolds. Douady-Oesterlé estimates for piecewise smooth maps on manifolds are given in [44]. The hyperbolic or quasi-hyperbolic structure was considered in dimension estimates in [10, 13] where also an entropy term into the estimate was introduced.

Various dimension upper bounds of invariant sets allow conclusions on the dynamical behavior of the system. The key step in the papers [29, 39, 45] is to prove that the Hausdorff dimension for the maximal compact invariant set is less than two. By a result of Smith ([45]) such a set contains no simple closed piecewise smooth invariant curves. In particular the system has no non-constant periodic orbits. On the base of such dimension estimates a generalization of the mentioned global stability results of Hartman-Olech and Borg, but also of other types of classical results from the Bendixson-Poincaré theory were derived in [29, 34, 35].

Parallel to Hausdorff dimension estimates a number of upper bounds for the box dimension of invariant sets were deduced ([3, 21, 22, 30, 38, 46]). The box dimension of a set is always not smaller than the Hausdorff dimension and gives important information about the possibility to use embedding homeomorphisms, which map the given invariant set orthogonal and one-to-one on a hyperplane in standard position ([22, 38]). Recently it was shown that such homeomorphisms can be chosen with Hölder-Lipschitz continuous inverse ([12]) which enables conclusions for dimension estimates.

Hausdorff and box dimension estimates for flow invariant sets show its effectivity if various types of local, global and uniform Lyapunov exponents are introduced ([7, 8, 25, 28, 46]). On the base of such Lyapunov exponents the Lyapunov dimension of a set was defined (Kaplan-Yorke formula [25, 42]) and it was conjectured that in typical cases this dimension coincides with the Hausdorff dimension.

Parallel to the dimension and stability investigation of invariant sets of flows and cascades various types of dimensions of an invariant measure have been developed ([7, 25, 41]). Defining for the invariant ergodic measure of a flow the Lyapunov exponents one can introduce the Lyapunov dimension of this measure which is an upper bound of the Hausdorff dimension of the measure. (The Hausdorff dimension of the measure is the largest lower bound of the Hausdorff dimension of the support of the measure ([25]).) As in the measure free case various stability properties of the underlying flow may be derived from the properties of the Lyapunov exponents of the measure. It is shown in [7] that if the invariant measure is ergodic and all Lyapunov exponents of the measure are negative, the support of this measure is a stable equilibrium point. If exactly one exponent is zero and the remaining ones are negative, the support is an equilibrium point or a stable limit cycle.

An important class of invariant sets of dynamical systems are strange attractors which have locally the structure of the product of a smooth (often one-dimensional) submanifold directed 'along the attractor' and a Cantor-like set 'transversal' to the attractor ([18, 41]). Thus, it is natural to investigate the stability and dimension properties of such attractors considering the intersection of the attractors with surfaces which are locally transversal to the attractor ([20, 26]). The use of transverse intersections (Poincaré sections) is well-known in stability theory investigations of flow orbits: contracting or expanding behavior in sections transverse to the flow line directions is the main reason for properties of stability or instability of the considered orbit ([29, 31, 32]).

The paper is organized as follows. In Section 2 we present a short review of basic facts on Riemannian geometry. We introduce the variational system written in normal variations, transversal to the evolution direction of the flow lines, which is natural to investigate in the case of attractors of differential equations. In Section 3 we give the definition of a special Carathéodory structure adapted for the dimension investigation of flow invariant sets. It is defined via covering elements which are tubular neighborhoods of arcs of smooth curves to approximate the fiber structure of the sets. The main results of the paper are contained in Section 4. For flow negatively invariant sets which do not contain singular points of the vector field an upper bound of the Hausdorff dimension is given. The estimates are derived by means of Carathéodory measures which are contractive under the flow and majorize the Hausdorff measure. These results generalize those from [26, 27] on Riemannian manifolds. The estimates are formulated in terms of the eigenvalues of the symmetric part of the generated operator of the associated system in normal variation. Assuming special properties of the stable and unstable manifolds of equilibrium points the results are generalized for vector fields having a finite number of such equilibrium points in the considered invariant set. The used Carathéodory measures show in many cases a better contracting behavior under the positive semi-flow than the Hausdorff measures do. Section 5 is concerned with Hausdorff dimension estimates of flow invariant sets with an equivariant tangent bundle splitting which are formulated in terms of uniform Lyapunov exponents. In Section 6 we end with a discussion of the effectivity of the obtained Hausdorff dimension estimates. In addition we obtain results about the asymptotic behavior of the dynamical system using the dimension bounds, which are closely related to results in [4, 19, 20].

2 The System in Normal Variation

In this section we introduce a modified variational equation for a vector field f which will be used for modeling the variation of time translated pieces of hypersurfaces orthogonal to a considered orbit. This idea originates from investigations on stability behavior of solutions of a differential equation (see [20, 31, 32]), where together with the movements of phase points along a trajectory one considers their movements in transversal direction. Projecting the covariant derivative of the vector field along a reference orbit into the $(n-1)$-dimensional tangent space lying orthogonal to the vector field in an arbitrary point of the orbit we get a variational equation describing the normal variation. For the first time this type of variational equation has been applied to dimensional estimates in [26, 27].

Let us recall some notation from linear algebra and differential geometry used later. If V and W are m-dimensional Euclidean spaces with scalar products $\langle\cdot,\cdot\rangle_V$ and $\langle\cdot,\cdot\rangle_W$, respectively, and $L\colon V\to W$ is a linear operator, then the adjoint operator $L^*\colon W\to V$ is the linear operator uniquely determined by the relation $\langle L\xi,\eta\rangle_W=\langle\xi,L^*\eta\rangle_V$ for all $\xi\in V$, $\eta\in W$. The *singular values* of the operator L are the eigenvalues of the positive semidefinite operator $(L^*L)^{\frac{1}{2}}\colon V\to V$. We denote them by $\sigma_1(L)\geq\cdots\geq\sigma_m(L)\geq 0$ ordered with respect to size and multiplicity. For $d\in\mathbb{R}$ let $\lfloor d\rfloor$ denote the largest integer less than d. For an arbitrary number $d\in[0,m]$ we define by

$$\omega_d(L)=\begin{cases}1 & \text{for}\quad d=0,\\ \sigma_1(L)\cdot\ldots\cdot\sigma_{\lfloor d\rfloor}(L)\sigma_{\lfloor d\rfloor+1}^{d-\lfloor d\rfloor}(L) & \text{for}\quad d\in(0,m],\end{cases}$$

the *singular value function of order d of L*. Let $\mathcal{E}$ be an ellipsoid in V and let $\sigma_1(\mathcal{E})\geq\ldots\geq\sigma_m(\mathcal{E})\geq 0$ denote the length of its semi-axes. For an arbitrary number $d\in[0,m]$ we introduce the *d-dimensional ellipsoid measure* by

$$\omega_d(\mathcal{E})=\begin{cases}1 & \text{for}\quad d=0,\\ \sigma_1(\mathcal{E})\cdot\ldots\cdot\sigma_{\lfloor d\rfloor}(\mathcal{E})\sigma_{\lfloor d\rfloor+1}^{d-\lfloor d\rfloor}(\mathcal{E}) & \text{for}\quad d\in(0,m].\end{cases}$$

For the linear operator $L\colon V\to W$ and the ball $B(O,r)$ of radius r around the origin O of V the image $LB(O,r)$ is an ellipsoid in W with length of semi-axes $\sigma_i(L)r$. For $d\in[0,m]$ it holds

$$\omega_d(LB(O,r))=\omega_d(L)\,r^d. \tag{2.1}$$

Consider now a Riemannian manifold (M,g) of dimension n $(n\geq 2)$ and, for simplicity, of class C^∞, which we call smooth. Denote by T_pM the tangent space at $p\in M$. The Christoffel symbols of second kind on (M,g) with respect to a chart $x\colon D(x)\to R(x)$

are given by the n^3 smooth functions $\Gamma_{ij}^k = \frac{1}{2}g^{ks}(g_{js,i} + g_{si,j} - g_{ij,s})$ (throughout this paper with summation on repeated indices), where $g_{kl,r} = \frac{\partial g_{kl}}{\partial x^r}$. Here and in the sequel let $f\colon M \to TM$ be a vector field of class C^2 on the n-dimensional Riemannian manifold M $(n \geq 2)$ and let us consider the corresponding differential equation

$$\dot{u} = f(u). \tag{2.2}$$

For simplicity we assume that the global flow $\varphi\colon \mathbb{R} \times M \to M$ of (2.2) exists. This flow φ can also be written as one-parameter family of C^2-diffeomorphisms $\{\varphi^t\}_{t\in\mathbb{R}}$ with $\varphi^t(\cdot) = \varphi(t,\cdot)$. In a chart x around p let $\{\partial_i(p)\}$ be the canonical basis of T_pM and $f(p) = f^i\partial_i(p)$ the representation of the vector field (2.2). The *covariant derivative* of f in p is the linear operator $\nabla f(p)\colon T_pM \to T_pM$ defined by $\nabla f(p)v = \nabla_i f^k v^i \partial_k(p) = \Big(\frac{\partial f^k}{\partial x^i}\, v^i + \Gamma_{ij}^k f^j v^i\Big)\partial_k(p)$ for all $v = v^i\partial_i(p) \in T_pM$. For the linear operator $\nabla f(p)\colon T_pM \to T_pM$ in the Euclidean space $(T_pM, \langle\cdot,\cdot\rangle_{T_pM})$ we denote by $\nabla f(p)^*$ the adjoint operator and by $S\nabla f(p) := \frac{1}{2}[\nabla f(p) + \nabla f(p)^*]$ the symmetric part of $\nabla f(p)$.

Let $c\colon [a,b] \to M$ be a piecewise smooth curve such that the restrictions $c|_{[t_j,t_{j+1}]}$ are smooth for any $j = 1,\dots,m-1$. Recall that the length $l(c)$ of c is defined as $l(c) = \sum\limits_{j=1}^{m-1} \int\limits_{t_j}^{t_{j+1}} \|\dot{c}(t)\|\,dt$. For a C^1-curve $c\colon [a,b] \to M$ let $x^i(t)$ be the local coordinates of $c(t)$ in the chart x. Let $F(t)$ be a vector field along c, i.e., $F(t) \in T_{c(t)}M$ for all $t \in [a,b]$. The absolute derivative $\frac{DF(t)}{dt} \in T_{c(t)}M$ of F along c is defined in the chart x by

$$\frac{DF(t)}{dt} \equiv \nabla_{\dot{c}}F(t) := \left(\frac{dF^k}{dt} + \Gamma_{ij}^k F^j \dot{c}^i\right)\partial_k(c(t)).$$

For a given C^1-curve $c\colon [a,b] \to M$ and $v \in T_{c(t_0)}M$ $(t_0 \in [a,b])$ there exists a unique vector field F_v along c such that F_v is parallel along c, i.e., $\nabla_{\dot{c}}F_v \equiv 0$ and $F_v(t_0) = v$. This defines for any $s,t \in [a,b]$ with $s < t$ the parallel transport $\tau_{c(s)}^{c(t)}\colon T_{c(s)}M \to T_{c(t)}M$ along c from $c(s)$ to $c(t)$ which relates to any $v \in T_{c(s)}M$ the vector $F_v(t) \in T_{c(t)}M$.

