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Fuzzy Differential Systems and the New Concept of Stability

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Abstract: The study of fuzzy differential systems is initiated and sufficient conditions, in terms of Lyapunov-like functions, are provided for the new concept of stability which unifies Lyapunov and orbital stabilities as well as includes new notions in between.

Keywords: *Fuzzy differential systems; new notion of stability; stability tests*

Mathematics Subject Classification (2000): 26E50, 34A19, 34D20, 34D99.

1 Introduction

Recently, the theory of fuzzy differential equations has been initiated and the basic results have been systematically investigated, including Lyapunov stability, in [2, 3, 6, 8, 10]. This study of fuzzy differential equations corresponds to scalar differential equations without fuzziness.

A new concept of stability that includes Lyapunov and orbital stabilities as well as leads to new notions of stability in between them is introduced in terms of a given topology of the function space [9] and sufficient conditions in terms of Lyapunov-like functions are provided for such concepts to hold relative to ordinary differential equations [5].

In this paper, we shall extend the notion fuzzy differential system employing the generalized metric space and then develop the new concept of stability theory proving sufficient conditions in terms of vector Lyapunov-like functions in the framework of fuzziness. For this purpose, we develop suitable comparison results to deal with fuzzy differential systems in terms of Lyapunov-like functions and then employing the comparison result offer sufficient conditions for the new concepts to hold. This new approach helps to understand the intricacies involved in incorporating fuzziness in the theory of differential equations.

2 Preliminaries

Let $P_k(R^n)$ denote the family of all nonempty compact, convex subsets of R^n. If $\alpha,\ \beta \in R$ and $A,\ B \in P_k(R^n)$, then

$$\alpha(A+B) = \alpha A + \alpha B, \quad \alpha(\beta A) = (\alpha\beta)A, \quad 1A = A$$

and if $\alpha,\ \beta \geq 0$, then $(\alpha+\beta)A = \alpha A + \beta A$. Let $I = [t_0, t_0 + a]$, $t_0 \geq 0$ and $a > 0$ and denote by $E^n = [u\colon R^n \to [0,1]$ such that u satisfies (i) to (iv) mentioned below]:

(i) u is normal, that is, there exists an $x_0 \in R^n$ such that $u(x_0) = 1$;

(ii) u is fuzzy convex, that is, for $x,\ y \in R^n$ and $0 \leq \lambda \leq 1$,

$$u(\lambda x + (1-\lambda)y) \geq \min[u(x), u(y)];$$

(iii) u is upper semicontinuous;

(iv) $[u]^0 = \overline{[x \in R^n\colon u(x) > 0]}$ is compact.

For $0 < \alpha \leq 1$, we denote $[u]^\alpha = [x \in R^n\colon u(x) \geq \alpha]$. Then from (i) to (iv), it follows that the α-level sets $[u]^\alpha \in P_k(R^n)$ for $0 \leq \alpha \leq 1$.

Let $d_H(A,B)$ be the Hausdorff distance between the sets $A,\ B \in P_k(R^n)$. Then we define

$$d[u,v] = \sup_{0 \leq \alpha \leq 1} d_H[[u]^\alpha, [v]^\alpha],$$

which defines a metric in E^n and (E^n, d) is a complete metric space. We list the following properties of $d[u,v]$:

$$d[u+w, v+w] = d[u,v] \quad \text{and} \quad d[u,v] = d[v,u],$$

$$d[\lambda u, \lambda v] = |\lambda| d[u,v],$$

$$d[u,v] \leq d[u,w] + d[w,v],$$

for all $u,\ v,\ w \in E^n$ and $\lambda \in R$.

For $x,\ y \in E^n$ if there exists a $z \in E^n$ such that $x = y + z$, then z is called the H-difference of x and y and is denoted by $x - y$. A mapping $F\colon I \to E^n$ is differentiable at $t \in I$ if there exists a $F'(t) \in E^n$ such that the limits

$$\lim_{h \to 0^+} \frac{F(t+h) - F(t)}{h} \quad \text{and} \quad \lim_{h \to 0^+} \frac{F(t) - F(t-h)}{h}$$

exist and are equal to $F'(t)$. Here the limits are taken in the metric space (E^n, d).

Moreover, if $F\colon I \to E^n$ is continuous, then it is integrable and

$$\int_a^b F = \int_a^c F + \int_c^b F.$$

Also, the following properties of the integral are valid. If $F,\ G\colon I \to E^n$ are integrable, $\lambda \in R$, then the following hold:

$$\int (F+G) = \int F + \int G;$$

$$\int \lambda F = \lambda \int F, \quad \lambda \in R;$$

$$d[F,G] \ \text{ is integrable;}$$

$$d\Big[\int F, \int G\Big] \leq \int d[F,G].$$

Finally, let $F\colon I \to E^n$ be continuous. Then the integral $G(t) = \int\limits_a^t F$ is differentiable and $G'(t) = F(t)$. Furthermore,

$$F(t) - F(t_0) = \int\limits_a^t F'(t).$$

See [2, 3, 8, 10] for details.

We need the following known [4] results from the theory of ordinary differential inequalities. Hereafter, the inequalities between vectors in R^d are to be understood componentwise.

Theorem 2.1 *Let $g \in C[R_+ \times R_+^d \times R_+^d, R^d]$, $g(t,w,\xi)$ be quasimonotone nondecreasing in w for each (t,ξ) and monotone nondecreasing in ξ for each (t,w). Suppose further that $r(t) = r(t,t_0,w_0)$ is the maximal solution of*

$$w' = g(t,w,w), \qquad w(t_0) = w_0 \ge 0, \tag{2.1}$$

existing on $[t_0,\infty)$. Then the maximal solution $R(t) = R(t,t_0,w_0)$ of

$$w' = g(t,w,r(t)), \qquad w(t_0) = w_0 \ge 0, \tag{2.2}$$

exists on $[t_0,\infty)$ and

$$r(t) \equiv R(t), \quad t \ge t_0. \tag{2.3}$$

Theorem 2.2 *Assume that the function $g(t,w,\xi)$ satisfies the conditions of Theorem 2.1. Then $m \in C[R_+, R_+^d]$ and*

$$D^+m(t) \le g(t,m(t),\xi), \quad t \ge t_0. \tag{2.4}$$

Then for all $\xi \le r(t)$, it follows that

$$m(t) \le r(t), \quad t \ge t_0.$$

3 Fuzzy Differential System

We have been investigating so far the fuzzy differential equation

$$u' = f(t,u), \quad u(t_0) = u_0, \tag{3.1}$$

where $f \in C[R_+ \times E^n, E^n]$, which corresponds to, without fuzziness, scalar differential equation [2, 3, 6, 8]. To consider the situation analogous to differential system, we need to prepare suitable notation. Let $u = (u_1, u_2, \dots, u_N)$ with $u_i \in E^n$ for each $1 \le i \le N$ so that $u \in E^{nN}$, where

$$E^{nN} = (E^n \times E^n \times \dots \times E^n), \quad N - \text{times}.$$

Let $f \in C[R_+ \times E^{nN}, E^{nN}]$ and $u_0 \in E^{nN}$. Then consider the fuzzy differential system

$$u' = f(t,u), \quad u(t_0) = u_0. \tag{3.2}$$

We have two possibilities to measure the new fuzzy variables u, u_0, f, that is,

(1) we can define $d_0[u,v] = \sum_{i=1}^{N} d[u_i, v_i]$, where u_i, $v_i \in E^n$ for each $1 \le i \le N$ and employ the metric space (E^{nN}, d_0), or
(2) we can define the generalized metric space (E^{nN}, D), where

$$D[u,v] = (d[u_1,v_1], d[u_2,v_2], \ldots, d[u_N,v_N]).$$

In any of the foregoing set-ups, one can prove existence and uniqueness results for (3.2) using the appropriate contraction mapping principles. See [1] for the details of generalized spaces and generalized contraction mapping principle.

We can now prove the needed comparison result in terms of suitable Lyapunov-like functions. For this purpose, we let

$$\Omega = [\sigma \in C^1[R_+, R_+] \colon \sigma(t_0) = t_0 \text{ and } w(t,\sigma,\sigma') \le r(t),\ t \ge t_0], \tag{3.3}$$

where $w \in C[R_+^2 \times R, R_+^d]$ and $r(t)$ is the maximal solution of (2.1).

Theorem 3.1 *Assume that for some $\sigma \in \Omega$, there exists a V such that $V \in C[R_+^2 \times E^{nN} \times E^{nN}, R_+^d]$ and*

$$|V(t,\sigma,u_1,v_1) - V(t,\sigma,u_2,v_2)| \le A[D[u_1,u_2] + D[v_1,v_2]],$$

where A is an $N \times N$ positive matrix. Moreover,

$$\begin{aligned} &D^+V(t,\sigma,u,v) \\ &= \limsup_{h\to 0^+} \frac{[V(t+h,\sigma(t+h),u+hf(t,u),v+hf(\sigma,v)\sigma') - V(t,\sigma,u,v)]}{h} \\ &\le g(t, V(t,\sigma,u,v), w(t,\sigma,\sigma')), \end{aligned}$$

where $g(t,w,\xi)$ satisfies the conditions of Theorem 2.1.

Then $V(t_0,\sigma(t_0),u_0,v_0) \le w_0$ implies

$$V(t,\sigma(t),u(t,t_0,u_0),v(\sigma(t),t_0,v_0)) \le r(t,t_0,w_0), \quad t \ge t_0.$$

Proof Let $u(t) = u(t,t_0,u_0)$, $v(t) = v(t,t_0,v_0)$ be the solutions of (3.2) and set $m(t) = V(t,\sigma(t),u(t),v(\sigma(t))$ so that $m(t_0) = V(t_0,\sigma(t_0),u_0,v_0)$. Let $w_0 = m(t_0)$. Then for small $h > 0$, we have, in view of the Lipschitz condition given in (i),

$$\begin{aligned} &m(t+h) - m(t) = V(t+\sigma,\sigma(t+h),u(t+h),v(\sigma(t+h))) \\ &- V(t,\sigma(t)u(t),v(\sigma(t))) + V(t+h,\sigma(t+h),u(t)+hf(t,u(t)), \\ &v(\sigma(t)) + hf(\sigma(t),v(\sigma)))\sigma'(t))) \\ &\le A[D[u(t+h),u(t)+hf(t,u(t))] + D[v(\sigma(t+h)), \\ &v(\sigma(t)) + hf(\sigma(t),v(\sigma(t)))\sigma'(t)]] + V(t+h,\sigma(t+h),u(t)+hf(t,u(t)), \\ &v(\sigma(t)) + hf(\sigma(t),v(\sigma(t)))\sigma'(t)) - V(t,\sigma(t),u(t),v(\sigma(t))). \end{aligned}$$

It therefore follows that

$$D^+m(t) = \limsup_{h\to 0^+} \frac{1}{h}\,[m(t+h) - m(t)] \le D^+V(t,\sigma(t),u(t),v(t))$$
$$+ A\limsup_{h\to 0^+} \frac{1}{h}\,[D[u(t+h),u(t)+hf(t,u(t))]$$
$$+ D[v(\sigma(t+h),v(\sigma(t)+hf(\sigma(t),v(\sigma(t))\sigma')]].$$

Since $u'(t)$, $v'(t)$ is assumed to exist, we see that $u(t+h) = u(t) + z(t)$, $v(\sigma(t+h)) = v(\sigma(t)) + \xi(\sigma(t))$, where $z(t)$, $\xi(\sigma(t))$ are the H-differences for small $h > 0$. Hence utilizing the properties of $D[u,v]$, we obtain

$$D[u(t+h),u(t)+hf(t,u(t))] = D[u(t)+z(t),u(t)+hf(t,u(t)))]$$
$$= D[z(t),hf(t,u(t))] = D[u(t+h)-u(t),hf(t,u(t))].$$

As a result, we get

$$\frac{1}{h}\,D[u(t+h),u(t)+hf(t,u(t))] = D\left[\frac{u(t+h)-u(t)}{h},\ f(t,u(t))\right]$$

and consequently

$$\limsup_{h\to 0^+} \frac{1}{h}\,D[u(t+h,u(t)+h(f(t,u(t))]$$
$$= \limsup_{h\to 0^+} D\left[\frac{u(t+h)-u}{h},\ f(t,u(t))\right] = D[u'(t),f(t,u(t))] = 0,$$

since $u(t)$ is the solution of (3.2). Similarly, we can obtain

$$\limsup_{h\to 0^+} \frac{1}{h}\,D[v(\sigma(t+h),v(\sigma(t))+hf(\sigma(t),v(\sigma(t))\sigma']$$
$$= D[v'(\sigma(t)),f(\sigma(t),v(\sigma(t))\sigma'(t)] = 0,$$

since $v(t)$ is the solution of (3.2). We have therefore the vector differential inequality

$$D^+m(t) \le g(t,m(t),w(t,\sigma(t),\sigma'(t))), \quad t \ge t_0.$$

Since $\sigma\in\Omega$, we then get

$$D^+m(t) \le g(t,m(t),r(t)), \quad t \ge t_0,$$

where $r(t)$ is the maximal solution of (2.1). By the theory of differential inequalities for systems [4] the claimed estimate (3.4) follows and the proof is complete.

Let us next introduce the new concept of stability. Let $v(t,t_0,v_0)$ bet the given unperturbed solution of (3.2) on $[t_0,\infty)$ and $u(t,t_0,u_0)$ be any perturbed solution of (3.2) on $[t_0,\infty)$ and $u(t,t_0,u_0)$ be any perturbed solution of (3.2) on $[t_0,\infty)$. Then the Lyapunov stability (LS) compares the phase space positions of the unperturbed and perturbed solutions at exactly simultaneous instants, namely

$$d_0[u(t,t_0,u_0), v(t,t_0,v_0)] < \epsilon, \quad t \geq t_0, \tag{LS}$$

which is a too restrictive requirement from the physical point of view. The orbital stability (OS), on the other hand, compares phase space positions of the same solutions at any two unrelated times, namely,

$$\inf_{s\in[t_0,\infty)} d_0[u(t,t_0,u_0), v(s,t_0,v_0)] < \epsilon, \quad t \geq t_0.$$

In this case, the measurement of time is completely irregular and therefore (OS) is too loose a demand.

We therefore need a new notion unifying (LS) and (OS) which would lead to concepts between them that could be physically significant. This is precisely what we plan to do below.

Let E denote the space of all functions from $R_+ \to R_+$, each function $\sigma(t) \in E$ representing a clock. Let us call $\sigma(t) = t$, the perfect clock. Let τ-be any topology in E. Given the solution $v(t,t_0,v_0)$ of (3.2) existing on $[t_0,\infty)$, we define following Massera [9], the new concept of stability as follows.

Definition 3.1 The solution $u(t,t_0,v_0)$ of (3.2) is said to be

(1) *τ-stable*, if, given $\epsilon > 0$, $t_0 \in R_+$, there exist a $\delta = \delta(t_0,\epsilon) > 0$ and an τ-neighborhood of N of the perfect clock satisfying $d_0[u_0,v_0] < \delta$ implies

$$d_0[u(t,t_0,u_0), v(\sigma(t),t_0,v_0)] < \epsilon, \quad t \geq t_0,$$

where $\sigma \in N$;

(2) *τ-uniformly stable*, if δ in (1) is independent of t_0.

(3) *τ-asymptotically stable*, if (1) holds and given $\epsilon > 0$, $t_0 \in R_+$, there exist a $\delta_0 = \delta_0(t_0) > 0$, a τ-neighborhood N of the perfect clock and a $T = T(t_0,\epsilon) > 0$ such that

$$d_0[u_0,v_0] < \delta_0 \quad \text{implies} \quad d_0[u(t,t_0,u_0), v(\sigma(t),t_0,v_0)] < \epsilon, \quad t \geq t_0 + T,$$

where $\sigma \in N$;

(4) *τ-uniformly asymptotically stable*, if δ_0 and T are independent of t_0.

We note that a partial ordering of topologies induces a corresponding partial ordering of stability concepts.

Let us consider the following topologies of E:

(τ_1) the discrete topology, where every set in E is open;

(τ_2) the chaotic topology, where the open sets are only the empty set and the entire clock space E;

(τ_3) the topology generated by the base

$$U_{\sigma_0,\epsilon} = [\sigma \in E\colon \sup_{t\in[t_0,\infty)} |\sigma(t) - \sigma_0(t)| < \epsilon];$$

(τ_4) the topology defined by the base

$$U_{\sigma_0,\epsilon} = [\sigma \in C^1[R_+,R_+]\colon |\sigma(t_0) - \sigma_0(t_0)| < \epsilon \quad \text{and} \quad \sup_{t\in[t_0,\infty)} |\sigma'(t) - \sigma_0'(t)| < \epsilon].$$

It is easy to see that (τ_3), (τ_4) topologies lie between (τ_1) and (τ_2). Also, an obvious conclusion is that if the unperturbed motion $v(t,t_0,v_0)$ is the trivial solution, then (OS) implies (LS).

4 Stability Criteria

In τ_1-topology, one can use the neighborhood consisting of solely the perfect clock $\sigma(t) = t$ and therefore, Lyapunov stability follows immediately from the existing results.

Define $B = B[t_0,v_0] = v([t_0,\infty),t_0,v_0)$ and suppose that B is closed. Then the stability of the set B can be considered the usual way in terms of Lyapunov functions [4, 7] since

$$\rho[u(t,t_0,u_0),B] = \inf_{s\in[t_0,\infty)} d_0[u(t,t_0,u_0),v(s,t_0,v_0)],$$

denoting the infimum for each t by s_t and defining $\sigma(t) = s_t$ for $t > t_0$, we see that $\sigma \in E$ in τ_2-topology. We therefore obtain orbital stability of the given solution $v(t,t_0,v_0)$ in terms of τ_2-topology.