Recall that a geodesic on (M,g) is a smooth curve $c\colon [a,b] \to M$ satisfying $\frac{D\dot{c}(t)}{dt} \equiv 0$. For any $p \in M$ and $v \in T_pM$ we denote the maximal geodesic with $\dot{c}(0) = v$ and $c(0) = p$ by $c_{p,v}$. Let $\mathcal{D}^1 \subset TM$ be the set of pairs $\{(p,v)\}$ with $p \in M$ and $v \in T_pM$ such that $c_{p,v}(1)$ exists. Then the *exponential map* $\exp\colon \mathcal{D}^1 \to M$ on (M,g) is given by $\exp((p,v)) = c_{p,v}(1)$ for all $(p,v) \in \mathcal{D}^1$ and $\exp_p$ is the restriction $\exp|_{T_pM\cap\mathcal{D}^1}$. It is well-known (see [24]) that $\mathcal{D}^1$ is open in TM, that $\exp\colon \mathcal{D}^1 \to M$ is smooth, and for any $p \in M$ there exists an open set $\mathcal{D}_p^1 \subset T_pM$ such that $\exp_p$ is a diffeomorphism on $\mathcal{D}_p^1$ and $\|d_{O_p}\exp_p\| = 1$.

The behavior of system (2.2) near a given solution $\varphi^{(\cdot)}(p)$ is described by the *variational equation*

$$\frac{Dy}{dt} = \nabla f(\varphi^t(p))y \tag{2.3}$$

(see [31, 39]). In local coordinates of a chart x around $\varphi^t(p)$ system (2.3) takes the form

$$\frac{Dy^k}{dt} = \frac{\partial f^k}{\partial x^i}\, y^i + \Gamma_{ij}^k f^j y^i = \nabla_i f^k y^i, \qquad k = 1,\dots,n.$$

For any $p \in M$ the differential $Y(t,p) = d_p\varphi^t$ is the operator solution of (2.3) with initial condition $Y(0,p) = \mathrm{id}_{T_pM}$.

All points $p \in M$ with $f(p) \neq O_p$ $(f(p) = O_p)$, where O_p denotes the origin of the tangent space T_pM, we call *regular* (*singular*) points of the vector field f. If p is a regular point we may consider the *system in normal variations* with respect to the solution $\varphi^{(\cdot)}(p)$ of (2.2) ([31])

$$\frac{Dz}{dt} = A(\varphi^t(p))z, \tag{2.4}$$

where the linear operator $A(p)\colon T_pM \to T_pM$ is given by

$$\begin{aligned} &A(p) = \nabla f(p) - B(p), \quad \text{where} \\ &B(p)v = 2\,\frac{f(p)}{\|f(p)\|^2}\,\langle f(p), S\nabla f(p)v\rangle \quad \text{for all} \quad v \in T_pM. \end{aligned} \tag{2.5}$$

The scalar product $\langle\cdot,\cdot\rangle$ and the associated norm $\|\cdot\|$ are taken in the tangent space T_pM. In coordinates of an arbitrary chart $x\colon D(x) \to R(x)$ around the regular point p the linear operator $A(p)$ is given by

$$A_i^k = \nabla_i f^k - \frac{2}{g_{mn}f^m f^n}\, f^k g_{jl} f^l S_i^j, \qquad k,i = 1,\dots,n,$$

where f^k and g_{jl} are the coordinates of the vector field f and the Riemannian metric tensor g in the chart x, respectively, and $S_i^j = \frac{1}{2}\big[g^{jk}\nabla_k f^p g_{pi} + \nabla_i f^j\big]$ is the representation in coordinates of the symmetric part $S\nabla f(p)$ of the covariant derivative of the vector field f in this chart. Note that for ODE's in $\mathbb{R}^n$ with standard metric the system in normal variations (2.4) coincides with the system in modified variations in [28, 29, 32]. Suppose that $p \in M$ is a regular point of f and $y(\cdot)$ is a solution of (2.3) along $\varphi^{(\cdot)}(p)$. This solution can be splitted for any $t \in \mathbb{R}$ into two orthogonal components as

$$y(t) = z(t) + \mu(t) f(\varphi^t(p)), \tag{2.6}$$

where $z(\cdot)$ is the solution of (2.4) with respect to $\varphi^{(\cdot)}(p)$ with initial condition $z(0) = y(0)$ and $\mu(\cdot)$ is a scalar valued C^1-function given by $\mu(t) = \langle y(t), f(\varphi^t(p))\rangle / \|f(\varphi^t(p))\|^2$.

For every regular point $p \in M$ of f we introduce the $(n-1)$-dimensional linear subspace

$$T^\perp(p) = \big\{v \in T_pM\colon \langle v, f(p)\rangle = 0\big\}$$

of the tangent space T_pM. Denote by $SA(p) := \frac{1}{2}[A(p) + A(p)^*]$ the symmetric part of the operator $A(p)$. A straight forward calculation shows that for all $v \in T^\perp(p)$ the following two relations

$$\langle f(p), SA(p)v\rangle = 0 \quad \text{and} \quad \langle v, A(p)v\rangle = \langle v, \nabla f(p)v\rangle \tag{2.7}$$

are satisfied. Hence, we have $SA(p)\colon T^\perp(p) \to T^\perp(p)$. Using this fact one can easily prove the first part of the following lemma.

Lemma 2.1 *For an arbitrary regular point $p \in M$ of the vector field (2.2) the eigenvalues of the operator $SA(p)\colon T_pM \to T_pM$ are the eigenvalues of the operator $SA(p)$*

which is restricted to the linear subspace $T^{\perp}(p)$ *and the value* $-\langle \nabla f(p) f(p), f(p) \rangle \backslash \|f(p)\|^2$. $\blacksquare$
Further we have

$$S\nabla f(p)z - \frac{f(p)}{\|f(p)\|^2} \langle f(p), S\nabla f(p)z \rangle = SA(p)z \quad \textit{for all} \quad z \in T^{\perp}(p).$$

In the following we denote at any regular point p of (2.2) the eigenvalues of the operator $SA(p)$ restricted to the subspace $T^{\perp}(p)$ by $\beta_1(p) \geq \dots \geq \beta_{n-1}(p)$, which are ordered with respect to size and multiplicity. By $Z(t,p)$ we denote the operator solution of (2.4) with initial condition $Z(0,p) = \mathrm{id}_{T^{\perp}(p)}$. For every $t \in \mathbb{R}$ the linear operator $Z(t,p)\colon T^{\perp}(p) \to T^{\perp}(\varphi^t(p))$ maps between the subspaces $T^{\perp}(p)$ and $T^{\perp}(\varphi^t(p))$ being orthogonal to the vector field in p and $\varphi^t(p)$, respectively. The next lemma will be needed in the sequel and can be proved analogously to [39].

Lemma 2.2 *Suppose that* $p \in M$ *is a regular point of the vector field (2.2) and* $Z(\cdot,p)$ *is the operator solution of (2.4). Let* $d \in (0, n-1]$. *Then for all* $t \geq 0$ *it holds*

$$\omega_d(Z(t,p)) \leq \exp\left\{ \int_0^t \big[\beta_1(\varphi^\tau(p)) + \ldots + \beta_{\lfloor d \rfloor}(\varphi^\tau(p)) + (d - \lfloor d \rfloor)\beta_{\lfloor d \rfloor + 1}(\varphi^\tau(p))\big]\, d\tau \right\}.$$

Let $B(O_p, r)$ denote the ball of radius r around the origin O_p of T_pM. For a regular point $p \in M$ of f let $B^{\perp}(O_p, r) = B(O_p, r) \cap T^{\perp}(p)$ be the ball in the subspace $T^{\perp}(p)$ centered in the origin O_p of T_pM with radius r. Fix p and r and consider for any $t \geq 0$ the ellipsoid $\mathcal{E}(t) = Z(t,p)B^{\perp}(O_p, r)$ in the subspace $T^{\perp}(\varphi^t(p))$. If $\sigma_1(\mathcal{E}(t)) \geq \dots \geq \sigma_{n-1}(\mathcal{E}(t))$ are the lengths of the semi-axes of $\mathcal{E}(t)$ and if d is an arbitrary number in $(0, n-1]$ we have by (2.1)

$$\omega_d(\mathcal{E}(t)) = \omega_d(Z(t,p))r^d. \tag{2.8}$$

Our aim is to describe the variation of time translated pieces of hypersurfaces, i.e., $(n-1)$-dimensional submanifolds, orthogonal to a considered orbit of (2.2). For this purpose we will use methods from [31, 32] developed there for stability investigations of flows on manifolds, in order to get information for the Hausdorff dimension of underlying flow invariant sets. Considering a non-equilibrium solution $\varphi^{(\cdot)}(p)$ of (2.2) with $p \in M$ the local transformation of small pieces of a hypersurface can be described by a *reparametrized local flow.* For $\delta > 0$ so small that $\exp_p$ is defined on $B(O_p, \delta)$ we consider the $(n-1)$-dimensional submanifold

$$B^{\perp}(p, \delta) := \exp_p(B^{\perp}(O_p, \delta))$$

of M through p which is local transversal at the point p to the trajectory of the vector field passing through the point p. Every point $u \in B^{\perp}(p, \delta)$ can be uniquely written in the form $u = \exp_p(rv)$, where $v \in T^{\perp}(p)$ is a vector of length $\|v\| = 1$ and $r \in [0, \delta)$ measures the arc length of the geodesic $c_{p,v}$ connecting p and u. This defines us a unique representation $u = u(r,v)$ of a point $u \in B^{\perp}(p, \delta)$.

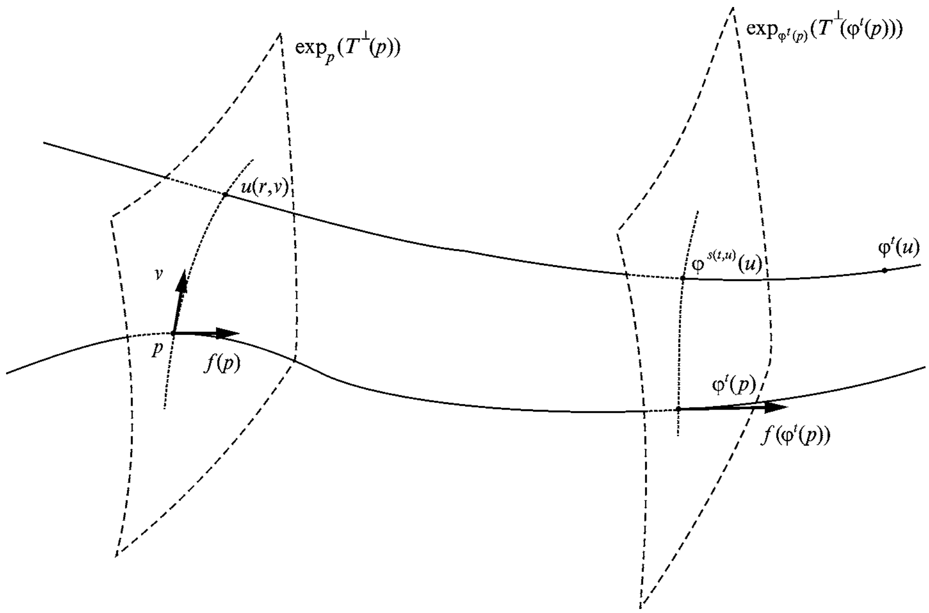

Figure 2.1. Reparametrization of the flow.

The main properties of the described reparametrization are summarized in the following two lemmata which proofs are similar to [20, 31], where a slightly different reparametrization is considered. Results on reparameterization for flows in $\mathbb{R}^n$ are given in [28,29,32]. $\blacksquare$

Lemma 2.3 *Suppose that $\varphi^{(\cdot)}(p)$ is a non-equilibrium solution of the C^2-vector field (2.2). Then for any finite number $T_0 > 0$ there exists a number $\varepsilon_1 > 0$ such that for every $u \in B^{\perp}(p, \varepsilon_1)$ there is a monotonously increasing differentiable function $s(\cdot, u)\colon \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $s(\cdot, p) = \mathrm{id}\,|_{[0,T_0]}$ and*

$$\left\langle \exp^{-1}_{\varphi^t(p)} \left(\varphi^{s(t,u)}(u)\right), f(\varphi^t(p)) \right\rangle = 0 \quad \textit{for all} \quad t \in [0, T_0]. \tag{2.9}$$

The next lemma states that for any regular point $p \in M$ of f for the locally defined reparametrized flow $\phi^t(\cdot) \equiv \phi(t, \cdot) := \varphi(s(t, \cdot), \cdot)$ the differential $d_p\phi^t$ of ϕ^t restricted to $T^{\perp}(p)$ satisfies (2.4). This provides the desired description of the variation of time translated pieces of hypersurfaces orthogonal to the considered orbit. For the proof again we refer to the method of [31].

Lemma 2.4 *Suppose that $\varphi^{(\cdot)}(p)$ is a non-equilibrium solution of (2.2) and the function $s(\cdot, \cdot)\colon [0, T_0] \times B^{\perp}(p, \varepsilon_1) \to \mathbb{R}_+$ as given in Lemma 2.3 defines a reparametrized local flow $\phi^t(u) := \varphi^{s(t,u)}(u)$. Then for all $t \in [0, T_0]$ there holds*

$$d_p\phi^t|_{T^{\perp}(p)} = Z(t, p),$$

where $Z(t, p)$ denotes the operator solution of (2.4) with $Z(0, p) = id_{T^{\perp}(p)}$.

We return to the Lemmata 2.3 and 2.4 in Section 4 where they are needed in the proof of Theorem 4.1.

3 Tubular Carathéodory Structure

In this section we define a special Carathéodory structure for flow negatively invariant sets on Riemannian manifolds. The outer measures which arise from this structure will majorize the Hausdorff measures and will be applied to obtain Hausdorff dimension estimates of flow-invariant sets on the manifold.

Carathéodory dimension structures were introduced by Pesin [41] (see also [42]) in order to give a general concept for most of the dimension-like characteristics of sets and measures. Such structures may be considered as a generalization of a well-known measure-theoretic construction of Carathéodory [5, 11]. The essential parts of such a structure are the following ([15]).

Let X be an arbitrary set, $\mathcal{F}$ be a family of subsets of X, $\mathbb{P} = [d^*, +\infty)$ for finite d^* or $\mathbb{P} = \mathbb{R}$ be a parameter set, and let $\xi\colon \mathcal{F} \times \mathbb{P} \to [0, \infty)$, $\eta\colon \mathcal{F} \times \mathbb{R} \to [0, \infty)$, and $\psi\colon \mathcal{F} \to [0, \infty)$ be functions. A sub-family $\mathcal{G} \subset \mathcal{F}$ is said to be an *ε-cover* of a set $Y \subset X$ if $Y \subset \bigcup\limits_{u \in \mathcal{G}} U$ and $\psi(\mathcal{G}) := \sup\{\psi(U) \mid U \in \mathcal{G}\} \le \varepsilon$ hold. The following conditions are assumed to be satisfied:

(A1) $\emptyset \in \mathcal{F}$, $\psi(\emptyset) = 0$, and $\xi(\emptyset, d) = 0$ for all $d \in \mathbb{P}$.
(A2) $\xi(U, s) = \eta(U, s - d)\xi(U, d)$ for all d, $s \in \mathbb{P}$ and all $U \in \mathcal{F}$.
(A3) For any $\Delta > 0$ there exists $\varepsilon > 0$ such that for all $U \in \mathcal{F}\backslash\{\emptyset\}$ with $\psi(U) \le \varepsilon$ we have $\eta(U, d) \le \Delta$ if $d > 0$ and $\eta(U, d) \ge \Delta^{-1}$ if $d < 0$.
(A4) For any subset $Y \subset X$ and for arbitrary $\varepsilon > 0$ there exists a countable ε-cover of Y.

In analogy to [42] we call such a collection $(\mathcal{F}, \mathbb{P}, \xi, \eta, \psi)$ which satisfies (A1)–(A4) a *Carathéodory (dimension) structure* on X. For a given Carathéodory structure $(\mathcal{F}, \mathbb{P}, \xi, \eta, \psi)$, an arbitrary set $Y \subset X$, $d \in \mathbb{P}$, and $\varepsilon > 0$ we define the *Carathéodory d-measure at level ε of Y with respect to* $(\mathcal{F}, \mathbb{P}, \xi, \eta, \psi)$ by

$$\mu_{\mathcal{C}}(Y, d, \varepsilon) = \inf_{\mathcal{G}} \sum_{U \in \mathcal{G}} \xi(U, d),$$

where the infimum is taken over all countable sub-collections $\mathcal{G} \subset \mathcal{F}$ being ε-covers of the set Y. For fixed Y and d the function $\mu_{\mathcal{C}}(Y, d, \varepsilon)$ is non-increasing with respect to ε. Therefore, there exists the limit

$$\mu_{\mathcal{C}}(Y, d) = \lim_{\varepsilon \to 0+0} \mu_{\mathcal{C}}(Y, d, \varepsilon)$$

which is called the *Carathéodory d-measure of Y with respect to* $(\mathcal{F}, \mathbb{P}, \xi, \eta, \psi)$. For arbitrary $d \in \mathbb{P}$ and arbitrary $\varepsilon > 0$ the functions $\mu_{\mathcal{C}}(\cdot, d, \varepsilon)$ and $\mu_{\mathcal{C}}(\cdot, d)$ are outer measures on X. It turns out that for any set $Y \subset X$ there exists a unique number $d_{\mathrm{cr}}(Y) \in \overline{\mathbb{P}}$ having the property that

$$\mu_{\mathcal{C}}(Y, d) = \begin{cases} 0 & \text{for} \quad d > d_{\mathrm{cr}}(Y) \\ +\infty & \text{for} \quad d < d_{\mathrm{cr}}(Y) \end{cases}$$

holds for $d \in \mathbb{P}$. This critical value $d_{\mathrm{cr}}(Y)$ is called *Carathéodory dimension* $\dim_{\mathcal{C}} Y$ *of Y with respect to the structure* $(\mathcal{F}, \mathbb{P}, \xi, \eta, \psi)$.

Note that our system of conditions (A1)–(A4) which leads to a Carathéodory structure is slightly different from the system in [41, 42]. In contrast to these works we assume that our family of objects in the Carathéodory construction depends on parameters which come from a (possibly proper) subset of $\mathbb{R}$.

For a standard Carathéodory structure let X be a separable metric space, $\mathcal{F}$ the family consisting of open balls $B(u,r)$ in X with center u and radius r and the empty set, $\mathbb{P}=\mathbb{R}_+$, $\xi(B(u,r),d)=r^d$, $\eta(B(u,r),s)=r^s$, $\psi(B(u,r))=r$, $\xi(\emptyset,d)=\psi(\emptyset)=0$, and $\eta(\emptyset,s)=1$ for each $u\in X$, $r>0$ and each $d\ge 0$, $s\in\mathbb{R}$. It is easy to see that such a system $(\mathcal{F},\mathbb{P},\xi,\eta,\psi)$ defines a Carathéodory structure on X. We denote by $\mu_H(\cdot,d,r)$, $\mu_H(\cdot,d)$ and $\dim_H$ the resulting Carathéodory measures and Carathéodory dimension which are in fact the Hausdorff d-measure at level r, the Hausdorff d-measure and the Hausdorff dimension, respectively. The concept of the Carathéodory dimension covers not only several dimension type characteristics of sets but also characteristics of dynamical systems such as topological pressure and topological entropy (see [41, 42]) or a dimension introduced for Poincaré recurrences ([1]).

Let (M,g) be a smooth n-dimensional Riemannian manifold and ρ the metric induced by g. For a piecewise smooth curve $\gamma\colon I\to M$ $(I\subset\mathbb{R}$ an interval) of finite length and arbitrary $\varepsilon>0$ we define the *ε-tubular neighborhood* $\Omega(\gamma,\varepsilon)$ of γ by

$$\Omega(\gamma,\varepsilon)=\bigcup_{u\in\gamma(I)}B(u,\varepsilon),$$

where $B(u,\varepsilon)=\{p\in M\mid \rho(u,p)<\varepsilon\}$ is again a metric ε-ball on M centered in the point u. For simplicity we call the ε-tubular neighborhood $\Omega(\gamma,\varepsilon)$ around the curve γ of length l shortly *tube of length l*.