To investigate the results corresponding to (τ_3) and (τ_4) topologies, we shall utilize the comparison Theorem 3.1 and modify suitably the proofs of standard stability results [4, 7].

Theorem 4.1 *Let the condition (i) of Theorem 3.1 be satisfied. Suppose further that*

(a) $b(d_0[u,v]) \le \sum_{i=1}^{d} v_i(t,\sigma,u,v) \le a(t,\sigma,d_0[u,v])$,

(b) $d(|t-\sigma(t)|) \le \sum_{i=1}^{d} w_i(t,\sigma,\sigma')$,

where $a(t,\sigma,\cdot)$, $b(\cdot)$ and $d(\cdot) \in \mathcal{K} = [a \in C[R_+,R_+]$, $a(0) = 0$ and $a(\eta)$ is increasing in $\eta]$.

Then the stability properties of the trivial solution of (2.1) imply the corresponding τ_3-stability properties of fuzzy differential system (3.2) relative to the given solution $v(t,t_0,v_0)$.

Proof Let $v(t) = v(t,t_0,v_0)$ be the given solution of (3.2) and let $0 < \epsilon$ and $t_0 \in R_+$ be given. Suppose that the trivial solution of (2.1) is stable. Then given $b(\epsilon) > 0$ and $t_0 \in R_+$, there exists a $\delta_1 = \delta_1(t_0,\epsilon) > 0$ such that

$$0 \le \sum_{i=1}^{d} w_{i0} < \delta_1 \quad \text{implies} \quad \sum_{i=1}^{d} w_i(t,t_0,w_0) < b(\epsilon), \quad t \ge t_0, \tag{4.1}$$

where $w(t,t_0,w_0)$ is any solution of (2.1). We set $w_0 = V(t_0,\sigma(t_0),u_0,v_0)$ and choose $\delta = \delta(t_0,\epsilon)$, $\eta = \eta(\epsilon)$ satisfying

$$a(t_0,\sigma(t_0),\delta) < \delta_1 \quad \text{and} \quad \eta = d^{-1}(b(\epsilon)). \tag{4.2}$$

Using (b) and the fact $\sigma \in \Omega$, we have

$$d(|t-\sigma|) \le \sum_{i=1}^{d} w_i(t,\sigma,\sigma') \le \sum_{i=1}^{d} r_i(t,t_0,w_0) \le \sum_{i=1}^{d} r_i(t,t_0,\delta_1) < b(\epsilon).$$

It then follows that $|t-\sigma(t)|<\eta$ and hence $\sigma\in N$. We claim that whenever

$$d_0[u_0,v_0]<\delta \quad \text{and} \quad \sigma\in N,$$

it follows that

$$d_0[u(t,t_0,u_0),v(\sigma(t),t_0,v_0]<\epsilon, \quad t\geq t_0.$$

If this is not true, there would exist a solution $u(t,t_0,u_0)$ and a $t_1>t_0$ such that

$$\begin{aligned} &d_0[u(t_1,t_0,u_0),v(\sigma(t_1),t_0,v_0]=\epsilon \quad \text{and}\\ &d_0[u(t,t_0,u_0,v(\sigma(t),t_0,v_0]\leq\epsilon \end{aligned} \tag{4.3}$$

for $t_0\leq t\leq t_1$. Then by Theorem 3.1, we get for $t_0\leq t\leq t_1$,

$$V(t,\sigma(t),u(t,t_0,u_0),v(t,t_0,v_0))\leq r(t,t_0,V(t_0,\sigma(t_0,u_0,v_0)),$$

where $r(t,t_0,w_0)$ is the maximal solution of (2.1). It then follows from (4.1), (4.3), using (a), that

$$b(\epsilon)=b(d_0[u(t_1),v(\sigma(t_1))])\leq\sum_{i=1}^{d}V_i(t_1,\sigma(t_1),u(t_1),v(\sigma(t_1))]$$

$$\leq\sum_{i=1}^{d}r_i(t_1,t_0,V(t_0,\sigma(t_0),u_0,v_0))\leq\sum_{i=1}^{d}r_i(t_1,t_0,a(t_0,\sigma(t_0),\delta_1))<b(\epsilon),$$

a contradiction, which proves τ_3-stability.

Suppose next that the trivial solution of (2.1) is asymptotically stable. Then it is stable and given $b(\epsilon)>0$, $t_0\in R_+$, there exist $\delta_{01}=\delta_{01}(t_0)>0$ and $T=T(t_0,\epsilon)>0$ satisfying

$$0\leq\sum_{i=1}^{d}w_{0i}<\delta_{10} \quad \text{implies} \quad \sum_{i=1}^{d}w_i(t,t_0,w_0)<b(\epsilon), \quad t\geq t_0+T. \tag{4.4}$$

The τ_3-stability yields taking $\epsilon=\rho>0$ and designating $\delta_0(t_0)=\delta(t_0,\rho)$

$$d_0[u_0,v_0]<\delta_0 \quad \text{implies} \quad d_0[u(t),v(\sigma(t))]<\rho, \quad t\geq t_0$$

for every σ such that $|t-\sigma|<\eta(\rho)$. This means that by Theorem 3.1

$$V(t,\sigma(t),u(t),v(t))\leq r(t,t_0,\delta_{10}), \quad t\geq t_0. \tag{4.5}$$

In view of (4.4), we find that

$$\sum_{i=1}^{d}r_i(t,t_0,\delta_{10})<b(\epsilon), \quad t\geq t_0+T,$$

which in turn implies

$$d[|(t-\sigma(t)|]\leq\sum_{i=1}^{d}w_i(t,\sigma,\sigma')\leq\sum_{i=1}^{d}r_i(t,t_0,\delta_{10})<b(\epsilon), \quad t\geq t_0+T.$$

Thus $|t-\sigma(t)| < d^{-1}b(\epsilon) = \eta(\epsilon)$, $t \ge t_0 + T$. Hence there exists a $\sigma \in N$ satisfying

$$d_0[u(t), v(\sigma(t))] \le \sum_{i=1}^{d} V_i(t, \sigma(t), u(t), v(\sigma(t)))$$

$$\le \sum_{i=1}^{d} r_i(t, t_0, \delta_{10}) < b(\epsilon), \quad t \ge t_0 + T,$$

which yields

$$d_0[u(t), v(\sigma(t))] < \epsilon, \quad t \ge t_0 + T,$$

whenever $d_0[u_0, v_0] < \delta_0$ and $\sigma \in N$. This proves τ_3-asymptotic stability of (3.2) and the proof is complete.

To obtain sufficient conditions for τ_4-stability, we need to replace (b) in Theorem 4.1 by

(c) $\quad d[|1-\sigma'(t)|] \le \sum_{i=1}^{d} w_i(t, \sigma, \sigma')$,

and then mimic the proof with suitable modifications. We leave the details to avoid monotony.

It would be interesting to obtain different sets of sufficient conditions as well as discover other topologies that would be of interest.

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Set Based Constant Reference Tracking for Continuous-Time Constrained Systems

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Abstract: In the paper study the possibility of tracking constant reference signals for a linear time-invariant dynamic system in the presence of state constraints. Resort to the theory of invariant sets due to its good capability of handling this kind of problem. Attention is placed on the determination of suitable sets for the attainable steady state values and of suitable control laws which guarantee that every possible output steady state value belonging to this set can be reached from any initial state belonging to a proper set. Then, based on recent results on the possibility of associating to these sets explicit smooth control laws, an explicit controller is derived which allows the system to asymptotically track constant reference signals and guarantees that no constraints violation occurs. Finally, an example of the implementation of the proposed control law will be reported.

Keywords: *Asymptotic stability domains; Lyapunov method; Lyapunov functions; non-linear systems; sets; uniform asymptotic stability.*

Mathematics Subject Classification (2000): 34C35, 34D05, 34D20, 34D45, 34H05, 54H20, 93C10, 93C15, 93C50, 93C60, 93D05, 93D20, 93D30.

1 Introduction

In most recent literature concerning linear time-invariant continuous-time dynamic systems much emphasis has been put on the constrained stabilization problem [1, 2, 3, 4, 5] but little has been done to derive stabilizing regulators which guarantee perfect asymptotic tracking of constant reference signal in the presence of state and control constraints. This problem can for instance be solved by recasting it as an l^1 problem, though this results in high complexity regulators due to the nature of the problem which in general results, according to [6], in being a multiblock problem. Another way to proceed is that of exploiting invariant regions as done in [7, 8, 9]. In [8] the authors have proposed a discrete-time reference governor which behaves significantly well in the presence of state and control constraints and whose expression is given in implicit form and can be derived from that of the "maximal output admissible set" [4] of a proper dynamic system. The mentioned governor acts as a nonlinear first order filter which limits the reference signal whenever the state is almost to exit from the maximal output admissible set. In this work we focus our attention on continuous-time systems with state constraints only and, instead of limiting instant by instant the reference signal, we provide a polyhedral set of signals the output can track. Then, exploiting some recent results concerning the possibility of "smoothing" polyhedral Lyapunov functions [10], we show how it is possible to associate a control law in explicit form to this set.

2 Notation

For a vector $x \in IR^n$ we denote by $\|x\|_\infty = \max_i |x_i|$. We call C-set a convex and compact set having the origin as an interior point. Given a C-set $\mathcal{S}$ we denote by $\partial\mathcal{S}$ and $int\mathcal{S}$ the border and interior of $\mathcal{S}$, respectively, and we denote the scaled set $\lambda\mathcal{S}$, for $\lambda \geq 0$, as $\lambda\mathcal{S} = \{y\colon y = \lambda x \ x \in \mathcal{S}\}$. Given a continuous function $\Psi\colon IR^n \to IR$ and $k \in IR$ we define the (possibly empty) closed set $\bar{\mathcal{N}}[\Psi, k]$ as $\bar{\mathcal{N}}[\Psi, k] = \{x \in IR^n\colon \Psi(x) \leq k\}$. We say that $\Psi\colon IR^n \to IR$ is a Gauge function if, for every $x, y \in IR^n$ it fulfills the following properties: $\Psi(x) > 0$, if $x \neq 0$, $\Psi(\lambda x) = \lambda\Psi(x)$, for every $\lambda \geq 0$, and $\Psi(x + y) \leq \Psi(x) + \Psi(y)$. If Ψ is a Gauge function, the set $\bar{\mathcal{N}}[\Psi, k]$ is a C-set for all $k > 0$. Any C-set $\mathcal{S}$ induces a Gauge function (the so-called Minkowski functional of $\mathcal{S}$) which is defined as $\Psi_{\mathcal{S}}(x) \doteq \inf\{\mu \geq 0\colon x \in \mu\mathcal{S}\}$ or, equivalently, as $\Psi_{\mathcal{S}}(x) \doteq \inf\{\mu \geq 0\colon \frac{x}{\mu} \in \mathcal{S}\}$. A polyhedral C-set $\mathcal{P} \in IR^n$ can be written as $\mathcal{P} = \{x\colon \max_{i=1,s} F_i x \leq 1\}$, or in compact form as $\mathcal{P} = \{x\colon Fx \leq \bar{1}\}$, where $F \in IR^{s\times n}$ is a full column rank matrix, $\bar{1}$ is the s-dimensional column vector $[1\,1 \cdots 1]^T$ and the inequality sign has to be intended component-wise. We will say that an homogeneous function $\Psi(x)$ from IR^n to IR^+ is a polyhedral function if it is the Minkowski functional of a polyhedral C-set. If $\mathcal{P} = \{x\colon Fx \leq \bar{1}\}$, then $\Psi_{\mathcal{P}}(x) = \max_i F_i x$.

3 Problem Statement

In this work we consider a continuous-time reachable and observable square dynamic system (that is with an equal number of inputs and outputs) in its standard form, say described by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t), \end{aligned} \tag{1}$$

where $A \in IR^{n\times n}$, $B \in IR^{n\times m}$ and the output matrix $C \in IR^{m\times n}$. The main additional requirement for this system is that the state never exceeds prescribed bounds represented by the C-set $\mathcal{X}$, say

$$x(t) \in \mathcal{X} \quad \text{for every} \quad t \geq 0.$$

Since a necessary and sufficient condition for the constant tracking problem to have a solution is that the system has no transmission zeros at the origin, we will work under the following assumption.

Assumption 3.1 *The pencil matrix*

$$A_c = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$$

is invertible.

For this kind of system the constrained stabilization problem is quite a well established subject [2, 1, 3]. If we assume $\mathcal{X}$ to be a polyhedral C-set we know that a stabilizing control law exists if and only if there exists a contractive set for (1) contained in $\mathcal{X}$. If we add the requirement on the output infinity norm not to exceed a prescribed value μ then the above statement must be slightly modified in the sense that the solution requires the determination of a contractive set for (1) contained in $\mathcal{X}^* = \mathcal{X} \bigcap \{x\colon \|Cx\|_\infty \leq \mu\}$ (this fact has been used in [11, 12] for the solution of l^1 problems with state feedback). In view of the reachability assumption it is easy to see that the afore-mentioned problem always has a solution (for instance a stabilizing linear regulator will do the job); nevertheless the interest in this kind of problem is usually mostly concerned with the criterion on the basis of which the stabilizing control law has to be chosen. One "natural" criterion is that of maximizing the domain of attraction to the origin included in the given set $\mathcal{X}$ as done in [2].

By exploiting this criterion we will consider the constrained tracking problem and we will take advantage of recent results [10] on the possibility of deriving suitable smooth controllers in explicit form for the solution of the constrained stabilization problem for tracking purposes. Before stating our problem it is worth recalling that, in view of Assumption 3.1 and of the constraints on the state, the set of admissible constant reference signals $\mathcal{Y}_R$ which the system will be able to track will be necessarily bounded. The problem we will focus our attention on can then be stated in the following way:

Problem 3.1 *Given the continuous-time dynamic system* (1) *and the state constraints set* $\mathcal{X}$ *find a state feedback control law* $u = \Phi(x)$ *and a set of reference signals* $\mathcal{Y}_R$ *such that for every constant reference signal* $\bar{y} \in \mathcal{Y}_R$ *the state evolution never exceeds the prescribed bounds for every* $t \geq 0$ *and such that* $\lim\limits_{t\to\infty} y(t) \to \bar{y}$.

4 Tracking a Constant Reference Signal

In the previous section without going into much detail we have stated our problem and we have mentioned the set $\mathcal{Y}_R$ of admissible values the reference signal $\bar{y}$ can assume. To see how it is possible to derive such a set we have first recall some results concerning the use of invariant regions for the solution of this kind of problem. As a first step we recall that, given the continuous-time system (1), its discrete-time Euler Approximating System (EAS) is defined as follows:

$$\begin{aligned} x(k+1) &= (I + \tau A)x(k) + \tau B u(k), \\ y(x) &= Cx(k). \end{aligned} \tag{2}$$

For continuous and discrete-time systems it is possible to furnish the following definitions of domain of attraction [2].

Definition 4.1 A region $\mathcal{P} \subset \mathcal{X}$ is a *domain of attraction* (β-contractive region) for system (1) if there exists a constant $\beta > 0$ (often referred to as speed of convergence) such that for every initial condition $x(0) \in \mathcal{P}$ there exists a piecewise continuous control function $u(\cdot)\colon IR \to IR^m$ such that the evolution corresponding to $u(t)$ is such that:

$$\Psi_{\mathcal{P}}(x(t)) \leq \Psi_{\mathcal{P}}(x(0))e^{-\beta t}$$

for every $t \geq 0$ (we recall that $\Psi_{\mathcal{P}}$ is the Minkowski functional induced by $\mathcal{P}$ on IR^n).

Definition 4.2 A region $\mathcal{P} \subset \mathcal{X}$ is a *domain of attraction* (λ-contractive region) for system (2) if there exists a constant $\lambda < 1$ (often referred to as contractivity) such that for every initial condition $x(0) \in \mathcal{P}$ there exists a sequence $u(k) \in IR^m$ such that the corresponding evolution is such that:

$$\Psi_{\mathcal{P}}(x(k)) \leq \Psi_{\mathcal{P}}(x(0))\lambda^k$$

for every $t \geq 0$.

It can be proven that the existence of a β-contractive set $\mathcal{P}$ for system (1) is equivalent to the existence, for every $x \in \mathcal{P}$, of a value v such that:

$$D^+\Psi_{\mathcal{P}}(x,v) \doteq \limsup_{\tau \to 0^+} \frac{\Psi_{\mathcal{P}}(x + \tau(Ax + Bv)) - \Psi_{\mathcal{P}}(x)}{\tau} \leq -\beta\Psi_{\mathcal{P}}(x) \tag{3}$$

(the introduction of the generalized Lyapunov derivative allows to deal with non smooth functionals, see [10] for details). In the discrete-time case the above condition, for the existence of a λ-contractive set for (2), translates in the following one-step contractivity requirement:

$$\Psi_{\mathcal{P}}(x + \tau(Ax + Bv)) \leq \lambda\Psi_{\mathcal{P}}(x). \tag{4}$$

It is well known that the systems under consideration, for a given β, admit a maximal β-contractive set $\mathcal{S}_\beta$ contained in $\mathcal{X}$ and that this set is in general not polyhedral. From [2] it is known that it is possible to approximate arbitrarily well the largest contractive set $\mathcal{S}_\beta \subset \mathcal{X}$ by means of a polyhedral set $\mathcal{P} \subset \mathcal{X}$ which results in being a domain of attraction for system (1) with a speed of convergence $\bar{\beta}$ arbitrarily close to the prescribed one and the control $u = \phi(x)$ can be expressed in feedback form, where $\phi(x)$ is Lipschitz on $\mathcal{P}$. It is straightforward that the same applies (with the cited replacement of the set $\mathcal{X}$ with $\mathcal{X}^*$) when output bounds have to be considered.

This approximation is derived and can be effectively computed by exploiting the relation existing between a continuous-time system of the form (1) and its discrete-time EAS (2), according to the next result.

Theorem 4.1 [13] *Suppose system* (1) *admits a β-contractive C-set $\mathcal{P} \subset \mathcal{X}$. Then for all $0 < \beta' < \beta$ there exists $\tau > 0$ such that $\mathcal{P}$ is λ'-contractive for the discrete-time system* (2) *with $0 < \lambda' = 1 - \tau\beta'$. Conversely, if system* (2) *admits a λ-contractive C-set $\mathcal{P}$ then $\mathcal{P}$ is β-contractive for system* (1) *with $\beta = \frac{(1-\lambda)}{\tau}$.*

Given the above definitions it is hence possible to define the set $\mathcal{Y}_R$ of admissible constant reference signals which the system will be able to track. Suppose a β-contractive set $\mathcal{P}$ has been found and consider the following equation:

$$A_c \begin{bmatrix} \bar{x} \\ \bar{u} \end{bmatrix} = \begin{bmatrix} 0 \\ \bar{y} \end{bmatrix}. \tag{5}$$

Since A_c is invertible the solution to the above set of equations can be written as

$$\bar{x} = K_{\bar{y}\bar{x}}\bar{y}, \tag{6}$$

$$\bar{u} = K_{\bar{y}\bar{u}}\bar{y}. \tag{7}$$

From (6) we see that all the admissible equilibrium states belong to the subspace $K_{\bar{y}\bar{x}}\bar{y}$ so that the admissible constant reference signals which do not lead to state constraints violation are given by

$$\mathcal{Y}_R = \{\bar{y}\colon K_{\bar{y}\bar{x}}\bar{y} \in \mathcal{P}\}, \tag{8}$$

while from (7) we know that to track an arbitrary constant signal $\bar{y} \in \mathcal{Y}_R$ the control value will have to converge to the value $\bar{u} = \bar{u}(\bar{y}) = K_{\bar{y}\bar{u}}\bar{y}$.

The next step for the solution of Problem 3.1 is that of determining a suitable control law such as to guarantee that the state constraints are never violated and the output converges to the given constant reference value $\bar{y}$. In view of Assumption 3.1 this amounts to requiring that $\lim\limits_{t\to\infty} x(t) = K_{\bar{y}\bar{x}}\bar{y}$.

To this aim consider a reference value $\bar{y} \in \alpha\mathcal{Y}_R$, $\alpha < 1$ (the need for the introduction of the parameter α will be clear in the sequel; the introduction of α basically amounts to discarding trackable signals corresponding to states belonging to the border of $\mathcal{P}$) and consider the following functional, which is the Gauge functional associated to the set $\mathcal{P}$ and centered in $\bar{x}(\bar{y})$

$$\Psi^{\bar{y}}(x) \doteq \inf\{\mu \ge 0\colon \bar{x}(\bar{y}) + \frac{1}{\mu}(x - \bar{x}(\bar{y})) \in \mathcal{P}\}.$$

The following lemma allows us to compute explicitly $\Psi^{\bar{y}}(x)$ whenever $\mathcal{P}$ is a polyhedral C-set.

Lemma 4.1 *If $\mathcal{P} = \{x\colon Fx \le \bar{1}\}$, then for every $\bar{y} \in \alpha\mathcal{Y}_R$, $\alpha < 1$, and $x \in \mathcal{P}$*

$$\Psi^{\bar{y}}(x) = \max_i \frac{F_i(x - \bar{x}(\bar{y}))}{1 - F_i\bar{x}(\bar{y})}. \tag{9}$$

Moreover $\Psi^{\bar{y}}(x) = 1$ whenever $x \in \partial\mathcal{P}$.

Proof It follows from simple algebra by first noting that, since $\bar{x}(\bar{y}) \in int\mathcal{P}$, the quantity $1 - F_i\bar{x}(\bar{y})$ is strictly greater than zero for every i. Hence

$$\begin{aligned}\Psi^{\bar{y}}(x) &\doteq \inf\left\{\mu \ge 0\colon \bar{x} + \frac{1}{\mu}(x - \bar{x}) \in \mathcal{P}\right\}\\ &= \inf\left\{\mu \ge 0\colon F_i\left(\bar{x} + \frac{1}{\mu}(x - \bar{x})\right) \le 1 \ \ \forall\, i\right\}\\ &= \inf\left\{\mu \ge 0\colon \frac{F_i(x - \bar{x})}{1 - F_i\bar{x}} \le \mu \ \ \forall\, i\right\}.\end{aligned}$$

The next lemma shows that the functional $\Psi^{\bar{y}}(x)$ just introduced, whenever $\mathcal{P}$ is a domain of attraction, can be regarded as a Lyapunov function for the dynamic of the error $e(t) = x(t) - \bar{x}(\bar{y})$ when the reference signal is a constant. For the sake of clarity and given the above-mentioned possibility of approximating the largest β-contractive set for system (1) by means of a polyhedral set, without lack of generality we will limit our attention to the case of polyhedral C-sets, although the next lemma can be proven true for any contractive C-set.

Lemma 4.2 *Let* $\mathcal{P} = \{x\colon Fx \le \bar{1}\}$ *be a* β*-contractive polyhedral* C*-set for system* (1) *and let* $\mathcal{Y}_R$ *be defined as in* (8). *Then for every constant value* $\bar{y} \in \alpha\mathcal{Y}_R$, $\alpha < 1$, *there exists* $0 < \beta_1 < \beta$ *and a state feedback control function* $u = \phi_1(x, \bar{x})$ *such that for every* $x(0) \in \mathcal{P}$ *the corresponding state evolution is such that*

$$\Psi^{\bar{y}}(x(t)) \le e^{-\beta_1 t}\Psi^{\bar{y}}(x(0)) \tag{10}$$

for every $t \ge 0$.

Proof Consider a constant reference value $\bar{y}$ and let $\bar{x}$ and $\bar{u}$ be the corresponding state and control values. Setting $e(t) = x(t) - \bar{x}$ and $v(t) = u(t) - \bar{u}$ leads to the following description of the error dynamics:

$$\dot{e} = \dot{x} - \dot{\bar{x}} = Ax + Bu - (A\bar{x} + B\bar{u}) = Ae + Bv. \tag{11}$$

Since $\Psi^{\bar{y}}(x) = \max_i \frac{F_i(x-\bar{x})}{1-F_i\bar{x}} = \max_i \frac{F_i}{1-F_i\bar{x}}\, e = \Psi_1(e)$, showing that (10) holds amounts to prove that $\mathcal{P}_1 = \left\{e\colon \frac{F_i}{1-F_i\bar{x}}\, e \le 1,\ \ i = 1, \ldots, s\right\}$ is a β_1-contractive domain for system (11). The latter, in view of Theorem 4.1, can be proven by determining τ and λ_1 such that $\mathcal{P}_1$ is λ_1 contractive for the discrete-time EAS of (11), say for every $e \in \mathcal{P}_1$ there exists v such that

$$\max_i \frac{F_i}{1 - F_i\bar{x}}\,(e + \tau(Ae + Bv)) \le \lambda_1 \max_j \frac{F_j}{1 - F_j\bar{x}}\, e. \tag{12}$$

Let us first consider $e \in \partial\mathcal{P}_1$ (hence $x \in \partial\mathcal{P}$). Expanding $v = u - \bar{u}$ in (12), the above requires, for every i, that

$$\begin{aligned} &\frac{F_i}{1 - F_i\bar{x}}\,(x - \bar{x} + \tau(Ax - A\bar{x} + Bu - B\bar{u})) \\ &\quad = \frac{F_i(x + \tau(Ax + Bu)) - 1}{1 - F_i\bar{x}} + 1 \le \lambda_1. \end{aligned} \tag{13}$$

From Theorem 4.1 for every $\beta' < \beta$ there exists τ such that $\mathcal{P}$ is $\lambda' = 1 - \tau\beta'$-contractive for the EAS of (1), say for every $x \in \partial\mathcal{P}$, there exists $\tilde{u}$ such that for every i

$$F_i(x + \tau(Ax + B\tilde{u})) \le 1 - \tau\beta'.$$

Hence, setting $u = \tilde{u}$ in (13), results in

$$\frac{F_i(x + \tau(Ax + B\tilde{u})) - 1}{1 - F_i\bar{x}} + 1 \le -\frac{\tau\beta'}{1 - F_i\bar{x}} + 1 \le \lambda_1,$$

for some $\lambda_1 < 1$ in view of the fact that $1 - F_i\bar{x} > 0$ for every i. The extension to the case of e (respectively $x = e + \bar{x}$) in the interior of $\mathcal{P}_1$ (resp. $\mathcal{P}$) is straightforward due to the homogeneity of $\Psi_1(e)$. In fact for every x in the interior of $\mathcal{P}$ the error e can be written as $e = x - \bar{x} = \gamma(x_1 - \bar{x}) = \gamma e_1$, with $e_1 \in \partial\mathcal{P}_1$, for a proper scaling factor $\gamma < 1$. The one step contractivity requirement (12) can then be rewritten as

$$\max_i \frac{F_i}{1 - F_i\bar{x}}\,\big(\gamma e_1 + \tau(\gamma Ae_1 + Bv)\big) \le \lambda_1\gamma. \tag{14}$$

Setting $v = \gamma v_1$ in (14) and dividing both terms by γ we get (13).

Now, since $\mathcal{P}_1$ is β_1-contractive for (11), it is possible to associate to $\mathcal{P}_1$ a Lipschitz continuous state feedback control law $\phi(e) = \phi(x - \bar{x})$. Going back from (11) to the original system (1) it is readily seen that $\phi_1(x, \bar{x}) = \bar{u} + \phi(x - \bar{x})$ is the desired control law.

The lemma just presented allows us partially to solve Problem 3.1 as it just states that whenever the initial condition lies in the set $\mathcal{P}$ and the reference signal is a constant value belonging to the interior of $\mathcal{Y}_R$ we can provide a Lipschitz continuous state feedback control function which guarantees that the corresponding state evolution belongs to $\mathcal{P}$ and asymptotically converges to the given steady state value. This might appear as an expected consequence of the existence of a contractive region (w.r.t. the origin) for system (1). Nevertheless, as we will see next, this way of proceeding allows us to determine an explicit feedback control law. Before going on with the next theorem we need to recall a result which is a restricted version of what has been presented in [14] concerning the possibility of deriving explicit continuous state feedback control law for the class of systems under consideration. This is obtained by smoothing the polyhedral function $\Psi_{\mathcal{P}}(x)$ so as to get, for a given positive integer $q > 0$, the Gauge function

$$\Psi_q(x) = \left(\sum_{i=1}^{s} \sigma_{2q}(F_i x) \right)^{\frac{1}{2q}} \tag{15}$$

with

$$\sigma_r(x) = \begin{cases} x^r & \text{if} \quad x \geq 0 \\ 0 & \text{otherwise} \end{cases}.$$

Introducing the function gradient

$$\nabla \Psi_q(x) = \left[\frac{\partial \Psi_q(x)}{\partial x_1}, \ \dots, \ \frac{\partial \Psi_q(x)}{\partial x_n} \right] = \Psi_q(x)^{(1-2q)} G_q(x) F,$$

where

$$G_q(x) = \left[\sigma_{2q-1}(F_1 x) \dots \sigma_{2q-1}(F_s x) \right],$$

the following result holds:

Theorem 4.2 [14] *Let $\mathcal{P} = \{x \colon Fx \leq \bar{1}\}$ be a β-contractive polyhedral C-set for system* (1). *Then for every $0 < \beta_1 < \beta$ there exists a positive integer q such that the set $\mathcal{P}_q = \{x \colon \Psi_q(x) \leq 1\}$ is β_1-contractive for system* (1). *Moreover it is possible to associate to $\Psi_q(x)$ the explicit smooth*[1] *state feedback control law*

$$u = \Phi(x) = -\mu_0 \Psi_q(x)^{2(1-q)} B^T F^T G_q(x), \tag{16}$$

where μ_0 is a finitely computable nonnegative constant.

In Lemma 4.2 it has been shown that the polyhedral function (9) is a Lyapunov function for the error whenever the reference signal belongs to the interior of $\mathcal{Y}_R$, but nothing has been said about the effective determination of a stabilizing control law (in

[1] We mean smooth for every $x \neq 0$.

the sense that we have proved its existence though not furnishing any expression for it), due to the lack of differentiability of (9).

The next theorem will provide us with the requested expression for the controller. To this aim we first "smooth", similarly to what we have done in (15), the expression given by (9) and centered in $\bar{x}(\bar{y})$ by taking $q < \infty$ sufficiently large so as to get the function

$$\Psi_q^{\bar{y}}(x) = \left(\sum_{i=1}^{s} \sigma_{2q} \left(\frac{F_i(x - \bar{x})}{1 - F_i \bar{x}} \right) \right)^{\frac{1}{2q}}. \tag{17}$$

Simple algebra shows that the gradient $\nabla \Psi_q^{\bar{y}}(x)$ of (17) is:

$$\nabla \Psi_q^{\bar{y}}(x) = \left(\Psi_q^{\bar{y}}(x) \right)^{(1-2q)} G_q^{\bar{y}}(x) F_{\bar{y}},$$

where

$$G_q^{\bar{y}}(x) = \left[\sigma_{2q-1} \left(\frac{F_1(x - \bar{x})}{1 - F_1 \bar{x}} \right) \dots \sigma_{2q-1} \left(\frac{F_s(x - \bar{x})}{1 - F_s \bar{x}} \right) \right]$$

and

$$F_{\bar{y}} = \begin{bmatrix} \frac{F_1}{1-F_1\bar{x}} \\ \cdots \\ \frac{F_s}{1-F_s\bar{x}} \end{bmatrix}.$$

These expressions allow us to introduce the next theorem.

Theorem 4.3 *Let $\mathcal{P} = \{x \colon Fx \le \bar{1}\}$ be a β-contractive polyhedral C-set contained in $\mathcal{X}$ for system* (1). *Then for every reference signal $\bar{y} \in \alpha \mathcal{Y}_R$, $\alpha < 1$, there exists $0 < \beta_1 < \beta$ and an integer q such that the control law*

$$\Phi(x, \bar{y}) = \bar{u}(\bar{y}) - \rho_0 \Psi_q^{\bar{y}}(x)^{2(1-q)} B^T F_{\bar{y}}^T G_q^{\bar{y}}(x), \tag{18}$$

where ρ_0 is a finitely computable nonnegative constant, is such that for every initial condition $x(0) \in \mathcal{P}$ the output of the corresponding evolution $y(t)$ asymptotically converges to $\bar{y}$ with speed equal to β_1 while assuring that $x(t) \in \mathcal{X}$ for every $t \ge 0$.

Proof From Lemma 4.2 we have that $\mathcal{P}_1 = \{e \colon \Psi_1(e) \le 1\}$, where $\Psi_1(e) = \max_i \frac{F_i}{1-F_i\bar{x}} e$, is a β_1-contractive set for system (11). The proof hence follows immediately by first recalling Theorem 4.2, which assures the existence of an explicit control law of the form (16) (which will result in being a function of $e = x - \bar{x}$), and by subsequently going back to the original system to obtain (18).

5 Example

Consider the following two dimensional system

$$\dot{x}(t) = \begin{bmatrix} -0.3 & 1 \\ -1 & -0.3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} -5 \\ 5 \end{bmatrix},$$

$$y(t) = \begin{bmatrix} 2 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

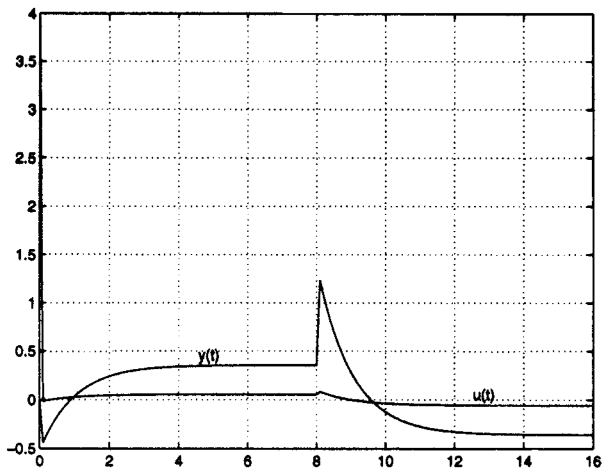

Figure 5.1. State space evolution.

with state constraints given by the set $\mathcal{X} = \{x\colon \|x\|_\infty \le 1\}$. A polyhedral 2-contractive set contained in $\mathcal{X}$ is $\mathcal{P} = \{x\colon \max_i F_i x \le 1\}$, where F is the following matrix

$$F = \begin{bmatrix} 0 & 1 \\ 0 & -1 \\ 1 & 0 \\ -1 & 0 \\ 1.391 & 1.540 \\ -1.391 & -1.540 \end{bmatrix}.$$

The resulting sets of admissible constant input and output values are $\mathcal{U}_R = [-0.073, 0.073]$ and $\mathcal{Y}_R = [-0.470, 0.470]$. We chose as a tracking value $\bar{y} = 0.358$ corresponding to $\alpha = 0.761$ and exploiting the results presented in Theorem 4.3 we determined the integer $q = 12$ such that the proposed control law (18) with $\rho_0 = 21.586$ guarantees asymptotic tracking of $\bar{y}$ for every $x_0 \in \mathcal{P}$ with speed of convergence $\beta_1 = 0.3$. Figure 5.1 depicts the state space evolution obtained starting from zero initial value and tracking value-equal to $\bar{y}$ for the first 8 seconds and $-\bar{y}$ for $t > 8$ together with different level surfaces of the Lyapunov functions $\Psi_{12}^{\bar{y}}$ and $\Psi_{12}^{-\bar{y}}$ (dotted) associated to the two tracking states $\bar{x}(\bar{y})$ and $-\bar{x}(\bar{y})$ which belong to the first and third quadrant and are indicated with a circled cross in the same figure.