For a given compact set $K\subset M$ and a given number $l_0>0$ we denote by $\Gamma=\{\gamma\}$ a family of piecewise smooth curves of a finite length $l(\gamma)=l_0$ such that for any $\varepsilon>0$ the following condition is satisfied:

(A) K is contained in the union of ε-tubular neighborhoods $\Omega(\gamma,\varepsilon)$ with $\gamma\in\Gamma$.

Condition (A) guarantees the existence of arbitrarily fine covers of the set K which are generated by the family Γ. For a family Γ satisfying (A) we define a family of subsets $\mathcal{F}$, a parameter set $\mathbb{P}$, and the functions $\xi\colon\mathcal{F}\times\mathbb{P}\to[0,\infty)$, $\eta\colon\mathcal{F}\times\mathbb{R}\to[0,\infty)$, and $\psi\colon\mathcal{F}\to[0,\infty)$ by

$$\begin{gathered}\mathcal{F}=\{\Omega(\gamma,\varepsilon)\cap K\mid\gamma\in\Gamma,\ \varepsilon>0\}\cup\{\emptyset\},\qquad \mathbb{P}=[1,+\infty),\\ \xi(\Omega(\gamma,\varepsilon)\cap K,d)=\varepsilon^{d-1},\quad \eta(\Omega(\gamma,\varepsilon)\cap K,s)=\varepsilon^s,\\ \psi(\Omega(\gamma,\varepsilon)\cap K)=\varepsilon\end{gathered}\tag{3.1}$$

for $\gamma\in\Gamma$, $\varepsilon>0$ with $\Omega(\gamma,\varepsilon)\cap K\neq\emptyset$, $\xi(\emptyset,d)=\psi(\emptyset)=0$, and $\eta(\emptyset,s)=1$ for all $d\in\mathbb{P}$, $s\in\mathbb{R}$.

Straight forward, one can verify that the collection $(\mathcal{F},\mathbb{P},\xi,\eta,\psi)$ defined via (3.1) with Γ satisfying (A) is a Carathéodory structure on K in the sense as considered above. In the sequel we will call such a structure simply a *Carathéodory structure with tubes of length l_0 on K* or *tubular Carathéodory structure on K*, if the underlying set K and the family Γ are clear from the context. The next proposition shows the relations between the Carathéodory measures and the Hausdorff measures, as well as between the Carathéodory dimension and the Hausdorff dimension, generated by this structure. For the proof we refer to [15, 16] and for the $\mathbb{R}^n$-case to [27].

Proposition 3.1 *Suppose that K is a compact set on the smooth n-dimensional Riemannian manifold (M,g). Suppose that $(\mathcal{F},\mathbb{P},\xi,\eta,\psi)$ is a tubular Carathéodory structure on K with tubes of length l_0 defined by (3.1) and with respect to this structure let be $\mu_{\mathcal{C}}(\cdot,d,\varepsilon)$, $\mu_{\mathcal{C}}(\cdot,d)$, and $\dim_{\mathcal{C}}$ the Carathéodory d-measure at level ε, the Carathéodory d-measure, and the Carathéodory dimension, respectively. Then there exist two numbers $k>0$ and $\varepsilon_0>0$ depending only on K such that for any set $Y\subset K$ and any $d\ge 1$ the inequality*

$$\mu_H(Y,d,\varepsilon)\le l_0k\mu_{\mathcal{C}}(Y,d,\varepsilon) \tag{3.2}$$

holds for all $\varepsilon\in(0,\varepsilon_0]$. Therefore, we have

$$\mu_H(Y,d)\le l_0k\mu_{\mathcal{C}}(Y,d) \quad \textit{and thus} \quad \dim_H Y\le \dim_{\mathcal{C}} Y.$$

Now we specify the family Γ of curves which will be used further for the considerations of sets being negatively invariant with respect to a flow. As in the previous section we consider the complete C^2-vector field $f\colon M\to TM$ on a smooth n-dimensional Riemannian manifold and the corresponding differential equation (2.2) with global flow $\{\varphi^t\}_{t\in\mathbb{R}}$. Let K and $\widetilde{K}$ be two compact sets in M satisfying

$$K\subset\varphi^t(K)\subset\widetilde{K} \quad \text{for all} \quad t\ge 0. \tag{3.3}$$

(A set K satisfying $K\subset\varphi^t(K)$ for all $t\ge 0$ is usually called *negatively invariant* with respect to the flow.) At first we suppose that the set K does not contain equilibrium points of (2.2).

To construct the family Γ we denote by Λ the set of all equilibrium points of (2.2) in $\widetilde{K}$ and set $e_1=\frac{1}{2}\,\mathrm{dist}\,(\Lambda,K)$, where $\mathrm{dist}\,(\Lambda,K)=\inf\limits_{u\in\Lambda,\,p\in K}\rho(u,p)$ is the usual metric distance between two sets in M, and define

$$\Phi:=\widetilde{K}\cap\bigcup_{p\in K}B(p,e_1). \tag{3.4}$$

With respect to the vector field f, the compact set $\widetilde{K}$ from (3.3), and the set Φ from (3.4) define the following coefficient

$$V(f,\widetilde{K},\Phi):=\frac{\max\limits_{u\in\widetilde{K}}\|f(u)\|_{T_uM}}{\min\limits_{u\in\Phi}\|f(u)\|_{T_uM}}, \tag{3.5}$$

which will be important for the proofs in Section 4. For any $p\in K$ we take a time $b_p>0$ such that $\varphi^t(p)\in\Phi$ for all $t\in[0,b_p]$. Further, since $d_p\varphi^t|_{t=0}=\mathrm{id}_{T_pM}$ we can suppose that $\|d_p\varphi^t\|\le 2$ holds for all $t\in[0,b_p]$. Since K is compact and contains no equilibrium points of f there exists a number $e_2>0$ such that for the length of the integral curve pieces it holds $l(\varphi(\cdot,p)|_{[0,b_p]})\ge e_2$ for any $p\in K$. We set

$$l_0:=\frac{1}{2}\min\{e_1,e_2\},$$

introduce for any $q\in K$ the number $\tau(q)>0$ satisfying $l(\varphi(\cdot,q)|_{[0,\tau(q)]})=l_0$, and define the set

$$\Gamma:=\{\varphi(\cdot,q)|_{[0,\tau(q)]}\mid q\in K\}. \tag{3.6}$$

Obviously this family Γ satisfies condition (A) and $(\mathcal{F}, \mathbb{P}, \xi, \eta, \psi)$ defined by (3.1) on the base of this family is a Carathéodory structure on K – a Carathéodory structure with tubes of length l_0 – which will be used in Section 4.

4 Dimension Estimates of Flow Negatively Invariant Sets

In the present section we derive upper bounds for the Hausdorff dimension of compact sets being negatively invariant with respect to the flow of the differential equation (2.2). Investigating the deformation of such a set under shift maps generated by the flow the deformation transversal to the flow lines is of great importance.

Our main result is the following theorem which generalizes the results of [26, 27] to vector fields on manifolds. Recall that for $d \in \mathbb{R}$ we denote by $\lfloor d \rfloor$ the largest integer less than d.

Theorem 4.1 *Let $f \colon M \to TM$ be the C^2-vector field (2.2) on the smooth n-dimensional $(n \geq 2)$ Riemannian manifold (M, g) satisfying the following conditions:*

(a) *The flow $\{\varphi^t\}_{t \in \mathbb{R}}$ of (2.2) satisfies (3.3) with respect to the compact sets K and $\widetilde{K}$ in M, where K does not contain equilibrium points of (2.2).*

(b) *For a regular point $p \in \widetilde{K}$ let $\beta_1(p) \geq \cdots \geq \beta_{n-1}(p)$ be the eigenvalues of the symmetric part $SA(p) = \frac{1}{2}[A(p) + A(p)^*]$ restricted to the subspace $T^{\perp}(p)$, where $A(p)$ is the operator from (2.5). There exist a number $d \in (0, n-1]$, a number $\Theta > 0$, and a time $T_0 > 0$ such that*

$$\int_0^{T_0} \left[\beta_1(\varphi^\tau(p)) + \cdots + \beta_{\lfloor d \rfloor}(\varphi^\tau(p)) + (d - \lfloor d \rfloor)\beta_{\lfloor d \rfloor + 1}(\varphi^\tau(p))\right] d\tau \leq -\Theta \tag{4.1}$$

is satisfied for all regular points $p \in \widetilde{K}$.

Then it holds $\dim_H K < d + 1$. If $d = 1$ we have $\dim_H K \leq 1$.

Before proving Theorem 4.1 we formulate some lemmata. The special flow line structure of sets which are flow negatively invariant allows us to obtain the dimension estimate. In order to describe the deformation under the map φ^t of tubular neighborhoods around an arc of a trajectory we investigate the evolution of time translated pieces of hypersurfaces lying transversal to the considered trajectory. In the next lemma we consider the influence of φ^t on arcs of a trajectory.

For an arbitrary piecewise smooth curve $c \colon [t_1, t_2] \to M$ we denote its length by $l(c)$.

Lemma 4.1 *Suppose that $\{\varphi^t\}_{t \in \mathbb{R}}$ is the flow of (2.2), Φ and $\widetilde{K}$ are compact sets in M, Φ does not contain any equilibrium points of (2.2), and $V(f, \widetilde{K}, \Phi)$ is the coefficient from (3.5). Let $p \in \Phi$ and let $c^t \colon [t_1, t_2] \to M$ be a restriction of the integral curve of (2.2) through p given by $c^t(\cdot) = \varphi(t + \cdot, p)|_{[t_1, t_2]}$ and satisfying $c^0([t_1, t_2]) \subset \Phi$ and $c^t([t_1, t_2]) \subset \widetilde{K}$ for all $t > 0$. Then the length $l(c^t)$ of such a restriction satisfies $l(c^t) \leq V(f, \widetilde{K}, \Phi) l(c^0)$ for all $t \geq 0$.*

Proof The statement follows immediately from

$$\begin{aligned} l(c^t) = \int_{t_1}^{t_2} \|\dot{\varphi}(\tau, \varphi^t(p))\| \, d\tau &= \int_{t_1}^{t_2} \frac{\|\dot{\varphi}(\tau + t, p)\|}{\|\dot{\varphi}(\tau, p)\|} \|\dot{\varphi}(\tau, p)\| \, d\tau \\ &\leq V(f, \widetilde{K}, \Phi) l(c^0). \end{aligned}$$

We consider now the family Γ of curves of length l_0 from (3.6) and the chosen Carathéodory structure $(\mathcal{F}, \mathbb{P}, \xi, \eta, \psi)$ on the compact set K with tubes of length l_0 from (3.1). The next lemma estimates the tubular measures $\mu_{\mathcal{C}}(\cdot, d, \varepsilon)$, generated with respect to $(\mathcal{F}, \mathbb{P}, \xi, \eta, \psi)$, of the flow-transformed set K. Its proof is based on the consideration of the deformation of tubular neighborhoods around trajectory pieces.

Lemma 4.2 *Suppose that $\{\varphi^t\}_{t\in\mathbb{R}}$ is the flow of (2.2) satisfying (3.3) with respect to the compact sets K and $\widetilde{K}$ in M, where K does not contain equilibrium points of (2.2). Suppose also that Φ, $V(f, \widetilde{K}, \Phi)$ and l_0 are given by (3.4), (3.5), and (3.6), respectively. For $p \in \widetilde{K}$ let $\alpha_1(p)$ be the largest eigenvalue of $S\nabla f(p)$, and for a regular point $p \in \widetilde{K}$ let $\beta_1(p) \geq \cdots \geq \beta_{n-1}(p)$ be the eigenvalues of $SA(p)|_{T^\perp(p)}$, where $A(p)$ is the operator from (2.5). Define for a number $d \in (0, n-1]$ and a time $T_0 > 0$ the values*

$$
\begin{aligned}
k &:= \max_{p\in K} \exp\Bigg\{ \int_0^{T_0} \big[\beta_1(\varphi^\tau(p)) + \cdots + \beta_{\lfloor d\rfloor}(\varphi^\tau(p)) \\
&\qquad + (d - \lfloor d\rfloor)\beta_{\lfloor d\rfloor+1}(\varphi^\tau(p))\big]\, d\tau \Bigg\}, \\
a &:= \exp\left[3l_0 \max_{p\in\widetilde{K}} \alpha_1(p)\, \frac{V(f, \widetilde{K}, \Phi)}{\min\limits_{p\in\Phi} \|f(p)\|_{T_pM}}\right], \\
\lambda &:= 2^6\sqrt{\lfloor d\rfloor + 1}\, a, \quad \text{and} \quad C := \big(3V(f, \widetilde{K}, \Phi) + 1\big)2^{\lfloor d\rfloor}\lambda^d.
\end{aligned}
\tag{4.2}
$$

Then for any $l > k$ there exists an $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$ the Carathéodory $(d+1)$-measure $\mu_{\mathcal{C}}(\cdot, d+1, \varepsilon)$ at level ε, generated with respect to the Carathéodory structure (3.1) with tubes of length l_0, satisfies the inequality

$$
\mu_{\mathcal{C}}(\varphi^{T_0}(K) \cap K, d+1, \lambda l^{1/d}\varepsilon) \leq Cl\mu_{\mathcal{C}}(K, d+1, \varepsilon). \tag{4.3}
$$

Proof Fix some $\gamma \in \Gamma$. For arbitrary $l > k$ we can choose an $\varepsilon_1 > 0$ such that the set $V := \bigcup\limits_{p\in K} B(p, \varepsilon_1)$ contains no equilibrium points of (2.2) and that the inequality

$$
\begin{aligned}
k' &:= \max_{u\in\overline{V}} \exp\Bigg\{ \int_0^{T_0} \big[\beta_1(\varphi^\tau(u)) + \cdots + \beta_{\lfloor d\rfloor}(\varphi^\tau(u)) \\
&\qquad + (d - \lfloor d\rfloor)\beta_{\lfloor d\rfloor+1}(\varphi^\tau(u))\big]\, d\tau \Bigg\} < l
\end{aligned}
\tag{4.4}
$$

is satisfied. We set

$$
\sigma := \max_{p\in\overline{V}} \exp\left\{ \int_0^{T_0} \beta_1(\varphi^\tau(p))\, d\tau \right\} \tag{4.5}
$$

and take a number $m > 0$ such that $k' < m^d$ and $\sigma \leq m$ are satisfied. Since $l > k'$ the equation

$$
\left[1 + \left(\frac{m^{\lfloor d\rfloor}}{k'}\right)^{1/(1-\lfloor d\rfloor)} \eta\right]^d k' = l
$$

uniquely defines a number $\eta > 0$.

Choose $\delta > 0$ such that for any $u \in \widetilde{K}$ the map $\exp_u$ maps the ball $B(O_u, \delta) \subset T_u M$ diffeomorphically onto the geodesic ball $B(u, \delta) \subset M$. Further with $\|d_{O_u} \exp_u\| = 1$ we can suppose that $\|d_v \exp_u\| \le 2$ and therefore

$$\rho(\exp_u v_1, \exp_u v_2) \le 2\rho(v_1, v_2)$$

holds for all $v,\, v_1,\, v_2 \in B(O_u, \delta)$.

To simplify the use of the reparametrized local flow we cover $\Omega(\gamma, r)$ by a set $S(\gamma_p, r)$ as follows. Let for some $p \in K$ and the associated time $t(p) > 0$ be $\gamma_p(\cdot) = \varphi(\cdot, p)|_{[0,t(p)]}$ the integral curve of lenght $2l_0$ such that $\gamma_p \supset \gamma$ and for any $r \in (0, l_0]$ the inclusion $\Omega(\gamma, r) \subset S(\gamma_p, r)$ holds, where

$$S(\gamma_p, r) := \bigcup_{u \in \gamma_p} B^{\perp}(u, r).$$

Let p and $t(p)$ be fixed in the sequel. We take now

$$\varepsilon_0(\gamma) < \frac{1}{4} \min\{\varepsilon_1, \delta, \operatorname{dist}(K, M \backslash V), l_0\}$$

small enough such that the following conditions are satisfied:

(1) The function $s\colon [0, \max\{T_0, t(p)\}] \times B^{\perp}(p, 4\varepsilon_0(\gamma)) \to \mathbb{R}_+$ as characterized in the Lemma 2.3 defines a local reparametrization of the flow φ by $\phi\colon [0, \max\{T_0, t(p)\}] \times B^{\perp}(p, 4\varepsilon_0(\gamma)) \to M$ with $\phi(t, \cdot) \equiv \phi^t(\cdot) := \varphi^{s(t,\cdot)}(\cdot)$ for $t \in [0, \max\{T_0, t(p)\}]$.

(2) $\phi^{T_0}(B^{\perp}(p, 4\varepsilon_0(\gamma))) \subset B(\varphi^{T_0}(p), \delta)$.

(3) The distance between the points $\phi^t(u)$ on an integral curve starting in $u = \exp_p(rv) \in B^{\perp}(p, \varepsilon_0(\gamma))$ and the reference orbit through p for a fixed $t \in [0, t(p)]$ is of the size

$$\rho(\varphi^t(p), \phi^t(u)) = \|d_p \phi^t\| \cdot r(1 + O(r))$$

as $r \to 0$. It holds $\|d_p \phi^t\| \le \|d_p \varphi^t\|$ and $\|d_p \varphi^t\| \le 2$ for any $t > 0$ such that $l(\varphi([0, t], p)) \le 2l_0$. Thus, for any $u \in B^{\perp}(p, \varepsilon_0(\gamma))$ it is $\rho(\varphi^t(p), \phi^t(u)) \le 4\rho(p, u)$ for any such t. We can assume analogous assumptions for the flow in reverse time-direction. Let for $\varepsilon_0(\gamma) > 0$ the following be satisfied: If $\gamma' = \phi([0, t(p)], u)$ is some arc of trajectory intersecting $S(\gamma_p, \varepsilon_0(\gamma))$ then γ' is completely contained in $S(\gamma_p, 4\varepsilon_0(\gamma))$ and satisfies $l(\gamma') \le 3l_0$.

(4) For any $u \in \widetilde{K}$ and for any time $\tau > 0$ such that the integral curve $\varphi([0, \tau], u)$ is of maximal lenght $3l_0 V(f, \widetilde{K}, \Phi)$ it holds

$$\sup_{q \in B(u, 16\sigma\varepsilon_0(\gamma))} \left\| \tau_{\varphi^t(q)}^{\varphi^t(u)} d_q \varphi^t \tau_u^q - d_u \varphi^t \right\| \le a \quad \text{for all} \quad t \in (0, \tau). \tag{4.6}$$

Suppose that it holds

$$\sup_{q \in B^{\perp}(p, 4\varepsilon_0(\gamma))} \left\| \tau_{\phi^{T_0}(q)}^{\phi^{T_0}(p)} d_q \phi^{T_0} \tau_p^q - d_p \phi^{T_0} \right\| \le \eta. \tag{4.7}$$

(5) For any $u = u(r,v) \in B^{\perp}(p, 4\varepsilon_0(\gamma))$ the deviation arising from the local reparametrization of the flow is of the form $s(T_0, u(r,v)) - T_0 = O(r)$ as $r \to 0$ which gives for the point $\phi^{T_0}(u) = \varphi^{s(T_0,u)-T_0}(\varphi^{T_0}(u))$ the representation

$$\exp^{-1}_{\varphi^{T_0}(u)}(\phi^{T_0}(u)) = O_{\varphi^{T_0}(u)} + f(\varphi^{T_0}(u))O(r) + o(r)$$

as $r \to 0$. The vector field C^2-varies on M. So we can suppose that for any point $u \in B^{\perp}(\varphi^{T_0}(p), \delta)$ for $\nu < 2^4\sqrt{\lfloor d\rfloor + 1}\,\sigma\varepsilon_0(\gamma)$ any set $(\varphi^{T_0} \circ \phi^{-T_0})B(u,\nu)$ is contained in a 2ν-tubular neighborhood of a curve $\varphi(\cdot, (\varphi^{T_0} \circ \phi^{-T_0})(u))|_{(-\tau,\tau)}$ of some finite length, say of length l_0.