Finally Figure 5.2 shows the evolution of the output as well as that of the control.

6 Conclusions

This work has dealt with perfect asymptotic tracking for state constrained dynamic systems. An alternative approach to the one proposed by Gilbert et al. [8], which is based on the concept of "maximal output admissible set" and recent results [10, 14], has been presented. This novel approach allows us to synthesize an explicit nonlinear state

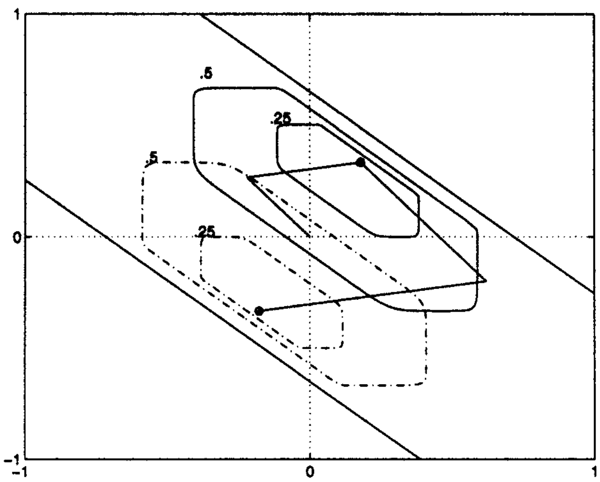

Figure 5.2. Control and output simulated plots.

feedback control law which guarantees perfect asymptotic tracking while maximizing the set of trackable signals which do not lead to state constraint violation.

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Optimal Design of Robust Control for Uncertain Systems: a Fuzzy Approach

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Abstract: The problem of designing controls for a linear dynamic system under input disturbance is considered. The input disturbance is bounded but the bound information is either deterministic or fuzzy. The control design is purely deterministic. However, the resulting system performance is interpreted differently, depending on the bound information. It may be deterministic or fuzzy (i.e. with a spectrum of outcome to various degrees). Finally, the optimal design problem of the control scheme, in which the cost is in quadratic form, is solved.

Keywords: *Uncertain systems; robust control; fuzzy approach.*

Mathematics Subject Classification (2000): 93C42, 92B12, 93B51, 93D05.

1 Introduction

Fuzzy theory was originally introduced to *describe information* (for example, the linguistic information) that is in lack of a sharp boundary with its environment (see [1]). However, it soon turned into the direction that mainly focuses on the use of *fuzzy reasoning* for control, estimation, decision making, etc. The application of fuzzy reasoning has enjoyed its advantage that it is model free. The designer's effort is mainly focused on tuning some parameters based on linguistic reasoning. It has been shown to be rather effective for a large amount of complex problems.

The current paper, on the other hand, proposes a rather different angle. It endeavors to explore applications of the original intention of fuzzy theory, namely, *information description*. In particular, we cast the framework within the context of control theory.

Granted that the probability theory is quite self-contained, criticism of its validity in describing the real world does exist. It is interesting to notice that Kalman [2], among others, despite his early devotion to the use of probability in mathematical system theory, is now critical on part of its foundation. Kalman contended that probability theory might not be all that suitable to describe the majority of randomness. In a sense, the link between a rather sophisticated mathematical tool and the physical world might be loose. We stress, however, that Kalman's recent comment on probability does not automatically assume him an advocate for fuzzy theory. His view on the latter has been unchanged (see [3] and [4]).

The *fuzzy* approach, as originally proposed by Zadeh [1] on the other hand, takes *the extent of occurrence* point of view. Historically, the merge between the probability theory and control/system theory, which can be traced back to the fifties, has been highly successful and received little criticism. In the state space framework, Kalman initiated the effort of looking into the estimation problem (see [5]) and control problem (see [6]) when a system is under stochastic noise. The effort has received tremendous attention. As it turns out, there is now a quite impressive arena on stochastic system and control theory (see, e.g., [7]) that can not be ignored by any practitioners.

In this work, we shall attempt to pursue a possible use of fuzzy description of uncertainty in robust control design. This may be viewed as an alternative proposal to combine the fuzzy theory and control theory. The objectives are two fold. First, we explore fuzzy descriptions of system performance should more information of the uncertainty (in the fuzzy sense) be provided. This adds more insight on the system performance. Furthermore, this also shows a way to view the system performance with human needs (which are often best described in a fuzzy sense). Second, we consider an optimal design of the robust control. The combined average system performance (over the fuzzy description) and control effort is to be extremized by an appropriate choice of a design parameter. This may be viewed as an analogous development to the LQG design in stochastic control.

2 Uncertain System and Robust Control

Consider the following uncertain system

$$\dot{x}(t) = Ax(t) + Bu(t) + Bv(x(t), t), \qquad x(t_0) = x_0, \tag{2.1}$$

where $t \in \mathbf{R}$ is the "time" (or more precisely, the independent variable), $x(t) \in \mathbf{R}^n$ is the state, $u(t) \in \mathbf{R}^m$ is the control, $v(x(t), t) \in \mathbf{R}^m$ is the (unknown) input disturbance, A, B are (known) constant matrices. The function $v(\cdot, t)$ is continuous. The function $v(x, \cdot)$ is Lebesgue measurable. The task is to choose the control u such that the state $x(t)$ of the controlled system of (2.1) enters a region around $x = 0$ after a finite time and remains there thereafter.

Assumption 2.1 The pair (A, B) is stabilizable.

Assumption 2.2 There is a known scalar $\overline{u} \geq 0$ such that

$$\max_{\substack{x \in \mathbf{R}^n \\ t \in \mathbf{R}}} \|v(x, t)\| \leq \overline{u}. \tag{2.2}$$

Choose constant $n \times n$ matrices $Q > 0$ and $R > 0$. Solve the following Riccati equation

$$A^T P + PA - PBR^{-1}B^T P + Q = 0 \tag{2.3}$$

for the solution $P > 0$, which is also an $n \times n$ matrix. Notice that the solution $P > 0$ exists and is unique if (A, B) is stabilizable. We propose the control u as follows:

$$u(t) = -\frac{1}{2} R^{-1} B^T P x(t) - \gamma B^T P x(t), \tag{2.4}$$

where $\gamma > 0$ is a scalar constant. The choice of γ will be made later.

Definition 2.1 Consider a dynamical system

$$\dot{\xi}(t) = f(\xi(t), t) \tag{2.5}$$

with $\xi(t_0) = \xi_0$. The solution of the system (suppose it exists) is *uniformly ultimately bounded* if for any $r > 0$ with $\|\xi_0\| \le r$, there are $\bar{d}(r) > 0$ and $\tilde{T}(\bar{d}(r), r) \ge 0$ such that

$$\|\xi(t)\| \le \bar{d}(r) \tag{2.6}$$

for all $t \ge t_0 + \tilde{T}(\bar{d}(r), r)$.

Theorem 2.1 [8] *Consider that the system (2.1) is subject to Assumptions 2.1 and 2.2. Suppose that the control (2.4) is applied. For each $\gamma > 0$, the resulting controlled system is uniformly ultimately bounded. Furthermore, the size of the ultimate boundedness region, i.e., $\bar{d}(r)$, can be made arbitrarily small by choosing a sufficiently large γ.*

There is a trade-off between the performance and the control effort. As a result, an optimal quest for the design may be interesting. It is also possible that, based on further understanding of the input disturbance, one is able to extract more information about its bound. We describe the information in the following.

Assumption 2.3 There is a scalar $\nu \ge 0$ such that

$$\max_{\substack{x \in \mathbf{R}^n \\ t \in \mathbf{R}}} \|v(x,t)\| \le \nu. \tag{2.7}$$

The membership value of ν in a region $U\colon = [\underline{u}, \overline{u}]$, $\overline{u} \ge \underline{u} \ge 0$, is prescribed by a fuzzy number N, whose membership function is $\mu_N\colon U \to [0,1]$.

The fuzzy description of the uncertainty bound, as shown in Assumption 2.3, enables us to pursue a fuzzy-based interpretation of the system performance. By Assumption 2.3, given that ν is in the fuzzy set N, the possibility that $\nu = u$, where $u \in [\underline{u}, \overline{u}]$, is given by $\mu_N(u)$.

For later purpose, we are also interested in the fuzzy number $N \cdot N$. This is discussed as follows. Let

$$v(x,t) = [\, v_1(x,t) \quad v_2(x,t) \quad \cdots \quad v_m(x,t) \,]^T. \tag{2.8}$$

It is possible that sometimes the designer only knows the fuzzy description of the bound of each component $v_i(x,t)$, $i = 1, 2, \cdots, m$. Suppose that $|v_i(x,t)| \le \nu_i$ for all x, t. The scalar ν_i belongs to a region $U_i\colon = [\underline{u}_i, \overline{u}_i]$, $\overline{u}_i \ge \underline{u}_i \ge 0$, which is the universe of discourse of a fuzzy number N_i. This fuzzy number is prescribed by a membership function $\mu_{N_i}\colon U_i \to [0,1]$.

With the membership function $\mu_{N_i}(\cdot)$ prescribed, one obtains its α-cuts $[\underline{u}_{i_\alpha}, \overline{u}_{i_\alpha}]$. The square of the α-cuts, that is, $[\underline{u}_{i_\alpha}, \overline{u}_{i_\alpha}] \cdot [\underline{u}_{i_\alpha}, \overline{u}_{i_\alpha}]$, is obtained (see [9]). The sum of all these α-cuts, i.e., $\sum_{i=1}^{m} [\underline{u}_{i_\alpha}, \overline{u}_{i_\alpha}] \cdot [\underline{u}_{i_\alpha}, \overline{u}_{i_\alpha}]$, also can be obtained for each α (see [9]). Finally, one may use decomposition theorem to reach the membership function for the fuzzy number $N \cdot N$.

If the designer already knows the membership function $\mu_N(\cdot)$, then it is easy to obtain the membership function of the fuzzy number $N \cdot N$. All it takes is to take the square of the α-cuts of $\mu_N(\cdot)$, summarize them, and then invoke the decomposition theorem. We now state the following fuzzy-based system performance.

Theorem 2.2 *Consider that the system (2.1) is subject to Assumptions 2.1 and 2.3. Suppose that the control (2.4) is applied. For any* $u \in [\underline{u}, \overline{u}]$ *and any* $r > 0$ *with* $\|x_0\| \le r$*, the possibility that*

$$\|x(t)\| \le \hat{d}(u) \quad \textit{for all} \quad t \ge t_0 + \tilde{T}$$

is given by $\mu_{N \cdot N}(u)$*, where*

$$\hat{d}(u) = \underline{\hat{d}}(u) + \epsilon, \tag{2.9}$$

$$\underline{\hat{d}}(u) := \sqrt{\frac{u^2}{2\gamma\lambda_m(Q)}}. \tag{2.10}$$

Proof By [8], for any $\nu = u$,

$$\dot{V} \le -\lambda_m(Q)\|x\|^2 + \frac{u^2}{2\gamma}. \tag{2.11}$$

This means that $\dot{V}$ is negative definite for all $\|x\|$ such that

$$\|x\| > \sqrt{\frac{u^2}{2\gamma\lambda_m(Q)}} =: \underline{\hat{d}}(u). \tag{2.12}$$

From Assumption 2.3, the possibility that $\nu = u$ is $\mu_N(u)$. Thus the possibility that $\dot{V}$ is negative for all $\|x\| > \underline{\hat{d}}(u)$ is $\mu_{N \cdot N}(u)$. By Theorem 2.1, for any $t \ge t_0 + \tilde{T}$, $\|x(t)\| \le \bar{d}$. Since $\bar{d} > \underline{\hat{d}}(u)$, this in turn shows that the possibility of $\|x(t)\| \le \hat{d}(u)$ is given by $\mu_{N \cdot N}(u)$.

Remark 2.1 The theorem asserts that, given the uniform ultimate boundedness result in Theorem 2.1, and the additional information provided by Assumption 2.3, one can further prescribe a possibility distribution that the state enters another region, which is in general of smaller size. This is a totally new aspect of the system performance, as compared with the previous work in robust control. The special way of incorporating fuzzy logic theory with control system analysis is believed to be the first time.

The input disturbance bound ν is often obtained via observed data and analyzed by the engineer. The observed data is, by nature, always limited. The source of the disturbance is unlikely to be exactly repeated. Hence any interpretation via the *frequency of occurrence*, as the number of repetitions approaches to infinity, suffers from a lack of basis. An alternative interpretation of the bound for circumstances like this would have to be fuzzy in its nature. For examples, one may need to adopt the fuzzy (linguistic) terms such as "*close to*" or "*very close to*" a (crisp) value.

The system performance is also often judged by the engineer in terms of the need of human being: One may choose a (crisp) set point and intend to have the performance to be "*close to*" or "*very close to*" it, after a finite time. These again fall into the fuzzy category. A typical example of this nature is the "comfort" control in Heating, Ventilating, and Air Conditioning (HVAC) (see, e.g., [10]). On top of this, the engineer also has the discretion to impose a hard bound (through, e.g., the prescription of the size of uniform ultimate boundedness region) on the performance, which must be met with absolutely no exceptions. All these can be addressed by the current framework.

3 Optimal Design of γ

The previous section shows a system performance which can be guaranteed by a deterministic control design. By the analysis, the size of the uniform ultimate boundedness region decreases as γ increases. As γ approaches to infinity, the size approaches to 0. This rather strong performance is accompanied by a (possibly) large control effort, which is reflected by γ. From the practical design point of view, the designer may be also interested in seeking an optimal choice of γ for a compromise among various conflicting criteria. This is associated with the minimization of a performance index.

We first explore more on the deterministic performance of the uncertain system. By the Rayleigh's principle,

$$\lambda_m(P)\|x\|^2 \le x^T P x = V \le \lambda_M(P)\|x\|^2 \tag{3.1}$$

and hence

$$-\|x\|^2 \le -\frac{1}{\lambda_M(P)}\, V. \tag{3.2}$$

With this into (2.11), we have

$$\dot V(t) \le -\frac{\lambda_m(Q)}{\lambda_M(P)}\, V(t) + \frac{\nu^2}{2\gamma}, \tag{3.3}$$

where $V_0 = V(t_0) = x_0^T P x_0$. This is a *differential inequality*. The following is needed for our analysis of (3.3).

Definition 3.1 [11] If $w(\psi, t)$ is a scalar function of the scalars ψ, t in some open connected set $\mathcal{D}$, we say a function $\psi(t)$, $t_0 \le t \le \bar t$, $\bar t > t_0$ is a *solution of the differential inequality*

$$\dot\psi(t) \le w(\psi(t), t) \tag{3.4}$$

on $[t_0, \bar t)$ if $\psi(t)$ is continuous on $[t_0, \bar t)$ and its derivative on $[t_0, \bar t)$ satisfies (3.4).

Theorem 3.1 [11] *Let $w(\phi, t)$ be continuous on an open connected set $\mathcal{D} \in \mathbf{R}^2$ and such that the initial value problem for the scalar equation*

$$\dot\phi(t) = w(\phi(t), t), \qquad \phi(t_0) = \phi_0 \tag{3.5}$$

has a unique solution. If $\phi(t)$ is a solution of (3.5) on $t_0 \le t \le \bar t$ and $\psi(t)$ is a solution of (3.4) on $t_0 \le t < \bar t$ with $\psi(t_0) \le \phi(t_0)$, then $\psi(t) \le \phi(t)$ for $t_0 \le t \le \bar t$.

Instead of exploring the solution of the differential inequality, which is often non-unique and not available, the theorem suggests that it may be feasible to study the upper bound of the solution. The reasoning is, however, based on that the solution of (3.5) is unique.

Theorem 3.2 [12] *Consider the differential inequality (3.4) and the differential equation (3.5). Suppose that for some constant $L > 0$, the function $w(\cdot)$ satisfies the Lipschitz condition*

$$|w(v_1, t) - w(v_2, t)| \le L|v_1 - v_2| \tag{3.6}$$

for all points (v_1, t), $(v_2, t) \in \mathcal{D}$. Then any function $\psi(t)$ that satisfies the differential inequality (3.4) for $t_0 \le t < \bar{t}$ satisfies also the inequality

$$\psi(t) \le \phi(t) \tag{3.7}$$

for $t_0 \le t \le \bar{t}$.

We consider the differential equation

$$\dot{r}(t) = -\frac{\lambda_m(Q)}{\lambda_M(P)}\, r(t) + \frac{\nu^2}{2\gamma}, \qquad r(t_0) = V_0. \tag{3.8}$$

The right-hand side satisfies the global Lipschitz condition with

$$L = \frac{\lambda_m(Q)}{\lambda_M(P)}. \tag{3.9}$$

We proceed with solving the differential equation (3.8). This results in

$$r(t) = \left(V_0 - \frac{\lambda_M(P)}{\lambda_m(Q)}\, \frac{\nu^2}{2\gamma}\right) \exp\left[-\frac{\lambda_m(Q)}{\lambda_M(P)}\,(t - t_0)\right] + \frac{\lambda_M(P)}{\lambda_m(Q)}\, \frac{\nu^2}{2\gamma}. \tag{3.10}$$

Therefore

$$V(t) \le r(t) \tag{3.11}$$

or

$$V(t) \le \left(V_0 - \frac{\lambda_M(P)}{\lambda_m(Q)}\, \frac{\nu^2}{2\gamma}\right) \exp\left[-\frac{\lambda_m(Q)}{\lambda_M(P)}\,(t - t_0)\right] + \frac{\lambda_M(P)}{\lambda_m(Q)}\, \frac{\nu^2}{2\gamma} \tag{3.12}$$

for all $t \ge t_0$. By the same argument, we also have, for any t_s and any $\tau \ge t_s$,

$$V(\tau) \le \left(V_s - \frac{\lambda_M(P)}{\lambda_m(Q)}\, \frac{\nu^2}{2\gamma}\right) \exp\left[-\frac{\lambda_m(Q)}{\lambda_M(P)}\,(\tau - t_s)\right] + \frac{\lambda_M(P)}{\lambda_m(Q)}\, \frac{\nu^2}{2\gamma}, \tag{3.13}$$

where $V_s = V(t_s) = x^T(t_s)Px(t_s)$. The time t_s is when the control scheme (2.4) starts to be executed. It does not need to be t_0.