Now let $r \le \varepsilon_0(\gamma)$. Suppose $\varphi^{T_0}(\Omega(\gamma, r)) \cap K \neq \emptyset$. The set $B(p, 4r)$ is contained in the open set V. Taylor's formula for the differentiable map ϕ^{T_0} provides ([39]) that for every $u \in B^{\perp}(p, 4r)$

$$\begin{aligned} &\big\| \exp^{-1}_{\phi^{T_0}(p)} \phi^{T_0}(u) - d_p\phi^{T_0}(\exp_p^{-1}(u))\big\| \\ &\le \sup_{q \in B(p,4r)} \big\|\tau^{\phi^{T_0}(p)}_{\phi^{T_0}(q)} d_q\phi^{T_0}\tau_p^q - d_p\phi^{T_0}\big\| \cdot \big\|\exp_p^{-1}(q)\big\| \end{aligned} \tag{4.8}$$

holds. Considering the image of $B^{\perp}(p, 4r)$ under ϕ^{T_0} with (4.7) we obtain the inclusion

$$\exp^{-1}_{\phi^{T_0}(p)}\big(\phi^{T_0}\big(B^{\perp}(p,4r)\big)\big) \subset d_p\phi^{T_0}\big(B^{\perp}(O_p, 4r)\big) + B^{\perp}\big(O_{\varphi^{T_0}(p)}, \eta 4r\big).$$

The set $d_p\phi^{T_0}\big(B^{\perp}(O_p, 4r)\big)$ is an ellipsoid with half-axes of length $4r\sigma_k(p)$, where $\sigma_k(p)$ $(k = 1, \dots, n-1)$ denote the singular values of the linear operator $d_p\phi^{T_0} \colon T^{\perp}(p) \to T^{\perp}(\varphi^{T_0}(p))$. Using the definition of k', Lemma 2.2 and (2.8) we conclude

$$\omega_d\big(d_p\phi^{T_0}(B^{\perp}(O_p, 4r))\big) \le (4r)^d k'. \tag{4.9}$$

By standard covering results (see e.g. [39]) an ellipsoid $\mathcal{E} \subset T^{\perp}(\varphi^{T_0}(p))$ can be found containing $d_p\phi^{T_0}\big(B^{\perp}(O_p, 4r)\big) + B\big(O_{\varphi^{T_0}(p)}, \eta 4r\big)$ and satisfying $\omega_d(\mathcal{E}) \le l(4r)^d$. Any set $\mathcal{E}$ can be covered by N balls of radius $R = \sqrt{\lfloor d\rfloor + 1}\,\sigma_{\lfloor d\rfloor+1}(\mathcal{E})$. The number N can be estimated from above by

$$N \le \frac{2^{\lfloor d\rfloor}\omega_d(\mathcal{E})}{\sigma_{\lfloor d\rfloor+1}(\mathcal{E})^d}\,.$$

Thus, any set $\exp_{\varphi^{T_0}(p)}(\mathcal{E})$ and therefore $\phi^{T_0}(B^{\perp}(p, 4r))$ can be covered by N geodesic balls in M of radius $2R$. Fixing such a cover $\{B(\widetilde{u}_j, 2R)\}_{j\ge 1}$, where $\widetilde{u}_j \in M$ $(j \ge 1)$, we choose in every set

$$K \cap B(\widetilde{u}_j, 2R) \cap B^{\perp}(\varphi^{T_0}(p), \delta)$$

a point u_j and obtain the cover $\{B_j\}_{j\ge1}$ of the set $\phi^{T_0}(B^{\perp}(p,4r)) \cap K$ with $B_j = B(u_j, 4R) \cap B^{\perp}(\varphi^{T_0}(p), \delta)$.

Now we consider the deviation arising from the reparametrization. By the property (5) any set $(\varphi^{T_0} \circ \phi^{-T_0})(B_j)$ is with precision $o(r)$ $(r \le \varepsilon_0(\gamma))$ contained in a

$4R$-neighborhood of the orbit trough u_j, or more precise, in an $8R$-neighborhood of a trajectory piece $\varphi(\cdot, (\varphi^{T_0} \circ \phi^{-T_0})(u_j))|_{(-\tau,\tau)}$ of length l_0.

By the choise of $\varepsilon_0(\gamma)$ any trajectory piece in $S(\gamma_p, 4r)$ which intersects $S(\gamma_p, r)$ is of maximal length $3l_0$. We shift the balls $B((\varphi^{T_0} \circ \phi^{-T_0})(u_j), 8R)$ along the flow lines. Thus, with the above and (4.6) the set $\varphi^{T_0}(S(\gamma_p, r))$ can be covered by N tubes of length $3l_0 V(f, \widetilde{K}, \Phi) + l_0$ and diameter $2a \cdot 8R$.

Covering each curve arc by curve arcs of length l_0 we conclude

$$\begin{aligned} &\mu_{\mathcal{C}}\Big(\varphi^{T_0}(\Omega(\gamma, r)) \cap K,\ d+1, 2^6 \sqrt{\lfloor d \rfloor + 1}\, l^{1/d} a r\Big) \\ &\le N(3V(f, \widetilde{K}, \Phi) + 1)\Big(2^6 a \sqrt{\lfloor d \rfloor + 1}\, \sigma_{\lfloor d \rfloor + 1}(\mathcal{E})\Big)^d \le C l r^d. \end{aligned} \tag{4.10}$$

Since Γ is the set of trajectory pieces starting in a point p in the compact set K we can pass to $\varepsilon_0 := \inf_{\gamma \in \Gamma} \varepsilon_0(\gamma) > 0$ such that the (4.10) holds for any $\Omega(\gamma, r)$ with $\gamma \in \Gamma$ and $r \le \varepsilon_0$. Let $\varepsilon \le \varepsilon_0$. For any $\nu > 0$ there exists a finite family $\{\Omega(\gamma_i, r_i)\}_{i \ge 1}$ with $\gamma_i \in \Gamma,\ r_i \le \varepsilon$ having the property that $\bigcup_i \Omega(\gamma_i, r_i) \supset K$ and $\sum_i r_i^d \le \mu_{\mathcal{C}}(K, d+1, \varepsilon) + \nu$. We obtain $\mu_{\mathcal{C}}(\varphi^{T_0}(K) \cap K, d+1, \lambda l^{1/d}\varepsilon) \le \sum_i \mu_{\mathcal{C}}(\varphi^{T_0}(\Omega(\gamma_i, r_i)) \cap K, d+1, \lambda l^{1/d}\varepsilon) \le Cl \sum_i r_i^d \le Cl(\mu_{\mathcal{C}}(K, d+1, \varepsilon) + \nu)$, where λ and C are defined by (4.2). Since ν has been chosen arbitrarily we obtain that (4.3) holds for any $\varepsilon \in (0, \varepsilon_0]$.

Although we are mainly interested in upper estimates of the Hausdorff dimension of flow negatively invariant sets we can deduce upper bounds of its Carathéodory dimension with respect to the chosen tubular Carathéodory structure.

Proposition 4.1 *Let the differential equation (2.2) satisfy the conditions of Theorem 4.1 with the number $d \in (0, n-1]$ in (4.1) and the negatively invariant set K. Then the Carathéodory dimension of K, determined with respect to the Carathéodory structure (3.1) on K consisting of tubes with length l_0 determined in (3.6), satisfies*

$$\dim_{\mathcal{C}} K < d + 1.$$

Proof It follows from (4.1) that for an arbitrarily small number $\varkappa \in (0,1)$ there exists some number $m = m(\varkappa) > 0$ such that

$$\begin{aligned} k := \sup_{p \in K} \exp \Bigg\{ \int_0^{mT_0} &\big[\beta_1(\varphi^\tau(p)) + \cdots + \beta_{\lfloor d \rfloor}(\varphi^\tau(p)) \\ &+ (d - \lfloor d \rfloor)\beta_{\lfloor d \rfloor + 1}(\varphi^\tau(p))\big]\, d\tau \Bigg\} \le \exp(-m\Theta) < \varkappa. \end{aligned} \tag{4.11}$$

Without loss of generality we can assume that this number k satisfies $\lambda k^{1/d} < 1$ and $Ck < 1$, where λ and C are the constants given in (4.2). We choose $l > k$ with $\lambda l^{1/d} < 1$ and $Cl < 1$. Lemma 4.2, applied to the map φ^{mT_0}, guarantees that for the chosen number l there exists a number $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$ the inequality

$$\mu_{\mathcal{C}}\big(\varphi^{mT_0}(K) \cap K, d+1, \lambda l^{1/d}\varepsilon\big) \le Cl\mu_{\mathcal{C}}(K, d+1, \varepsilon) \tag{4.12}$$

holds. Let $\varepsilon \in (0, \varepsilon_0]$ be arbitrarily small. Since K is compact the value $\mu_{\mathcal{C}}(K, d+1, \varepsilon)$ is finite. Since K is negatively invariant with respect to φ^{mT_0} we have $K = \varphi^{mT_0}(K) \cap K$. Using inequality $\lambda l^{1/d} < 1$ we conclude $\mu_{\mathcal{C}}(K, d+1, \varepsilon) < CL\mu_{\mathcal{C}}(K, d+1, \varepsilon)$. From this we follow that the equality $\mu_{\mathcal{C}}(K, d+1, \varepsilon) = 0$ holds for every $\varepsilon \in (0, \varepsilon_0]$. We see that $\mu_{\mathcal{C}}(K, d+1) = 0$. This implies $\dim_{\mathcal{C}} K \leq d+1$. Since (4.11) holds true if we slightly reduce d we conclude $\dim_{\mathcal{C}} K < d+1$.

Proof of Theorem 4.1 Applying Proposition 4.1 and Proposition 3.1 we obtain $\dim_H K < d+1$. If condition (4.1) is also satisfied for $d = 1$ it is satisfied for all $d \in (0, n-1]$. Thus, $\dim_H K < d+1$ for all $d \in (0, n-1]$ and we obtain $\dim_H K \leq 1$. This proves the Theorem.

Let us again consider compact sets K and $\widetilde{K}$ in M satisfying (3.3) with respect to the flow of (2.2). We may now assume that the set K possesses equilibrium points and satisfies the following condition:

(S) The set K contains at most a finite number of equilibrium points of (2.2). Every such equilibrium point possesses a local stable manifold with dimension at least $n-1$. Trajectories starting in local unstable manifolds or local center manifolds of such an equilibrium point in K converge for $t \to +\infty$ to an asymptotically stable equilibrium point of (2.2) in $\widetilde{K}$.

The special structure of equilibrium points satisfying (S) allows us to obtain the following theorem. The reason for this is that in some sense in open and flow positively invariant neighborhoods of these points the flow preserves its contracting property with respect to the Hausdorff measure ([16]).

Theorem 4.2 *Let $f\colon M \to TM$ be a C^2-vector field (2.2) on the smooth n-dimensional Riemannian manifold (M, g). Suppose that the flow $\{\varphi^t\}_{t\in\mathbb{R}}$ of (2.2) satisfies (3.3) and condition* (S) *with respect to compact sets K and $\widetilde{K}$ in M. Suppose also that condition (b) of Theorem 4.1 is satisfied. Then the conclusion of Theorem 4.1 holds.*

In the following statement we denote for a differentiable function $v\colon U \subset M \to \mathbb{R}$, U an open set, by $L_f v(p)$ the Lie derivative of v in p in direction of the vector field f.

Corollary 4.1 *Suppose that the flow $\{\varphi^t\}_{t\in\mathbb{R}}$ of (2.2) satisfies (3.3) and condition* (S) *with respect to compact sets K and $\widetilde{K}$ in M.*

Denote by Λ the set of equilibrium points of (2.2) in M. For $p \in M\backslash\Lambda$ let $\beta_1(p) \geq \ldots \geq \beta_{n-1}(p)$ be the eigenvalues of the symmetric part $SA(p)$ restricted to the subspace $T^{\perp}(p)$, where $A(p)$ is the operator from (2.5), and let $v\colon M\backslash\Lambda \to \mathbb{R}$ be a C^1-function. Suppose also that for a number $d \in (0, n-1]$ there exist a number $\Theta > 0$ and a time $T_0 > 0$ such that

$$\int_0^{T_0} \big[\beta_1(\varphi^\tau(p)) + \cdots + \beta_{\lfloor d\rfloor}(\varphi^\tau(p)) + (d - \lfloor d\rfloor)\beta_{\lfloor d\rfloor+1}(\varphi^\tau(p)) + L_f v(\varphi^\tau(p))\big]\, d\tau \leq -\Theta \tag{4.13}$$

holds for all regular points $p \in \widetilde{K}$. Then the conclusion of Theorem 4.1 holds.

Proof As mentioned above, on open and flow positively invariant neighborhoods of equilibrium points of (2.2) which satisfy (S) the flow preserves its contracting property with respect to the Hausdorff measure. So it remains to show that for any compact, flow negatively invariant set $K_1 \subset K$ which does not contain equilibrium points of (2.2) it holds $\dim_H K_1 < d+1$. On $M \backslash \Lambda$ we introduce a new metric tensor by $\hat{g}(p) := \exp\left(\frac{2v(p)}{d}\right) g(p)$ for $p \in M\backslash\Lambda$. On K_1 the Riemannian metric $\hat{g}$ is equivalent to g. Changing to the metric $\hat{g}$ does not alter the Hausdorff dimension of the compact set K_1. Consider the operator $\hat{A}(p)$ from (2.5), the symmetric part $S\hat{A}(p)$ of $\hat{A}(p)$, the operator $\hat{\nabla} f(p)$, and $S\hat{\nabla} f(p)$, which are defined regarding to the scalar product in T_pM induced by the metric $\hat{g}$. As in [39] one shows that $S\hat{\nabla} f(p) = S\nabla f(p) + \frac{L_f v(p)}{d}\,\mathrm{id}_{T_pM}$. Using (2.7) we obtain that for a regular point $p \in M$ the eigenvalues $\hat{\beta}_i(p)$ of the operator $S\hat{A}(p)|_{T^\perp(p)}$ are related to the eigenvalues $\beta_i(p)$ $(i = 1, \ldots, n-1)$ with respect to the original metric g by $\hat{\beta}_i(p) = \beta_i(p) + \frac{L_f v(p)}{d}$. Therefore,

$$\begin{aligned} &\hat{\beta}_1(p) + \cdots + \hat{\beta}_{\lfloor d \rfloor}(p) + (d - \lfloor d \rfloor)\hat{\beta}_{\lfloor d \rfloor + 1}(p) \\ &= \beta_1(p) + \cdots + \beta_{\lfloor d \rfloor} + (d - \lfloor d \rfloor)\beta_{\lfloor d \rfloor + 1}(p) + L_f v(p) \end{aligned}$$

guarantees (4.13) and thus (4.1) of Theorem 4.1. Hence $\dim_H K_1 < d+1$.

Corollary 4.2 *Consider a 2-dimensional Riemannian manifold M. Suppose that the flow $\{\varphi^t\}_{t\in\mathbb{R}}$ of (2.2) satisfies (3.3) and condition* (S) *with respect to compact sets K and $\widetilde{K}$ in M. If* div $f(p) < 0$ *holds for any regular points $p \in \widetilde{K}$ then* $\dim_H K \le 1$.

Proof For the operator $A(p)$ from (2.5) it holds $\mathrm{tr}\,(SA(p)|_{T^\perp(p)}) = \mathrm{tr}\,\nabla f(p) - \langle \nabla f(p) f(p), f(p)\rangle / \|f(p)\|^2$. We define the C^1-function v on the set of all regular points p in M by $v(p) = \frac{1}{2}\ln\|f(p)\|^2$. The statement follows with Corollary 4.1.

5 Flow Invariant Sets with an Equivariant Tangent Bundle Splitting

The considered outer measures defined via tube covers show in many cases a better contraction behavior under the flow operator of a vector field in positive time direction than conventional outer measures defined via a covering of balls do. Using such an approach for a class of generalized hyperbolic flows on n-dimensional Riemannian manifolds we may improve upper Hausdorff dimension estimates which are obtained with methods from [39] (or from [45] for the $\mathbb{R}^n$).

Consider again the vector field $f\colon M \to TM$ from (2.2) on the smooth n-dimensional Riemannian manifold (M, g). Let us introduce a property of flow-invariant sets which may be considered as a generalized hyperbolic structure. We say that a flow-invariant compact set $K \subset M$ possesses an *equivariant tangent bundle splitting* (which for simplicity consists of only two components) $T_K M = E^1 \oplus E^2$ with respect to the flow $\{\varphi^t\}_{t\in\mathbb{R}}$ if for any $p \in K$ and $i = 1, 2$ the space $E^i_p = E^i \cap T_pM$ is an n_i-dimensional subspace of T_pM such that $n_1 + n_2 = n$ and $d_p\varphi^t(E^i_p) = E^i_{\varphi^t(p)}$ hold for any $p \in K$ and $t \in \mathbb{R}$. Recall that an *Anosov flow* on K is a flow without equilibria for which among other properties there exists an equivariant tangent bundle splitting $T_K M = E^1 \oplus E^2$, where $E^2_p = \mathrm{span}\,\{f(p)\}$ for each $p \in K$. For $d \in (0, n - n_2]$ and $t \in \mathbb{R}$ we introduce the *singular value function of order d of φ^t on K with respect to the splitting $E^1 \oplus E^2$* which is defined by

$$\omega^{E^1,E^2}_{d,K}(\varphi^t) := \sup_{p\in K} \omega_d\big(d_p\varphi^t|_{E^1(p)}\big).$$

Since $\omega_{d,K}^{E^1,E^2}(\varphi^t)$ is a sub-exponential function the limit

$$\nu_d := \lim_{t\to\infty} \frac{1}{t} \ln \omega_{d,K}^{E^1,E^2}(\varphi^t)$$

exists for any $d \in (0, n-n_2]$ ([46]). We call the numbers

$$\nu_1^u := \nu_1, \quad \nu_i^u := \nu_i - \nu_{i-1} \quad \text{for} \quad i = 1, \dots, n-n_2$$

the *uniform Lyapunov exponents of* $\{\varphi^t\}$ *with respect to the splitting* $E^1 \oplus E^2$. Let us investigate the splitting $T_K M = E^2 \oplus E^2$ such that $E^1 = T^\perp$ with $E_p^1 = T^\perp(p)$ and $E^2 = T^\parallel$ with $E_p^2 = T^\parallel(p) = \operatorname{span}\{f(p)\}$.

With the help of Lemma 2.1 one shows that for any regular point $p \in M$ satisfying

$$\langle S\nabla f(p)z, f(p)\rangle = 0 \quad \text{for all} \quad z \in T^\perp(p) \tag{5.1}$$

the $n-1$ eigenvalues $\beta_1(p), \dots, \beta_{n-1}(p)$ of $SA(p)|_{T^\perp(p)}$, with the operator $A(p)$ from (2.5), coincide with $n-1$ eigenvalues of $S\nabla f(p)$. The subspace $T^\parallel(p)$ is the eigenspace of the remaining nth eigenvalue $\overline{\alpha}(p) = \langle \nabla f(p)f(p), f(p)\rangle / \|f(p)\|^2$ of $S\nabla f(p)$.

We consider now two compact sets K and $\widetilde{K}$ in M without equilibrium points of (2.2) satisfying (3.3) and suppose that (5.1) is satisfied for any $p \in \widetilde{K}$. By $\alpha_1(p) \ge \dots \ge \alpha_n(p)$ denote the eigenvalues of $S\nabla f(p)$. For that case Theorem 3.1 from [39] states that if for some $d \in (0, n]$ the inequality

$$\alpha_1(p) + \ldots + \alpha_{\lfloor d\rfloor}(p) + (d - \lfloor d\rfloor)\alpha_{\lfloor d\rfloor+1}(p) < 0$$

holds for all $p \in \widetilde{K}$, the estimate $\dim_H K < d$ is true. For the C^1-function $v\colon \widetilde{K} \to \mathbb{R}$ given by $v(p) = \frac{1}{2}\ln\|f(p)\|^2$ we have $L_f v(p) = \langle \nabla f(p)f(p), f(p)\rangle/\|f(p)\|^2 = \overline{\alpha}(p)$ for each $p \in \widetilde{K}$. If $\overline{\alpha}(p) \ge 0$ holds for all $p \in \widetilde{K}$ then

$$\begin{aligned}
&\alpha_1(p) + \dots + \alpha_{\lfloor d\rfloor}(p) + (d - \lfloor d\rfloor)\alpha_{\lfloor d\rfloor+1}(p) \\
&= \beta_1(p) + \dots + \beta_{\lfloor d\rfloor-1}(p) + (d - \lfloor d\rfloor)\beta_{\lfloor d\rfloor}(p) + L_f v(p).
\end{aligned}$$

With this Corollary 4.1 gives an upper bound of $\dim_H K$ which is less than or equal to the upper bound we would get applying Theorem 3.1 from [39]. If $d = 2$ then Corollary 4.1 gives the better estimate $\dim_H K \le 1$.

One easily shows that a compact, flow-invariant set K without equilibrium points possesses an equivariant tangent bundle splitting $T^\perp \oplus T^\parallel$ if and only if (5.1) holds for any $p \in K$. Obviously the flow $\{\varphi^t\}_{t\in\mathbb{R}}$ on K then is already reparametrized globally if one considers the reparametrization described in Lemma 2.3. For that case the assumptions of Theorem 4.1 can be weaken if we consider the long-time behavior.