By the Rayleigh's principle $V(\tau) \ge \lambda_m(P)\|x(\tau)\|^2$, the right-hand side of (3.13) provides an upper bound of $\lambda_m(P)\|x(\tau)\|^2$. This in turn leads to an upper bound of $\|x(\tau)\|^2$. For each $\tau \ge t_s$, let

$$\eta(\nu, \gamma, \tau, t_s) := \left(V_s - \frac{\lambda_M(P)}{\lambda_m(Q)}\, \frac{\nu^2}{2\gamma}\right) \exp\left[-\frac{\lambda_m(Q)}{\lambda_M(P)}\,(\tau - t_s)\right], \tag{3.14}$$

$$\eta_\infty(\nu, \gamma) := \frac{\lambda_M(P)}{\lambda_m(Q)}\, \frac{\nu^2}{2\gamma}. \tag{3.15}$$

Notice that for each ν, γ, t_s, $\eta(\nu, \gamma, \tau, t_s) \to 0$ as $\tau \to \infty$.

One may relate $\eta(\nu, \gamma, \tau, t_s)$ to the transient portion and $\eta_\infty(\nu, \gamma)$ the steady state portion of the system performance. Since there is no knowledge of the input disturbance $v(x, t)$ except its possible bound, it is only realistic to refer to $\eta(\nu, \gamma, \tau, t_s)$ and $\eta_\infty(\nu, \gamma)$ while analyzing the system performance. We also notice that both $\eta(\nu, \gamma, \tau, t_s)$ and $\eta_\infty(\nu, \gamma)$ are dependent on ν. The value of ν is not known except that it lies within a set U (i.e., the universe of discourse) to the degree that is defined by $\mu_N(\cdot)$.

Definition 3.2 For any function $f\colon [\underline{u},\overline{u}] \to \mathbf{R}$, the D-operation $D[f(\nu)]$ is defined as follows:

$$D[f(\nu)] = \frac{\int_{\underline{u}}^{\overline{u}} f(\nu)\mu_N(\nu)d\nu}{\int_{\underline{u}}^{\overline{u}} \mu_N(\nu)d\nu}. \tag{3.16}$$

Remark 3.1 In a sense, the D-operation $D[f(\nu)]$ takes an average value of $f(\nu)$ over $\mu_N(\nu)$. In the special case that $f(\nu) = \nu$, this is reduced to the well-known center-of-gravity defuzzification method (see, e.g., [13]). If N is crisp (i.e., $\mu_N(\nu) = 1$ for all ν), then $D[f(\nu)] = f(\nu)$. This is reduced to the classical case.

Lemma 3.1 *For any crisp constant $a \in \mathbf{R}$,*

$$D[af(\nu)] = aD[f(\nu)]. \tag{3.17}$$

We now propose the following performance index: For any t_s, let

$$\begin{aligned} J(\gamma, t_s) &:= D\Big[\int_{t_s}^{\infty} \eta^2(\nu,\gamma,\tau,t_s)\,d\tau\Big] + \alpha D[\eta_\infty^2(\nu,\gamma)] + \beta\gamma^2 \\ &=: J_1(\gamma,t_s) + J_2(\gamma) + J_3(\gamma), \end{aligned} \tag{3.18}$$

$\alpha, \beta > 0$. The performance index consists of three parts. The first part $J_1(\gamma,t_s)$ may be interpreted as the average (via the D-operation) of the overall transient performance (via the integration) from time t_s. The second part $J_2(\gamma)$ may be interpreted as the average (via the D-operation) of the steady state performance. The third part $J_3(\gamma)$ is due to the control cost. Both α and β are weighting factors. The weighting of J_1 is normalized to be unity.

Remark 3.2 A standard LQG (i.e., linear-quadratic-Gaussian) problem in stochastic control is to minimize a performance index which is the average (via the expectation value operation in probability) of the overall state and control accumulation. The current optimal design of γ may be viewed as a parallel problem, though not equivalent, in the fuzzy setting. However, one can not be too careful in distinguishing the difference. For example, the Gaussian probability distribution implies that the uncertainty is unbounded (although a higher bound is predicted by a lower probability). In the current consideration, the uncertainty bound is always finite.

Let $\kappa := \lambda_M(P)/\lambda_m(Q)$. One can show that

$$\begin{aligned} \int_{t_s}^{\infty} \eta^2(\nu,\gamma,\tau,t_s)\,d\tau &= \Big(V_s - \frac{\lambda_M(P)}{\lambda_m(Q)}\,\frac{\nu^2}{2\gamma}\Big)^2 \int_{t_s}^{\infty} \exp\Big[-2\frac{\lambda_m(Q)}{\lambda_M(P)}(\tau - t_s)\Big]\,d\tau \\ &= \Big(V_s - \kappa\frac{\nu^2}{2\gamma}\Big)^2\,\frac{\kappa}{2}. \end{aligned} \tag{3.19}$$

Taking the D-operation,

$$\begin{aligned} D\Big[\int_{t_s}^{\infty} \eta^2(\nu,\gamma,t,t_s)\,dt\Big] &= D\bigg[\Big(V_s - \kappa\frac{\nu^2}{2\gamma}\Big)^2\frac{\kappa}{2}\bigg] \\ &= \Big(V_s - \frac{V_s\kappa}{\gamma}\,D[\nu^2] + \frac{\kappa^2}{4\gamma^2}\,D[\nu^4]\Big)\frac{\kappa}{2}. \end{aligned} \tag{3.20}$$

Next, we analyze the cost $J_2(\gamma)$:

$$D[\eta_\infty^2(\nu,\gamma)] = D\bigg[\Big(\frac{\lambda_M(P)}{\lambda_m(Q)}\Big)^2\Big(\frac{\nu^2}{2\gamma}\Big)^2\bigg] = \frac{\kappa^2}{4\gamma^2}\,D[\nu^4]. \tag{3.21}$$

With (3.20) and (3.21) into (3.18),

$$\begin{aligned} J(\gamma,t_s) &= \Big(V_s - \frac{V_s\kappa}{\gamma}\,D[\nu^2] + \frac{\kappa^2}{4\gamma^2}\,D[\nu^4]\Big)\frac{\kappa}{2} + \alpha\frac{\kappa^2}{4\gamma^2}\,D[\nu^4] + \beta\gamma^2 \\ &=: \kappa_1 - \frac{\kappa_2}{\gamma} + \frac{\kappa_3}{\gamma^2} + \alpha\frac{\kappa_4}{\gamma^2} + \beta\gamma^2, \end{aligned} \tag{3.22}$$

where $\kappa_1 := \frac{\kappa}{\gamma}V_s$, $\kappa_2 := \frac{\kappa^2}{2}V_s D[\nu^2]$, $\kappa_3 := \frac{\kappa^4}{4}D[\nu^4]$, $\kappa_4 := \frac{\kappa^2}{4}D[\nu^4]$.

The optimal design problem is then the following constrained optimization problem: For any t_s,

$$\min_{\gamma} J(\gamma,t_s) \quad \text{subject to} \quad \gamma > 0. \tag{3.23}$$

For any t_s, taking the first order derivative of J with respect to γ:

$$\frac{\partial J}{\partial\gamma} = \frac{\kappa_2}{\gamma^2} - 2\,\frac{\kappa_3}{\gamma^3} - 2\alpha\,\frac{\kappa_4}{\gamma^3} + 2\beta\gamma = \frac{1}{\gamma^3}\left(\kappa_2\gamma - 2\kappa_3 - 2\alpha\kappa_4 + 2\beta\gamma^4\right). \tag{3.24}$$

That

$$\frac{\partial J}{\partial\gamma} = 0 \tag{3.25}$$

leads to

$$\kappa_2\gamma - 2\kappa_3 - 2\alpha\kappa_4 + 2\beta\gamma^4 = 0 \tag{3.26}$$

or

$$\kappa_2\gamma + 2\beta\gamma^4 = 2(\kappa_3 + \alpha\kappa_4). \tag{3.27}$$

Equation (3.27) is a scalar quartic equation. For simplicity, in the rest of discussion, we shall rule out the trivial possibility of $\underline{u} = \overline{u} = 0$, which results in $D[\nu^2] = 0$ and $D[\nu^4] = 0$. In other words, we only consider $D[\nu^2] > 0$ and $D[\nu^4] > 0$ and hence $\kappa_3 > 0$ and $\kappa_4 > 0$ (notice that $\kappa > 0$). This in turn means that the solutions (there are two) γ to (3.27) are not identical to zero.

To observe the constraint $\gamma > 0$, we now restrict ourselves to only the positive solution of (3.27). For the $\gamma > 0$ that solves (3.27),

$$\begin{aligned} \frac{\partial^2 J}{\partial\gamma^2} &= -\frac{3}{\gamma^4}\left(\kappa_2\gamma^2\kappa_3 - 2\alpha\kappa_4 + 2\beta\gamma^4\right) + \frac{1}{\gamma^3}\left(\kappa_2 + 8\beta\gamma^3\right) \\ &= \frac{1}{\gamma^3}\left(\kappa_2 + 8\beta\gamma^3\right) > 0. \end{aligned} \tag{3.28}$$

The positive solution of the scalar quartic equation (3.27), which depends on V_s, solves the constrained minimization problem (3.23). By the continuity of the left-hand side of (3.27) on γ, the solution $\gamma > 0$ to (3.27) always exists. In addition, since the left-hand side of (3.27) is strictly increasing in γ, the solution $\gamma > 0$ to (3.27) is unique. We summarize the main result as follows.

Theorem 3.3 *Consider that the system (2.1) is subject to Assumptions 2.1 and 2.3. Suppose that the control (2.4) is applied. For given V_s, the unique solution $\gamma > 0$ of (3.27) minimizes the performance index (3.18).*

The solutions of the quartic equation (3.27) depend on the *cubic resolvent* (see [14])

$$z^3 + (-4r)z - q^2 = 0, \tag{3.29}$$

where

$$r = -\frac{1}{\beta}\big(\kappa_3 + \alpha\kappa_4\big), \qquad q = \frac{\kappa_2}{2\beta}.$$

Let $p_1 := -4r,\ p_2 := -q^2$. The discriminant D of the cubic resolvent is given by

$$D = \left(\frac{p_1}{3}\right)^3 + \left(\frac{p_2}{2}\right)^2. \tag{3.30}$$

Since $r < 0,\ D > 0$. The solutions of the cubic resolvent are given by

$$z_1 = u + v, \tag{3.31}$$

$$z_2 = -\frac{(u+v)}{2} + (u-v)i\sqrt{\frac{3}{2}}, \tag{3.32}$$

$$z_3 = -\frac{(u+v)}{2} - (u-v)i\sqrt{\frac{3}{2}}, \tag{3.33}$$

where

$$u = \left(-\frac{p_2}{2} + \sqrt{D}\right)^{\frac{1}{3}}, \tag{3.34}$$

$$v = \left(-\frac{p_2}{2} - \sqrt{D}\right)^{\frac{1}{3}}. \tag{3.35}$$

The cubic resolvent possesses one real solution and two complex conjugate solutions. This in turn implies that the quartic solution has two real solutions and one pair of complex conjugate solutions. The maximum real solution, which is positive, of the quartic equation is given by

$$\gamma = \frac{1}{2}\big(\sqrt{z_1} + \sqrt{z_2} + \sqrt{z_3}\big). \tag{3.36}$$

With z_1, z_2, and z_3 into (3.36), a lengthy but straightforward algebra shows that the positive solution of the quartic equation is given by

$$\gamma = \frac{1}{2}\left(\sqrt{u+v} + \sqrt{7u^2 + 7v^2 - 10uv}\,\cos\frac{\theta}{2}\right), \tag{3.37}$$

where

$$\theta = \tan^{-1} \frac{\sqrt{\frac{3}{2}}(u-v)}{-\frac{1}{2}(u+v)}. \tag{3.38}$$

Remark 3.3 The calculation of γ in (3.37) requires V_s which depends on $x(t_s)$. In implementations, this can be obtained via on-line feedback of the state. Notice that t_s is the starting time of the execution of the control. It does not need to be identical to the initial time t_0. The control starts to activate as soon as it receives the feedback signal $x(t_s)$. The control scheme, which minimizes the performance index (3.18), also only depends on t_s, not t_0. Certainly, the controlled system with $x(t_s)$ the initial state is uniformly ultimately bounded.

By using (3.27), the cost J in (3.22) can be rewritten as

$$\begin{aligned} J &= \kappa_1 - \frac{\kappa_2}{\gamma} + \frac{\kappa_3}{\gamma^2} + \alpha\frac{\kappa_4}{\gamma^2} + \beta\gamma^2 \\ &= \kappa_1 - \frac{1}{\gamma^2}\left(\kappa_2\gamma + 2\beta\gamma^4\right) + \kappa_3\gamma^2 + \alpha\,\frac{\kappa_4}{\gamma^2} + 3\beta\gamma^2 \\ &= \kappa_1 - \frac{1}{\gamma^2}\left(\kappa_3 + \alpha\kappa_4 + 3\beta\gamma^4\right). \end{aligned} \tag{3.39}$$

With (3.37), the minimum cost is given by

$$J_{\min} = \kappa_1 - \frac{4}{(\sqrt{z_1}+\sqrt{z_2}+\sqrt{z_3})^2}\left(\kappa_3 + \alpha\kappa_4 + \frac{3}{8}\,\beta\big(\sqrt{z_1}+\sqrt{z_2}+\sqrt{z_3}\big)^4\right). \tag{3.40}$$

Remark 3.4 Combining the previous results, the robust control scheme (2.4) using the optimal design of $\gamma > 0$ renders the closed-loop system uniformly ultimately bounded (with the initial state $x(t_s)$). In addition, there is a possibility distribution associated with the size of the region that the state will enter.

4 Conclusions

The incorporation of uncertainty, which is described in a fuzzy sense, into a robust control framework is introduced. This is believed to the first attempt for such a merge. As to the prescription of the desirable performance, it is often the designer's discretion. Since in practice it is in fact more realistic to prescribe the performance in a fuzzy sense (such as "close to", "very close to"), the current framework fits in well with both the need (the performance) and the given (uncertainty).

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Stability and Hopf Bifurcation in Differential Equations with One Delay

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Abstract: A class of parameter dependent differential equations with one delay is considered. A decomposition of the parameter space into domains where the corresponding characteristic equation has a constant number of zeros with positive real part is provided. The local stability analysis of the zero solution and the computation of all Hopf bifurcation points with respect to the delay is given.

Keywords: *Nonlinear delay differential equations; zeros of quasi-polynomials; local stability; Hopf bifurcation*

Mathematics Subject Classification (2000): 34K18, 34K20.

1 Introduction

Local stability and bifurcation analysis of systems of nonlinear differential equations with one time delay of the following type

$$\dot{x}(t) = Ax(t) + Bx(t-\tau) + F(x(t), x(t-\tau)), \tag{1.1}$$

where $\tau \geq 0;\ A,\, B \in \mathbb{R}^{n,n},\ F \in C^k(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n),\ k \geq 1,\ F(0,0) = DF(0,0) = 0$, often leads to the consideration of quasi-polynomials $\Phi_{\tau,\lambda}\colon \mathbb{C} \to \mathbb{C};\ \tau \geq 0,\ \lambda \in \mathbb{C}$, given by

$$\Phi_{\tau,\lambda}(s) := (s+1)\, \exp(\tau s) - \lambda. \tag{1.2}$$

In this context it is of particular relevance to know how the zeros of $\Phi_{\tau,\lambda}$ are distributed in the complex plane, whether they lie in the left or right half plane, and finally, how they depend on the parameters τ and λ.

The objective of this work is to divide the τ-halfline and the λ-plane into domains where $\Phi_{\tau,\lambda}$ has a constant number of zeros with positive real part and to investigate the local stability of the zero solution and the Hopf bifurcation points of systems given by (1.1) with appropriate matrices A and B.

Systems of type (1.1) occur in several fields of science. For example, they model electro-optical circuits which display bistability and chaotic behavior (see [12, 17]), they describe dynamical processes in neural networks (see [1, 22]), they model protein synthesis (see [2]) and they arise in the study of white blood-cell production (see [21]). Interested readers may find further applications, for example, in [15, pp.1–8]; [13, pp.72–81], [18, pp.1–34], [19, pp.1–17].

The problem to estimate the zeros of (1.2) with positive real part, the stability analysis of equilibria and the computation of Hopf bifurcation points of (1.1) has attracted the interest of several authors. For instance, Hayes [16] discusses quasi-polynomial equations equivalent to $\Phi_{\tau,\lambda}(s) = 0$ with $\tau > 0$ and $\lambda \in \mathbb{R}$ (see also [5, pp.444–446], [6]). El'sgolts and Norkin [11, pp.134–136] give a partition of the (A, B)-plane consisting of regions where the corresponding characteristic quasi-polynomials of the linear approximation of (1.1) with $n = 1$ and $A, B \in \mathbb{R}$ has a constant number of zeros with positive real part (see also [9, pp.305–309], [19, pp.56, 57]). Braddock and Van den Driessche [7] estimate the domains in λ-plane, where corresponding quasi-polynomials of the form $\Phi(s) = (s + \mu)\exp(\tau s) - \lambda$ have no zeros with positive real part and discuss the local stability of the trivial solution $x(t) = 0$ of (1.1). Bélair [4] also investigates the local stability of the trivial solution of (1.1) with $A = -I_n$, and proves the existence of a Hopf bifurcation point in the one dimensional case $n = 1$ with $B < 0$. Godoy and dos Reis [14] explore (1.1) with $n = 2$, $A = -I_2$ and B having eigenvalues in $\mathbb{C} \setminus \{\mathbb{R} \cup i\mathbb{R}\}$, and provide a partition of the τ-halfline $(\tau \geq 0)$ in segments where the corresponding characteristic quasi-polynomials of the linear approximation of (1.1) have a constant number of zeros with positive real part (for the case that B has eigenvalues in $\mathbb{C} \setminus \mathbb{R}$, see [3]).

In this work we extend the results above in the following way. For given $\tau \geq 0$ $(\lambda \in \mathbb{C})$ we divide the λ-plane (τ-halfline) into regions (intervals) with constant number of zeros with positive real part of the corresponding quasi-polynomials $\Phi_{\tau,\lambda}(s)$ (Section 2). We investigate the local τ-dependent stability of the zero solution of (1.1) for a large class of matrices A and B (Section 3), and we compute all Hopf bifurcation points of (1.1) with τ as bifurcation parameter (Section 4).

2 Zeros of $\Phi_{\tau,\lambda}$ with Positive Real Part

Consider the quasi-polynomial equation

$$\Phi_{\tau,\lambda}(s) = (s + 1)\exp(\tau s) - \lambda = 0 \tag{2.1}$$

for given $\tau > 0$ and $\lambda \in \mathbb{C}$. The primary objective of this section is to divide the λ-plane into regions by a planar curve with following properties. Points λ lying on the curve represent quasi-polynomials $\Phi_{\tau,\lambda}$ having at least one pure imaginary root, and points in each region correspond to quasi-polynomials with the same number of zeros having positive real part, counted by their multiplicity. This method is well known as D-decomposition (just as D-subdivision or D-partition) (see [11, pp.132–138], [19, pp.55–60]). Then, as consequence of the D-decomposition of the λ-plane, we get a D-decomposition of the τ-halfline.

Let us first state a few elementary results on the roots of (2.1).

Lemma 2.1

a) *$s \in \mathbb{C}$ is a zero of $\Phi_{\tau,\lambda}$ if and only if $\bar{s}$ is a zero of $\Phi_{\tau,\bar{\lambda}}$.*

b) *For $|\lambda| \le 1$ equation (2.1) has no solution with positive real part.*
For $|\lambda| > 1$ equation (2.1) has a finite number of solutions with positive real part. Furthermore, if such solutions exist, they belong to the open and bounded set

$$S_\lambda := \Big\{ s \in \mathbb{C} \mid 0 < \operatorname{Re} s < |\lambda| - 1 \quad and \quad |\operatorname{Im} s| < \sqrt{|\lambda|^2 - 1} \Big\}. \tag{2.2}$$

c) *Any root s of (2.1) with $\tau s \ne -(1+\tau)$ is simple.*

Proof a) Part a) is evident.

b) For all $s \in \mathbb{C}$ with $|s+1| \ge |\lambda|$ and $\operatorname{Re} s > 0$ it holds

$$|s+1| > |\lambda \exp(-\tau s)|. \tag{2.3}$$

This implies that equation (2.1) has no roots with $|s+1| \ge |\lambda|$ and $\operatorname{Re} s > 0$. So all roots of (2.1) with positive real part have to satisfy $|s+1| < |\lambda|$. We set $S_\lambda := \{s \in \mathbb{C} \colon |s+1| < |\lambda|,\ \operatorname{Re} s > 0\}$. Because S_λ is a bounded and connected subset of $\mathbb{C}$, the analytic function $\Phi_{\tau,\lambda}$ has only a finite number of zeros s with $\operatorname{Re} s > 0$ (see [8, p.78]). For $|\lambda| \le 1$ the set S_λ is empty and consequently (2.1) has no roots with positive real part.

For $|\lambda| > 1$ it follows

$$S_\lambda = \Big\{ s \in \mathbb{C} \mid 0 < \operatorname{Re} s < |\lambda| - 1 \quad \text{and} \quad |\operatorname{Im} s| < \sqrt{|\lambda|^2 - 1} \Big\}.$$

c) For $\tau = 0$ the only root $s = \lambda - 1$ is simple. If $\tau > 0$ the assertion follows from

$$\frac{d}{ds}\,\Phi_{\tau,\lambda}(s) = [\tau(s+1) + 1]\,\exp(\tau s) \ne 0 \tag{2.4}$$

for any $s \in \mathbb{C} \setminus \Big\{ -\frac{1+\tau}{\tau} \Big\}$.

2.1 D-decomposition of the $\boldsymbol{\lambda}$-plane

Let us now consider the planar curve mentioned above. Equation (2.1) has a pure imaginary root $s = i\omega$ if and only if

$$\lambda = (i\omega + 1)\,\exp(i\omega\tau) =: K_\tau(\omega). \tag{2.5}$$

In the following we summarize a few useful properties of the function K_τ (see Figure 2.1).

Lemma 2.2 *For $\tau > 0$ let $K_\tau \colon \mathbb{R} \to \mathbb{C}$ be the function defined by (2.5). Then:*

a) *K_τ describes a spiral in $\mathbb{C}$ with decreasing radius and argument for $\omega \in (-\infty, 0]$ and increasing radius and argument for $\omega \in [0, \infty)$. Moreover the curve described by K_τ is convex and lies symmetrically to the $\operatorname{Re}\lambda$-axis, i.e. $K_\tau(\omega) = \lambda \Leftrightarrow K_\tau(-\omega) = \bar{\lambda}$.*

b) *For $\omega, \tilde{\omega} \in \mathbb{R}$ and $\omega \ne \tilde{\omega}$ with $\lambda = K_\tau(\omega) = K_\tau(\tilde{\omega})$ it follows that $\omega = -\tilde{\omega}$ and $\lambda \in \mathbb{R}$.*

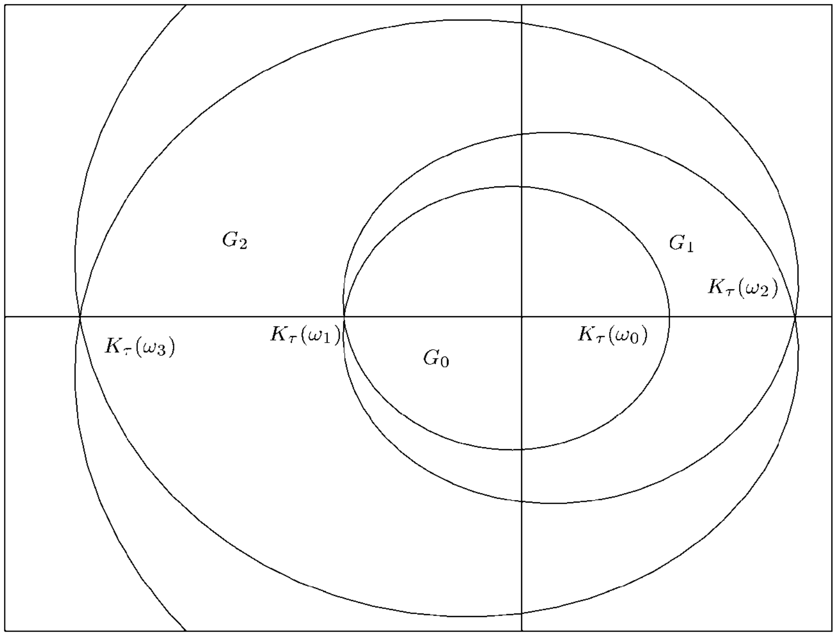

Figure 2.1. D-decomposition of the λ-plane.

Proof Part a) follows from (2.5), see also (2.6), (2.9), (2.10) below.

Now suppose that there exist $\omega, \tilde{\omega} \in \mathbb{R}$, $\omega \neq \tilde{\omega}$, with $K_\tau(\omega) = K_\tau(\tilde{\omega})$. Equation (2.5) yields $|K_\tau(\omega)|^2 = 1+\omega^2 = 1+\tilde{\omega}^2 = |K_\tau(\tilde{\omega})|^2$ and so $\omega = -\tilde{\omega}$. With a) we obtain $\lambda = K_\tau(\omega) = K_\tau(-\omega) = \bar{\lambda} \in \mathbb{R}$ and the proof is complete.

Every $\lambda \in \mathbb{C}$ can be written in polar coordinates, namely

$$\lambda = \rho e^{i\theta}, \tag{2.6}$$

where $\rho \geq 0$ is the radius and θ the argument of λ. Inserting (2.6) into (2.5) yields

$$(1+i\omega) = \rho e^{i(\theta-\omega\tau)}. \tag{2.7}$$

From (2.7) we obtain following conditions for θ and ρ

$$\omega\tau - \theta \in \left(2k\pi - \frac{\pi}{2}, 2k\pi + \frac{\pi}{2}\right), \quad k \in \mathbb{Z}, \tag{2.8}$$

$$\sqrt{1+\omega^2} = \rho = |\lambda|, \tag{2.9}$$

$$\omega = \tan(\theta - \omega\tau). \tag{2.10}$$

The next lemma deals with solutions of (2.10). We first set

$$I_k(\tau,\theta) := \left(\frac{1}{\tau}\left(2k\pi + \theta - \frac{\pi}{2}\right), \frac{1}{\tau}\left(2k\pi + \theta + \frac{\pi}{2}\right)\right), \quad k \in \mathbb{Z}, \ \tau > 0. \tag{2.11}$$

Lemma 2.3 *For any given $\tau > 0$, $\theta \in [0, 2\pi)$ and $k \in \mathbb{Z}$, equation (2.10) has a unique solution $\omega_k(\tau,\theta) \in I_k(\tau,\theta)$ with the following properties:*

$$\omega_k(\tau,\theta) \in \left(\frac{1}{\tau}\left(2k\pi + \theta - \frac{\pi}{2}\right), \frac{1}{\tau}(2k\pi+\theta)\right) \quad \textit{for} \quad k > 0, \tag{2.12}$$

$$\omega_0(\tau,\theta) \in \left(0, \frac{\theta}{\tau}\right) \quad \textit{for} \quad \theta \neq 0 \quad \textit{and} \quad \omega_0(\tau,0) = 0, \tag{2.13}$$

$$\omega_k(\tau,\theta) \in \left(\frac{1}{\tau}(2k\pi+\theta), \frac{1}{\tau}\left(2k\pi + \theta + \frac{\pi}{2}\right)\right) \quad \textit{for} \quad k < 0. \tag{2.14}$$

Proof $\tan(\theta - \omega\tau)$ is a decreasing function of $\omega \in I_k(\tau,\theta)$ with $\tan(\theta-\omega\tau) > 0$ for $\omega \in \left(\frac{1}{\tau}\left(2k\pi+\theta-\frac{\pi}{2}\right), \frac{1}{\tau}(2k\pi+\theta)\right)$, $\tan(\theta-\omega\tau) = 0$ for $\omega \in \frac{1}{\tau}(2k\pi+\theta)$ and $\tan(\theta-\omega\tau) < 0$ for $\omega \in \left(\frac{1}{\tau}(2k\pi+\theta), \frac{1}{\tau}\left(2k\pi+\theta+\frac{\pi}{2}\right)\right)$. This yields the assertions of the lemma.

For the construction of the regions with constant number of zeros of $\Phi_{\tau,\lambda}$ having positive real part, we need the intersection points of the curve K_τ with the $\operatorname{Re}\lambda$-axis. These intersection points are given by (2.10) with $\theta = 0$, if $k = 2l$ and $\theta = \pi$, if $k = 2l+1$, $l \in \mathbb{N}_0$. Because of symmetry properties of $K_\tau(\omega)$, see Lemma 2.2, we only consider the case $\omega \geq 0$. From Lemma 2.3 we obtain:

Lemma 2.4 *For $\tau > 0$ there is an increasing sequence of real numbers $0 = \omega_0^R < \omega_1^R < \dots$, where $\omega_k^R \in I_l(\tau,\theta)$ with $\theta = 0$ if $k = 2l$ and $\theta = \pi$ if $k = 2l+1$, $l \in \mathbb{N}_0$, such that*

a) $K_\tau(\omega_k^R) \in \mathbb{R}$ *and, if* $\omega \neq \omega_k^R$, $K_\tau(\omega) \notin \mathbb{R}$ *for any* $k \in \mathbb{N}_0$,
b) $(K_\tau(\omega_{2l}^R))_{l\in\mathbb{N}_0}$ *is an unbounded strictly increasing sequence with* $K_\tau(\omega_0^R) = 1$,
c) $\left(K_\tau(\omega_{2l+1}^R)\right)_{l\in\mathbb{N}_0}$ *is an unbounded strictly decreasing sequence with* $K_\tau(\omega_1^R) < -1$.

Using the sequence $(\omega_k^R)_{k\in\mathbb{N}_0}$ we now define segments of the curve described by K_τ lying in the upper and lower half of the λ-plane:

$$C_{\tau,k}^{\pm} := \{\lambda \in \mathbb{C}\colon \lambda = \operatorname{Re} K_\tau(\omega) \pm i|\operatorname{Im} K_\tau(\omega)|,\ \omega \in [\omega_k^R, \omega_{k+1}^R]\} \tag{2.15}$$

and $G_{\tau,k}$ as the region bounded by $C_{\tau,k}^+$ and $C_{\tau,k}^-$:

$$G_{\tau,k} := \{\mu \in \mathbb{C}\colon \operatorname{Re}\mu = \operatorname{Re}\lambda,\ -\operatorname{Im}\lambda < \operatorname{Im}\mu < \operatorname{Im}\lambda,\ \lambda \in C_{\tau,k}^+\} \tag{2.16}$$

for given $k \in \mathbb{N}_0$ and $\tau > 0$. Further we set

$$G_{\tau,-1} := \emptyset. \tag{2.17}$$

We summarize a few useful properties of the regions $G_{\tau,k}$ (see Figure 2.1) in the following.

Lemma 2.5 *Assume $\tau > 0$. For any $k \in \mathbb{N}_0$ the regions $G_{\tau,k}$ are bounded, connected and open subsets of the λ-plane, symmetric to the $\operatorname{Re}\lambda$-axis, satisfying*

a) $0 \in G_{\tau,k} \subset G_{\tau,k+1}$,
b) $G_{\tau,k+1} \setminus \overline{G_{\tau,k}} \neq \emptyset$,
c) $(\overline{G_{\tau,k+2}} \setminus G_{\tau,k+1}) \cap (\overline{G_{\tau,k+1}} \setminus G_{\tau,k}) = \partial G_{\tau,k+1} = C^+_{\tau,k+1} \cup C^-_{\tau,k+1}$,
d) $(\overline{G_{\tau,k+2}} \setminus G_{\tau,k+1}) \cap (\overline{G_{\tau,k}} \setminus G_{\tau,k-1}) = \partial G_{\tau,k+1} \cap \partial G_{\tau,k} = \{K_\tau(\omega^R_{k+1})\} \subset \mathbb{R}$,
e) $G_{\tau,0} \cap \{\lambda \in \mathbb{C}\colon\ \operatorname{Re}\lambda \geq 1\} = \emptyset$.

Proof By construction (see (2.16) and (2.15)) we obtain the boundness, connectivity and openness of $G_{\tau,k}$. Lemma 2.1a and 2.2 provide the symmetry.

For $x \in \left[\omega^R_k, \omega^R_{k+1}\right)$ and $y \in \left[\omega^R_{k+1}, \omega^R_{k+2}\right)$ $(k \in \mathbb{N}_0)$ we have $x < y$ and (2.9) implies $|K(x)| < |K(y)|$. The definition of $G_{\tau,k}$ and $C^\pm_{\tau,k}$, $k \in \mathbb{N}_0$, yield the assertions a), b), c) and d). $K_\tau\left(\omega^R_{k+1}\right) \in \mathbb{R}$ follows from Lemma 2.4a.

Since $\frac{dK_\tau}{d\omega}(0) = i(1+\tau)$, the curve K_τ is tangent to the straight line $\{\lambda \in \mathbb{C}\colon\ \operatorname{Re}\lambda = 1\}$ at $\lambda = 1$. The convexity (see Lemma 2.2) of K_τ and the definition (see (2.16)) of $G_{\tau,k}$ implies part e).

Proposition 2.1 *Let $\tau > 0$ and $k \in \mathbb{N}_0$. By passing from region $G_{\tau,k}$ into region $G_{\tau,k+1} \setminus \overline{G_{\tau,k}}$ along the positive $\operatorname{Im}\lambda$-axis exactly one root of (2.1) with positive real part appears.*

Proof Lemma 2.2 and 2.5 provide the existence of an unbounded strictly increasing sequence of positive real numbers $(\beta^I_k)_{k\in\mathbb{N}_0}$ such that

$$\partial G_{\tau,k} \cap \{i\,\beta \in \mathbb{C}\colon\ \beta > 0\} = \{i\beta^I_k\}$$

for $k \in \mathbb{N}_0$. Suppose $\lambda = i\,\beta^I_k$. First we consider the case $k = 2l$, $l \in \mathbb{N}_0$. For $\lambda = i\,\beta^I_k$ (2.1) has a root $s_{0,k} = i\omega^I_k$, with $2l\pi < \omega^I_k\tau < 2l\pi + \frac{\pi}{2}$. Notice that $\omega^I_k = \omega_l(\tau, \frac{\pi}{2})$ with ω_l as in Lemma 2.3. $s_{0,k}$ is the only root s of (2.1) for $\lambda = i\beta^I_k$ with $\operatorname{Re}s = 0$ (see Lemma 2.2 and 2.3).