Proposition 5.1 *Let $f\colon M \to TM$ be the C^2-vector field from (2.2) on the n-dimensional Riemannian manifold (M, g). Suppose that $K \subset M$ is a compact and flow-invariant set without equilibrium points of (2.2) and that K possesses an equivariant tangent bundle splitting $T_K M = T^\perp \oplus T^\parallel$ with respect to the flow. Let $D \in \{0, \dots, n-1\}$ be the smallest number such that $\nu_1^u + \dots + \nu_D^u + \nu_{D+1}^u < 0$. Then it holds*

$$\dim_H K \le D + \frac{\nu_1^u + \dots + \nu_D^u}{|\nu_{D+1}^u|} + 1.$$

Proof Take an arbitrary number $d \in \Big(D + \frac{\nu_1^u + \cdots + \nu_D^u}{|\nu_{D+1}^u|}, n-1\Big]$. Then it holds $\nu_d = \nu_1^u + \cdots + \nu_{\lfloor d \rfloor}^u + (d - \lfloor d \rfloor)\nu_{\lfloor d \rfloor + 1}^u < 0$. Fix some $\varepsilon \in (0, \nu_d)$. By definition of ν_d there is a finite number $T_0 > 0$ such that $\frac{1}{T_0} \ln \omega_{d,K}^{T^\perp, T^\parallel}(\varphi^{T_0}) < \nu_d + \varepsilon$, i.e., $\omega_{d,K}^{T^\perp, T^\parallel}(\varphi^{T_0}) < \exp(T_0(\nu_d + \varepsilon)) < 1$. Theorem 4.1 basically uses properties of the singular value function which is estimated from above applying Lemma 2.2. Thus, the proposition can be proved applying analogous arguments and using $\omega_{d,K}^{T^\perp, T^\parallel}(\varphi^{T_0}) = \sup_{p \in K} \omega_d\big(d_p \varphi^{T_0}|_{T^\perp(p)}\big)$.

Example 5.1 Consider the vector field in $\mathbb{R}^2$ given by

$$\dot{\theta} = a \sin \theta, \qquad \dot{x} = -x + b \tag{5.2}$$

(with parameters $a \geq 1$, $b \neq 0$), being in the first coordinate periodic with period 2π. The arising dynamical system can be interpreted as a dynamical system on the flat cylinder Z of all equivalence classes $[u]$, $u \in \mathbb{R}^2$, being a smooth 2-dimensional Riemannian manifold with the standard metric for factor manifolds. Every solution of (5.2) is bounded in the second coordinate. Obviously, the set $K = \{z \in Z | z = [u], u = (\theta, 0), \theta \in \mathbb{R}\}$ is compact and flow-invariant with respect to (5.2). The variational system (2.3) and the system in normal variations (2.4) with respect to any solution $(\theta(t), 0)$ in K are given by

$$\dot{y} = \begin{pmatrix} a\cos\theta(t) & 0 \\ 0 & -1 \end{pmatrix} y \quad \text{and} \quad \dot{z} = \begin{pmatrix} -a\cos\theta(t) & 0 \\ 0 & -1 \end{pmatrix} z,$$

respectively. Thus, $\beta_1(z) = -1$ for any $z \in K$ and condition (4.1) is satisfied with $d = 1$ and $\Theta = T = 1$. By Theorem 4.1 we conclude that $\dim_H K \leq 1$. Note that in the present situation other available theorems [39, 45] are not applicable since the divergence of the right-hand side of (5.2) gives the expression $a\cos\theta - 1$ which is, in contrast to the assumptions of Theorem 3.1 from [39], not always negative.

6 Generalizations of the Theorems of Hartman-Olech and Borg

In this section we show that for certain vector fields in $\mathbb{R}^3$ the methods of the present paper provide always more effective conditions for upper Hausdorff bounds than those which work without projection onto transversal submanifolds (e.g. [39, 45]). In addition to this we improve for these systems results about the structure of ω-limit sets, which are closely related to results in [4, 19, 20].

Consider an arbitrary C^2-vector field f in $\mathbb{R}^3$ with the standard Euclidean metric, i.e., the differential equation

$$\dot{x} = f(x). \tag{6.1}$$

Suppose that for (6.1) the global flow $\{\varphi^t\}_{t \in \mathbb{R}}$ exists. Let K and $\widetilde{K}$ be two compact sets in $\mathbb{R}^3$ satisfying $K \subset \varphi^t(p) \subset \widetilde{K}$ for all $t \geq 0$. For that case for any $x \in \mathbb{R}^3$ the covariant derivative $\nabla f(x)$ can be identified with the Jacobi matrix $Df(x)$ of f in x. Suppose that f possesses in $\widetilde{K}$ a finite number of equilibrium points and that for any such equilibrium point all eigenvalues of $Df(x)$ have negative real part.

Consider the symmetric part $SDf(x) = \frac{1}{2}\big(Df(x) + Df(x)^*\big)$ of $Df(x)$. As in the previous sections for any regular p of f define the hyperplanes $T^{\perp}(x) = \{z \in \mathbb{R}^3 \mid f(x)^* z = 0\}$, where $f(x)^*$ denotes the transposed vector. Let the linear operator $SA(x)\colon T^{\perp}(x) \to T^{\perp}(x)$ be given by

$$SA(x) = SDf(x) - \frac{f(x)f(x)^*}{\|f(x)\|^2}\, SDf(x) = \left(I - \frac{f(x)f(x)^*}{\|f(x)\|^2}\right) Df(x)$$

(compare with Lemma 2.1). Denote the eigenvalues of $SDf(x)$, ordered with respect to size and multiplicity, by $\alpha_1(x) \ge \alpha_2(x) \ge \alpha_3(x)$. Suppose that $\beta_1(x) \ge \beta_2(x)$ are the eigenvalues of $SA(x)$ restricted to the subspace $T^{\perp}(x)$ and suppose further that $\beta_1(x)$ and $\beta_2(x)$ are not eigenvalues of $S\nabla f(x)$. It is easy to see that $\beta_1(x)$ and $\beta_2(x)$ are the zeros of the equation $f(x)^*\big(\beta_i(x)I - SDf(x)\big)^{-1} f(x) = 0$. We introduce the polynomial

$$\det(\beta I - Df(x)) \equiv \beta^3 + \delta_2(x)\beta^2 + \delta_1(x)\beta + \delta_0(x). \tag{6.2}$$

Let $x \in \widetilde{K}$. Note that we have $\delta_2(x) = -(\alpha_1(x) + \alpha_2(x) + \alpha_3(x))$, $\delta_1(x) = \alpha_1(x)\alpha_2(x) + \alpha_2(x)\alpha_3(x) + \alpha_1(x)\alpha_3(x)$ and $\delta_0(x) = -\alpha_1(x)\alpha_2(x)\alpha_3(x)$. From this with elementary calculations (see [16]) it follows that the eigenvalues $\beta_i(x)$ $(i = 1, 2)$ of $SA(x)$ are the zeros of the polynomial

$$\beta^2 + \big[\delta_2(x) + \Delta_1(x)\big]\beta + \big[\delta_1(x) + \delta_2(x)\Delta_1(x) + \Delta_2(x)\big],$$

where

$$\begin{aligned} \Delta_1(x) &= \frac{1}{\|f(x)\|^2}\, f(x)^* Df(x) f(x) \quad \text{and} \\ \Delta_2(x) &= \frac{1}{\|f(x)\|^2}\, f(x)^* Df(x)^2 f(x). \end{aligned} \tag{6.3}$$

Using this fact one sees immediately that the assumptions of Corollary 4.1 are satisfied for (6.1) if we suppose for the auxiliary function $v(x) = \frac{1}{2}\ln\|f(x)\|^2$, defined on the set of all regular points of $\mathbb{R}^3$, the following conditions: There exists a continuous function $s\colon \widetilde{K} \to [0, d_1]$ with $d_1 \in (0, 1]$ such that for any regular point $x \in \widetilde{K}$ with $h(x) := \frac{1-s(x)}{1+s(x)}$ the inequalities

$$\begin{gathered} \delta_2(x) - h(x)\Delta_1(x) > 0 \quad \text{and} \\ \frac{1}{4h(x)^2}\big(\delta_2(x) - h(x)\Delta_1(x)\big)^2 > \frac{1}{4}\big(\delta_2(x) - \Delta_1(x)\big)^2 - \delta_1(x) - \Delta_2(x) \end{gathered}$$

hold. As a corollary we get that if the inequalities

$$\begin{aligned} &\delta_2(x) - \Delta_1(x) > 0 \quad \text{and} \\ &\delta_1(x) + \Delta_2(x) > 0 \end{aligned} \tag{6.4}$$

are satisfied for all regular points x of f on $\widetilde{K}$ then by Corollary 4.1 it holds that $\dim_H K \le 1$. Further, the set K consists of a finite number of equilibrium points and closed trajectories of (6.1). This can be easily shown using coverings of appropriated tubular neighborhoods. Note that the last result is closely related to results in [4, 19, 20]. If in addition to this the set $\widetilde{K}$ is positively invariant with respect to the flow of (6.1), connected, and if $\widetilde{K}$ contains exactly one equilibrium point being asymptotically stable, then $\widetilde{K}$ is contained in the basin of attraction of this equilibrium point.

The Hartman-Olech condition ([20]) requires that $\alpha_1(x) + \alpha_2(x) < 0$ for all regular points $x \in \widetilde{K}$. This is one of the most effective sufficient condition which guarantees that in the present situation the set $\widetilde{K}$ is contained in the basin of attraction of an equilibrium. Note that this is always sufficient for the condition (6.4).

Let us formulate a further corollary from Theorem 4.2 for the case $M = \mathbb{R}^3$. Suppose now that $\delta_2(x) > 0$ for all regular points $x \in \widetilde{K}$ and that there exists a continuous function $s\colon \widetilde{K} \to [0, d_1)$ with $d_1 \in (0, 1]$ such that the inequalities

$$\begin{gathered}\frac{1+s(x)}{1-s(x)}\,\delta_2(x) - \Delta_1(x) \geq 0 \quad \text{and}\\ \frac{s(x)}{(1-s(x))^2}\,\delta_2(x)^2 - \frac{s(x)}{1-s(x)}\,\delta_2(x)\Delta_1(x) + \delta_1(x) + \Delta_2(x) \geq 0\end{gathered} \tag{6.5}$$

hold for all regular $x \in \widetilde{K}$. It follows from Corollary 4.1 that $\dim_H K < 2 + d_1$. It is well-known (see [39, 45]) that a sufficient condition for the dimension estimate $\dim_H K < 2 + d_1$ is the inequality

$$\alpha_1(x) + \alpha_2(x) + d_1\alpha_3(x) < 0 \quad \text{for all} \quad x \in \widetilde{K}. \tag{6.6}$$

It is easy to show ([16]) that our condition (6.5) is always satisfied supposed that (6.6) is satisfied.

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