Now consider the case $k = 2l+1$, $l \in \mathbb{N}_0$. For $\lambda = i\,\beta^I_k$ (2.1) has a root $s_{0,k} = -i\omega^I_k$, with $(2l+1)\pi < \omega^I_k\tau < (2l+1)\pi + \frac{\pi}{2}$. Notice that $\omega^I_k = \omega_l(\tau, \frac{3}{2}\pi)$ with ω_l as in Lemma 2.3. $s_{0,k}$ is the only root s of (2.1) for $\lambda = i\beta^I_k$ with $\operatorname{Re}s = 0$ (see Lemma 2.2 and 2.3).

In both cases there holds

$$\sin\tau(-1)^k\omega^I_k > 0. \tag{2.18}$$

Since $s_{0,k}$ is a simple root of (2.1) (see Lemma 2.1c) the implicit function theorem (see [10]) provides the existence of $\delta > 0$ and a unique differentiable function

$$s\colon\ \left(\beta^I_k - \delta,\, \beta^I_k + \delta\right) \to \mathbb{C},$$

where $s(\beta)$ solves equation (2.1) for $\lambda = i\,\beta$ and $s(\beta^I_k) = i(-1)^k\omega^I_k$. Moreover it holds

$$\frac{ds(\beta^I_k)}{d\beta} = \frac{\tau\beta^I_k + \sin\tau(-1)^k\omega^I_k + i\,\cos\tau(-1)^k\omega^I_k}{|\cos\tau(-1)^k\omega^I_k - i\left(\tau\beta^I_k + \sin\tau(-1)^k\omega^I_k\right)|^2}. \tag{2.19}$$

Using (2.18) this yields

$$\frac{d\,\operatorname{Re}s(\beta^I_k)}{d\beta} = \frac{\tau\beta^I_k + \sin\tau(-1)^k\omega^I_k}{|\cos\tau(-1)^k\omega^I_k - i\left(\tau\beta^I_k + \sin\tau(-1)^k\omega^I_k\right)|^2} > 0. \tag{2.20}$$

Therefore we can choose δ sufficiently small such that

$$\operatorname{Re} s(\beta)\begin{cases} <0 & \text{for} \quad \beta_k^I-\delta<\beta<\beta_k^I, \\ =0 & \text{for} \quad \beta=\beta_k^I, \\ >0 & \text{for} \quad \beta_k^I<\beta<\beta_k^I+\delta. \end{cases}$$

On the other hand we know that $i(-1)^k\omega_k^I$ is the only solution with zero real part of (2.1) for $\lambda = i\beta_k^I$ (see Lemma 2.2) and that the real part of every solution of (2.1) is bounded above (see Lemma 2.1b). So the assertion of the proposition is proved.

Lemma 2.6 *Let $k \in \mathbb{N}_0$. For every $\lambda \in G_{\tau,k} \setminus \overline{G_{\tau,k-1}}$ the number of zeros with positive real parts (counted by their multiplicities) of (2.1) is constant.*

Proof First recall that all solutions with positive real part are in the open and bounded set S_λ (see Lemma 2.1b). Let $S := \bigcup\limits_{\lambda\in G_{\tau,k}\setminus\overline{G_{\tau,k-1}}} S_\lambda$. S is an open and bounded set. By definition it holds $|\Phi_{\tau,\lambda}(z)|>0$ for all $z\in\partial S$. By Theorem 9.17.4 of [10, p.243], an application of Rouche's theorem, the number of zeros with positive real part is constant for all $\lambda \in G_{\tau,k} \setminus \overline{G_{\tau,k-1}}$.

We are now in a position to state the main result of this section.

Theorem 2.1 *Let $k \in \mathbb{N}_0$, $\tau > 0$. For any given $\lambda \in G_{\tau,k}\setminus\overline{G_{\tau,k-1}}$ the number of zeros with positive real parts (counted by their multiplicities) of (2.1) is exactly k.*

Proof The theorem is proved by induction on $k \in \mathbb{N}_0$. First notice that $0 \in G_{\tau,0}$ and that (2.1) with $\lambda = 0$ has no solution with positive real part. Consequently for all $\lambda \in G_{\tau,0}$ equation (2.1) has no solution with positive real part (see Lemma 2.6).

Suppose that (2.1) for $\lambda \in G_{\tau,k}\setminus\overline{G_{\tau,k-1}}$ has exactly $k \in \mathbb{N}_0$ solutions with positive real part. Proposition 2.1 yields that (2.1) has exactly $k+1$ solutions with positive real part for $\lambda \in G_{\tau,k+1}\setminus\overline{G_{\tau,k}}$. The theorem is proved.

2.2 D-decomposition of the τ-halfline.

Now we want to use the preceding results to give an D-decomposition of the τ-halfline. For any given $\tau > 0$ and $\theta \in [0, 2\pi)$ we define a sequence $\big(\lambda_k(\tau,\theta)\big)_{k\in\mathbb{N}_0}$ by

$$C^+_{\tau,k} \cap \{\lambda\in\mathbb{C}\colon \lambda=\rho e^{i\theta},\ \rho\ge 0\} = \{\lambda_k(\tau,\theta)\}, \quad k\in\mathbb{N}_0 \quad \text{if} \quad \theta\in[0,\pi],$$
$$C^-_{\tau,k} \cap \{\lambda\in\mathbb{C}\colon \lambda=\rho e^{i\theta},\ \rho\ge 0\} = \{\lambda_k(\tau,\theta)\}, \quad k\in\mathbb{N}_0 \quad \text{if} \quad \theta\in(\pi,2\pi).$$

Lemma 2.7 *For $\tau > 0$ it holds*

a) *For any $k \in \mathbb{N}_0$ ($k \in \mathbb{N}$) and $\theta \in (0, 2\pi)$ ($\theta \in [0, 2\pi)$) $|\lambda_k(\tau,\theta)|$ is a decreasing function of $\tau > 0$. $\lambda_0(\tau, 0) = 1$ for all $\tau > 0$.*

b) $\lim\limits_{\tau\to 0+} |\lambda_k(\tau,\theta)| = \infty$ *provided $k > 0$ or $k = 0$ and $\theta \in \left[\frac{\pi}{2}, \frac{3}{2}\pi\right]$*

$\lim\limits_{\tau\to 0+} \lambda_0(\tau,\theta) = 1 + i\tan\theta$ *if $\theta \in \left[0,\frac{\pi}{2}\right)\cup\left(\frac{3}{2}\pi, 2\pi\right)$.*

c) $\lim\limits_{\tau\to\infty} \lambda_k(\tau,\theta) = e^{i\theta}$.

Proof Suppose $\theta \in (0,\pi)$. By construction of $C^+_{\tau,k}$, there is $\omega_k(\tau,\theta) \in (\omega_k^R, \omega_{k+1}^R)$ such that $\lambda_k(\tau,\theta) = K_\tau(\omega_k(\tau,\theta))$ if $k = 2l$ and $\overline{\lambda_k(\tau,\theta)} = K_\tau(\omega_k(\tau,\theta))$ if $k = 2l+1$, $l \in \mathbb{N}_0$. Now consider $\omega_k(\tau,\theta)$ as function of $\tau > 0$. By differentiating (2.10) with respect to τ we obtain:

$$\frac{d\omega_k(\tau,\theta)}{d\tau} = -\frac{\omega_k(\tau,\theta)(1+\omega_k^2(\tau,\theta))}{1+\tau(1+\omega_k^2(\tau,\theta))} < 0. \tag{2.21}$$

Consequently $\omega_k(\tau,\theta)$ is a decreasing function of $\tau > 0$, and thus, by (2.9), $|\lambda_k(\tau,\theta)|$ is also a decreasing function of $\tau > 0$. This proves part a) with $\theta \in (0,\pi)$. Part a) with $\theta \in (\pi, 2\pi)$ follows by symmetry (see Lemma 2.2). For $k = 0$ and $\theta = 0$ there holds $\lambda_0(\tau, 0) = 1$. The cases $(\theta = 0,\ k \in \mathbb{N})$ and $(\theta = \pi,\ k \in \mathbb{N}_0)$ can be proved in a similar way.

Equations (2.8) and (2.9) provide b) and part c) follows from (2.8), (2.9) and (2.12).

Using the lemma above we obtain

Lemma 2.8 *For any* $\tau_1, \tau_2 > 0$ *with* $\tau_1 < \tau_2$ *there holds*

a) $G_{\tau_2,k} \subsetneqq G_{\tau_1,k}$ *for any* $k \in \mathbb{N}_0$*;*
b) $\partial G_{\tau_1,0} \cap \partial G_{\tau_2,0} = \{1\}$ *and* $\partial G_{\tau_1,k} \cap \partial G_{\tau_2,k} = \emptyset$*, for* $k \in \mathbb{N}$.

To complete the discussion about the τ-dependence of the regions $G_{\tau,k}$ we consider the limiting cases $\tau = 0$ and $\tau \to \infty$.

Lemma 2.9 *Let* $\tau = 0$*. Equation (2.1) has exactly one solution, namely* $s = \lambda - 1$.

For $\tau \in (0,\infty)$ we set $z = \tau s$. From (2.1) for $\tau \to \infty$ we obtain

$$\Phi_\lambda(z) := \exp(z) - \lambda = 0. \tag{2.22}$$

It is easy to prove, that

Lemma 2.10 *For* $|\lambda| < 1$ Φ_λ *has only zeros with negative real part, and for* $|\lambda| > 1$ Φ_λ *has only zeros with positive real part.* z *is a zero of* Φ_λ *with* $\operatorname{Re} z = 0$ *if and only if* $|\lambda| = 1$.

Remark 2.1 For any $\tau > 0$ there holds

$$G_{\infty,0} \subsetneqq G_{\tau,0} \subsetneqq G_{0,0}, \tag{2.23}$$

where

$$G_{\infty,0} := \{\lambda \in \mathbb{C}\colon |\lambda| < 1\} \quad \text{and} \quad G_{0,0} := \{\lambda \in \mathbb{C}\colon \operatorname{Re}\lambda < 1\}. \tag{2.24}$$

In order to be able to state the main results on the D-decomposition of the τ-halfline we define positive real numbers $\tau_k(\lambda)$ for $\lambda \in \mathbb{C}$, $|\lambda| > 1$, such that $\lambda \in \partial G_{\tau,k}$ if and only if $\tau = \tau_k(\lambda)$ for $k \in \mathbb{N}_0$ $(k \in \mathbb{N})$ if $\operatorname{Re}\lambda < 1$ $(\operatorname{Re}\lambda \geq 1)$. For $\operatorname{Re}\lambda \geq 1$ we set $\tau_0(\lambda) := 0$. Moreover let $\tau_{-1}(\lambda) := 0$.

As a consequence of Lemma 2.7 we obtain

Proposition 2.2 *If* $\lambda \in \mathbb{C} \setminus \mathbb{R}$ *with* $|\lambda| > 1$ *then* $(\tau_k(\lambda))_{k\in\mathbb{N}_0}$ *is an unbounded and strictly increasing sequence.*

If $\lambda \in \mathbb{R}$ *with* $\lambda > 1$ *then* $(\tau_k(\lambda))_{k\in\mathbb{N}_0}$ *is an unbounded and increasing sequence with* $\tau_{2k-1}(\lambda) = \tau_{2k}(\lambda) < \tau_{2k+1}(\lambda)$.

If $\lambda \in \mathbb{R}$ *with* $\lambda < -1$ *then* $(\tau_k(\lambda))_{k\in\mathbb{N}}$ *is an unbounded and increasing sequence with* $\tau_{2k}(\lambda) = \tau_{2k+1}(\lambda) < \tau_{2k+2}(\lambda)$.

Remark 2.2 One can compute $\tau_k(\lambda)$ explicitly. Because of the symmetry properties of $K_\tau(\omega)$, see Lemma 2.2, it is sufficient to consider $\mathbb{C} \ni \lambda = |\lambda|e^{i\theta}$ with $\operatorname{Im}\lambda \geq 0$, i.e. $\theta \in [0, \pi]$. It holds

$$\tau_{2k}(\lambda) = \frac{2k\pi + \theta - \arctan\left(\sqrt{|\lambda|^2 - 1}\right)}{\sqrt{|\lambda|^2 - 1}}$$

for $k \in \mathbb{N}_0$ $(k \in \mathbb{N})$ if $\operatorname{Re}\lambda \leq 1$ $(\operatorname{Re}\lambda > 1)$ and

$$\tau_{2k+1}(\lambda) = \frac{2(k+1)\pi - \theta - \arctan\left(\sqrt{|\lambda|^2 - 1}\right)}{\sqrt{|\lambda|^2 - 1}}$$

for $k \in \mathbb{N}_0$. Note that $\arctan\left(\sqrt{\lambda^2 - 1}\right) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Theorem 2.2

a) *Let* $\lambda \in \mathbb{C} \setminus \mathbb{R}$ *with* $|\lambda| > 1$. *For* $\operatorname{Re}\lambda < 1$ $(\operatorname{Re}\lambda \geq 1)$ *and* $\tau \in (\tau_{k-1}(\lambda), \tau_k(\lambda)]$, $k \in \mathbb{N}_0$ $(k \in \mathbb{N})$ *the number of zeros with positive real part of (2.1) counted by their multiplicities is exactly* k.

b) *Let* $\lambda \in \mathbb{R}$. *For* $\lambda > 1$ *and* $\tau \in (\tau_{2k}(\lambda), \tau_{2k+2}(\lambda)]$, $k \in \mathbb{N}_0$, *equation (2.1) has exactly* $2k+1$ *solutions with positive real part. For* $\lambda < -1$ *and* $\tau \in (\tau_{2k-1}(\lambda), \tau_{2k+1}(\lambda)]$, $k \in \mathbb{N}_0$, *equation (2.1) has exactly* $2k$ *solutions with positive real part.*

Proof Theorem 2.1, Lemma 2.8 and the definition of $\tau_k(\lambda)$ yield the assertions.

3 Stability of Delay Differential Equations

We consider the following system of delay differential equations:

$$\dot{x}(t) = Ax(t) + Bx(t-\tau) + F(x(t), x(t-\tau)), \tag{3.1}$$

where $\tau > 0$; $A, B \in \mathbb{R}^{n,n}$, $F \in C^k(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$, $k \geq 1$ and $F(0,0) = DF(0,0) = 0$. It follows $\bar{x} = 0$ is an equilibrium point of (3.1).

3.1 Characteristic equation

The linear part of the system (3.1) is given by

$$\dot{x}(t) = Ax(t) + Bx(t-\tau), \tag{3.2}$$

where $\tau \geq 0$; $A, B \in \mathbb{R}^{n,n}$. The corresponding characteristic equation satisfies:

$$\det(sI - A - B\exp(-\tau s)) = 0. \tag{3.3}$$

We are interested in special matrices A and B, for which it is possible to study the properties of the solutions of the characteristic equation above by help of (2.1).

Definition 3.1 We say the matrices $A, B \in \mathbb{R}^{n,n}$ satisfy *condition (C)* if there is a regular (unitary) matrix $M \in \mathbb{C}^{n,n}$ such that $A = M(D_A + T_A)M^{-1}$ and $B = M(D_B + T_B)M^{-1}$, where $D_A = \operatorname{diag}(-p_1, \ldots, -p_n) \in \mathbb{R}^{n,n}$, with $p_i > 0$, $i \in \{1, \ldots, n\}$, $D_B = \operatorname{diag}(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^{n,n}$, and T_A, T_B are upper triangular with all diagonal entries equal to zero.

Example 3.1 If $A = \operatorname{diag}(-p, \ldots, -p) \in \mathbb{R}^{n,n}$, with $p > 0$, and $B \in \mathbb{R}^{n,n}$ is a general matrix, or if $A \in \mathbb{R}^{n,n}$ is a matrix with n real negative eigenvalues and $B = \operatorname{diag}(\lambda, \ldots, \lambda) \in \mathbb{R}^{n,n}$, with $\lambda \in \mathbb{R}$, then A and B satisfy the condition (C) (see [4, 7]).

Using the multiplicativity of the determinant function we prove

Lemma 3.1 *Let $A, B \in \mathbb{R}^{n,n}$ satisfy condition (C). Then:*

$$\det(sI - A - B\exp(-\tau s)) = \exp(-\tau s)\prod_{i=1}^{n}\big[(s + p_i)\exp(\tau s) - \lambda_i\big].$$

Remark 3.1

a) Setting $\acute{s} = \frac{s}{p}$, $\acute{\tau} = p\tau$ and $\acute{\lambda} = \frac{\lambda}{p}$ into $(s + p)\exp(s\tau) - \lambda = 0$ we obtain

$$\Phi_{\acute{\tau},\acute{\lambda}}(\acute{s}) := (\acute{s} + 1)\exp(\acute{s}\acute{\tau}) - \acute{\lambda} = 0. \tag{3.4}$$

b) From a) and Lemma 3.1. It follows: If $A, B \in \mathbb{R}^{n,n}$ satisfy (C), equation (3.3) can be reduced to n simpler equations of type (3.4) with $\acute{\tau} = p_i\tau$ and $\acute{\lambda} = \frac{\lambda_i}{p_i}$, $i \in \{1, \ldots, n\}$.

3.2 τ-dependent stability

In the following we study the τ-dependent stability properties of the trivial equilibrium $\bar{x} = 0$ of system (3.1).

Theorem 3.1 *Suppose the matrices $A, B \in \mathbb{R}^{n,n}$ satisfy (C). Then*

a) *If $|\lambda_i| \le p_i$ and $\lambda_i \ne p_i$ for all $i \in \{1, \ldots, n\}$, then $\bar{x} = 0$ is asymptotically stable for any $\tau \ge 0$.*

b) *If there is $l \in \{1, \ldots, n\}$ such that $\operatorname{Re}\lambda_l \ge p_l$ ($\operatorname{Re}\lambda_l > p_l$) and $\lambda_l \ne p_l$, then $\bar{x} = 0$ is unstable for any $\tau > 0$ ($\tau \ge 0$).*

c) *Suppose $\operatorname{Re}\lambda_i < p_i$ for all $i \in \{1, \ldots, n\}$. Further we suppose there exist $l \in \{1, \ldots, n\}$ such that $|\lambda_l| > p_l$. Then there is $0 < \tau^s$, such that $\bar{x} = 0$ is asymptotically stable for $0 \le \tau < \tau^s$ and unstable for $\tau > \tau^s$.*

Proof The case $\tau = 0$ is covered by Lemma 2.9. In the sequel we suppose $\tau > 0$. $|\lambda_i| \le p_i$ and $\lambda_i \ne p_i$ for all $i \in \{1, \ldots, n\}$ yields $\frac{\lambda_i}{p_i} \in G_{\infty,0} \subset G_{\tau p_i,0}$ (see Lemma 2.8a) for arbitrary $\tau > 0$. It follows that for any $\tau > 0$ the characteristic equation (3.3) has only roots with negative real part (see Theorem 2.1). Standard results on stability in first approximation (see [11, pp.160, 161]) prove part a.

If there is $l \in \{1, \ldots, n\}$ such that $\operatorname{Re}\lambda_l \ge p_l$ and $\lambda_l \ne p_l$, then there holds $\frac{\lambda_l}{p_l} \notin \overline{G_{\tau p_l,0}}$ (see Lemma 2.5d) for arbitrary $\tau > 0$. This implies (see Theorem 2.1) that for any τ the characteristic equation (3.3) has at least one root with positive real part, and thus part b is proved.

Now let $l \in \{1,\dots,n\}$ be such that $|\lambda_l| > p_l$ and $\operatorname{Re}\lambda_l < p_l$. By Lemmas 2.7 and 2.8 there exist a $\tau_l^s > 0$ such that

$$\frac{\lambda_l}{p_l} \in G_{\tau p_l,0}, \qquad \tau < \tau_l^s,$$
$$\frac{\lambda_l}{p_l} \in \partial G_{\tau p_l,0}, \quad \tau = \tau_l^s,$$
$$\frac{\lambda_l}{p_l} \notin \overline{G_{\tau p_l,0}}, \qquad \tau > \tau_l^s.$$

We set

$$\tau^s = \min\{\tau_l^s \colon l \in \{1,\dots,n\} \text{ with } |\lambda_l| > p_l > \operatorname{Re}\lambda_l\}.$$

Consequently the characteristic equation (3.3) has only roots with negative real part if $\tau < \tau^s$ and at least one root with positive real part if $\tau > \tau^s$.

Remark 3.2 τ^s in Theorem 3.1c is defined by

$$\tau^s = \min\left\{\frac{\theta_l - \arctan(\frac{1}{p_l}\sqrt{|\lambda_l|^2 - p_l^2})}{\sqrt{|\lambda_l|^2 - p_l^2}} \colon\ l \in \{1,\dots,n\} \text{ with } |\lambda_l| > p_l > \operatorname{Re}\lambda_l\right\},$$

where $\theta_l \in [0, 2\pi)$ such that $\lambda_l = |\lambda_l|\, e^{i\theta_l}$ and $\arctan\Big(\frac{1}{p_l}\sqrt{|\lambda_l|^2 - p_l^2}\Big) \in \Big(-\frac{\pi}{2}, \frac{\pi}{2}\Big)$.

For the sake of completeness we consider the limiting case $\tau \to \infty$. For $\tau \in (0,\infty)$ we set $t' = \frac{t}{\tau}$ and $y(t') = x(t'\tau)$. Then (3.1) becomes

$$\frac{1}{\tau}\,\dot y(t') = Ay(t') + By(t'-1) + F(y(t'), y(t'-1)).$$

For $\tau \to \infty$ we obtain

$$Ay(t') + By(t'-1) + F(y(t'), y(t'-1)) = 0. \tag{3.5}$$

Theorem 3.2 *Suppose the matrices A, $B \in \mathbb{R}^{n,n}$ satisfy (C). Then*

a) *If $|\lambda_i| < p_i$ for all $i \in \{1,\dots,n\}$, then $\bar x = 0$ as solution of equation (3.5) is asymptotically stable.*

b) *If there is $l \in \{1,\dots,n\}$ such that $|\lambda_l| > p_l$, then $\bar x = 0$ is unstable.*

Proof Since A is a regular matrix, equation (3.5) can be rewritten as

$$y(t) = Cy(t-1) + g(y(t'-1)),$$

where $C = -A^{-1}B$ and $g \in C^1(U,V)$; $U, V \subset \mathbb{R}^n$ neighborhoods of $\bar x = 0$, an appropriate function with $g(0) = Dg(0) = 0$. The eigenvalues μ_i of C satisfy $\mu_i = \frac{\lambda_i}{p_i}$, $i \in \{1,\dots,n\}$. From Lemma 2.10 and Remark 3.1: $|\mu_i| < 1$ if $|\lambda_i| < p_i$ for all $i \in \{1,\dots,n\}$ and if there is $l \in \{1,\dots,n\}$ such that $|\lambda_l| > p_l$ then $|\mu_l| > 1$. The Theorem about Stability by First Approximation for Difference Equations (see [20, p.104]) completes the proof.

4 Hopf Bifurcation

In this section we derive sufficient conditions for the occurrence of Hopf bifurcation points in (3.1) with bifurcation parameter τ.

Theorem 4.1 *Suppose that*

A1: *Matrices A and B satisfy condition (C).*

A2: *There are $i_0, i_1 \in \{1,\dots,n\}$ with $i_0 \neq i_1$ if $\lambda_{i_0} \in \mathbb{C}\setminus\mathbb{R}$ and $i_0 = i_1$ if $\lambda_{i_0} \in \mathbb{R}$, such that $\frac{\overline{\lambda_{i_1}}}{p_{i_1}} = \frac{\lambda_{i_0}}{p_{i_0}}$ and $p_{i_0} < |\lambda_{i_0}|$. Assume $\operatorname{Im}\lambda_{i_0} \geq 0$. Set*

$$\tau_{2k}^H := \frac{2k\pi + \theta_{i_0} - \arctan\left(\frac{1}{p_{i_0}}\sqrt{|\lambda_{i_0}|^2 - p_{i_0}^2}\right)}{\sqrt{|\lambda_{i_0}|^2 - p_{i_0}^2}}$$

for $k \in \mathbb{N}_0$ if $\operatorname{Re}\lambda_{i_0} < p_{i_0}$ and $k \in \mathbb{N}$ if $\operatorname{Re}\lambda_{i_0} \geq p_{i_0}$, and

$$\tau_{2k+1}^H := \frac{2(k+1)\pi - \theta_{i_0} - \arctan\left(\frac{1}{p_{i_0}}\sqrt{|\lambda_{i_0}|^2 - p_{i_0}^2}\right)}{\sqrt{|\lambda_{i_0}|^2 - p_{i_0}^2}}$$

for $k \in \mathbb{N}_0$, where $\theta_{i_0} \in [0,\pi)$ such that $\lambda_{i_0} = |\lambda_{i_0}|\, e^{i\theta_{i_0}}$.

A3: *For any $i \in \{1,\dots,n\}\setminus\{i_0,i_1\}$, for which there exist $l \in \mathbb{N}_0$ such that $\frac{\lambda_i}{p_i} \in \partial G_{\tau_k^H p_i, l}$, it follows $\frac{1}{p_i}\sqrt{|\lambda_i|^2 - p_i^2} \neq \mathbb{N}$.*

Then a Hopf bifurcation takes place at $\tau = \tau_k^H$ for $k \in \mathbb{N}_0$ if $\operatorname{Re}\lambda_{i_0} < p_{i_0}$ respectively $k \in \mathbb{N}$ if $\operatorname{Re}\lambda_{i_0} \geq p_{i_0}$.

Proof The Theorem is proved by verifying the hypotheses (H1) and (H2) of the Hopf Bifurcation Theorem (see [15, pp.331–333]). If $\tau = \tau_k^H$, equations (2.8) – (2.10) and (2.16) yield $\frac{\lambda_{i_0}}{p_{i_0}} \in \partial G_{\tau_k^H p_{i_0}, k}$. Lemma 3.1 and Remark 3.1 provide that $s_0 = i p_{i_0}\omega(\tau_k^H p_{i_0}, \theta_{i_0})$ is a purely imaginary root of the characteristic equation (3.3), where $\omega(\tau_k^H p_{i_0}, \theta_{i_0})$ is the unique solution of equation (2.10) in $I_k(\tau_k^H p_{i_0}, \theta_{i_0})$ (see Lemma 2.3). From Lemma 2.1c we obtain $i\omega(\tau_k^H p_{i_0}, \theta_{i_0})$ is a simple root of (2.1) for $\lambda = \frac{\lambda_{i_0}}{p_{i_0}}$ and $\tau = \tau_k^H p_{i_0}$, and consequently s_0 is a simple root of the characteristic equation (3.3) for $\tau = \tau_k^H p_{i_0}$. Further we get by (A3) that there are no other roots $s \neq s_0, \overline{s_0}$ of the characteristic equation (3.3) for $\tau = \tau_k^H$ which satisfy $s = m s_0$ with $m \in \mathbb{Z}$. This verifies hypothesis (H1) in [15, pp.331–333].

Since s_0 is a simple root the implicit function theorem (see [10]) provides the existence of $\delta > 0$ and a differentiable function $s\colon (\tau_k^H - \delta, \tau_k^H + \delta) \to \mathbb{C}$ with $s(\tau_k^H) = s_0$ and $s(\tau)$ solves (3.3). Moreover one can compute

$$\frac{d\,\operatorname{Re} s}{d\tau}(\tau_k^H) = p_{i_0}\,\frac{\omega^2(\tau_k^H p_{i_0}, \theta_{i_0})}{(1 + \tau_k^H p_{i_0})^2 + (\tau_k^H p_{i_0})^2\,\omega^2(\tau_k^H p_{i_0}, \theta_{i_0})} > 0.$$

Thus, hypothesis (H2) in [15, pp.331–333] is satisfied.

Remark 4.1 If $n = 1$ and $n = 2$ with $\lambda_{i_0} \in \mathbb{C}\setminus\mathbb{R}$, respectively, condition (A3) in Theorem 4.1 is always satisfied.

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Periodic Solutions of a Singular Lagrangian System Related to Dispersion-Managed Fiber Communication Devices

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Abstract: We prove the existence of periodic solutions to a certain singular Lagrangian system that describes the evolution of the optical pulse width and chirp for so-called dispersion-managed solitons.

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1 Introduction and Main Results

In data communication systems like transoceanic transmission along a fiber cable, there is increasing demand to achieve transmission rates as high as possible, mainly to the extensive use of the internet. To do so, a recent approach is to utilize non-linear light-wave communications with suitable periodic amplifications to compensate for loss and dispersive effects. The transmission of such optical signal is described by

$$i\Psi_z - \frac{1}{2}\beta_2(z)\Psi_{tt} + \sigma(z)|\Psi|^2\Psi = iG(z)\Psi, \tag{1}$$

see [6, 8, 9]. Here $\Psi = \Psi(z,t)$ is some complex-valued envelope function of the original electric field, t is time, and z is the longitudinal coordinate of the fiber cable, which should be thought of to be a periodic variable, since both amplification and dispersion repeat periodically. Moreover, $G(z)$ accounts for both loss and amplification in the fiber, whereas $\beta_2(z)$ is related to the dispersion; $\sigma(z)$ is some additional function.

The transformation $\Psi(z,t) = A(z,t)\exp\Big(\int^z G(z')\,dz'\Big)$ removes the term on the right-hand side of (1) to yield the nonlinear Schrödinger equation

$$iA_z + d(z)A_{tt} + c(z)|A|^2A = 0, \tag{2}$$

with coefficient functions $c(z)$ and $d(z)$ being periodic of some period $L > 0$. It is then well-accepted that the central part of the desired pulse-shaped solution to (2) is described to leading order by

$$A(z,t) = \frac{Q(t/T(z))}{\sqrt{T(z)}}\,\exp\left(i\,\frac{M(z)}{T(z)}\,t^2\right), \tag{3}$$

see the references cited above, and also [2, 4, 5]; the function $Q(x)$ is an input pulse which often is taken as $Q(x) = C_0 \exp(-x^2/2)$, and $M(z)$ resp. $T(z)$ describe the optical pulse width resp. the chirp (time-dependent phase) of the breathing central part of the optical soliton. Most importantly for our purposes, $T(z)$ and $M(z)$ are L-periodic solutions to

$$\frac{dT}{dz} = 4d(z)M, \qquad \frac{dM}{dz} = \frac{d(z)C_1}{T^3} - \frac{c(z)C_2}{T^2}, \tag{4}$$

with fixed constants

$$C_1 = \frac{\int |Q'(x)|^2\,dx}{\int x^2|Q(x)|^2\,dx}, \qquad C_2 = \frac{\int |Q(x)|^4\,dx}{4\int x^2|Q(x)|^2\,dx}.$$

It is hence of fundamental importance for the whole approach to deduce whether or not periodic solutions of (4) do exist. In some of the papers cited above, this problem is studied numerically for the dispersion map $d(z)$ taken as an L-periodic step function,

$$d(z) = \begin{cases} d_+ & : \quad 0 \le z \le L/4,\ \ 3L/4 \le z \le L \\ -d_- & : \quad L/4 < z < 3L/4 \end{cases}, \tag{5}$$

with $d_+,\ d_- > 0$; the function $c(z)$ was chosen to be constant as is physically reasonable in case the compensation period is much larger than the amplification distance. Taking $d(z)$ as in (5) corresponds to a transmission line consisting of two pieces of fibers with opposite dispersion. Eq. (4), even with dispersion map as in (5), poses interesting mathematical problems, but despite that there is a large mathematical literature on singular Lagrangian problems, cf. e.g. [1, 3] and many others, it does not seem that there are general results that apply to a system as (4), which is Hamiltonian with

$$\mathcal{H}(T,M,z) = 2d(z)M^2 + \frac{d(z)C_1}{2T^2} - \frac{c(z)C_2}{T}.$$

As we are interested in periodic solutions of period L (the "fixed period problem"), it would be natural to consider the action functional $\mathcal{I}$ corresponding to (4) which is here

$$\mathcal{I}(T,M) = \int_0^L \left[T(z)\,\frac{dM}{dz}(z) - \mathcal{H}(T(z),M(z),z)\right]dz$$

for M, T in a suitable function space. A critical point of $\mathcal{I}$ then would provide a solution to (4), but it is not clear how the necessary assumptions on $\mathcal{I}$ can be verified to apply some minimax-argument.

The following theorem is our main result.

Theorem 1.1 *Assume $c(z) = c > 0$ is a constant and $d(z)$ is given by (5). Then (4) has a periodic solution of period L if $d_+ > d_-$.*

The proof of Theorem 1.1 is rather elementary and possible through direct calculation and estimates. Rather than this we would have preferred to give a more functional analytical proof that also works for non-explicit dispersion maps, but such an approach was not clear to us. Nevertheless, the same proof also yields some results for a dispersion map which has the more general form

$$d(z) = \begin{cases} d_+ & : \quad 0 \le z \le L_1, \;\; L - L_1 \le z \le L \\ -d_- & : \quad L_1 < z < L - L_1 \end{cases}, \tag{6}$$

for some $L_1 \in (0, L/2)$; see Theorem 2.4.

Theorem 1.1 discusses the case of a dispersion map with positive average dispersion $\langle d \rangle = \frac{1}{L} \int\limits_0^L d(z)\, dz = \frac{1}{2}(d_+ - d_-)$, cf. [7] for some results in the same direction. Due to numerical observations in [8, 9] there should also exist periodic solutions for the zero-average case $d_+ = d_-$, at least if those values are sufficiently large. If the average dispersion is negative, $d_+ < d_-$, then it will be seen below by means of a symmetry argument that again a periodic solution $T(z)$, $M(z)$ of (4) can be found. However, it is of no practical relevance for the original problem, since it will be negative contrary to what is needed in the ansatz (3). The situation for negative average dispersion currently is rather unclear.

2 Existence of Periodic Solutions

In this section we carry out the proof of Theorem 1.1. First we rewrite (4), introducing $t = z$, $a_+ = 4d_+\sqrt{C_1}$, $a_- = 4d_-\sqrt{C_1}$, $b = cC_2/\sqrt{C_1}$ and $q(t) = T(z)$. Then (4) reads as

$$\ddot{q} = \begin{cases} \frac{a_+^2}{4q^3} - \frac{a_+ b}{q^2} = -V_+'(q) & : \quad 0 \le t \le L/4, \;\; 3L/4 \le t \le L \\ \frac{a_-^2}{4q^3} + \frac{a_- b}{q^2} = -V_-'(q) & : \quad L/4 < t < 3L/4 \end{cases}, \tag{7}$$

where

$$V_+(q) = \frac{a_+^2}{8q^2} - \frac{a_+ b}{q} \quad \text{and} \quad V_-(q) = \frac{a_-^2}{8q^2} + \frac{a_- b}{q}.$$

Throughout we assume $b > 0$, and we also introduce the corresponding energies

$$H_+(q, \dot{q}) = \frac{1}{2}\,\dot{q}^2 + V_+(q) \quad \text{and} \quad H_-(q, \dot{q}) = \frac{1}{2}\,\dot{q}^2 + V_-(q).$$

For the proof of Theorem 1.1, from $d_+ > d_-$ we have the hypothesis

$$a_+ > a_-. \tag{8}$$

It should be noted that the transformation $\bar{q}(t) = -q(L/2 + t)$ changes the rôles of a_+ and a_- in (7). However, since the solution q will be positive under assumption (8), it turns out that for the negative dispersion case $a_+ < a_-$ the function $\bar{q}$ is negative and hence cannot play the rôle of $T(z)$, cf. the corresponding remarks in the introduction.

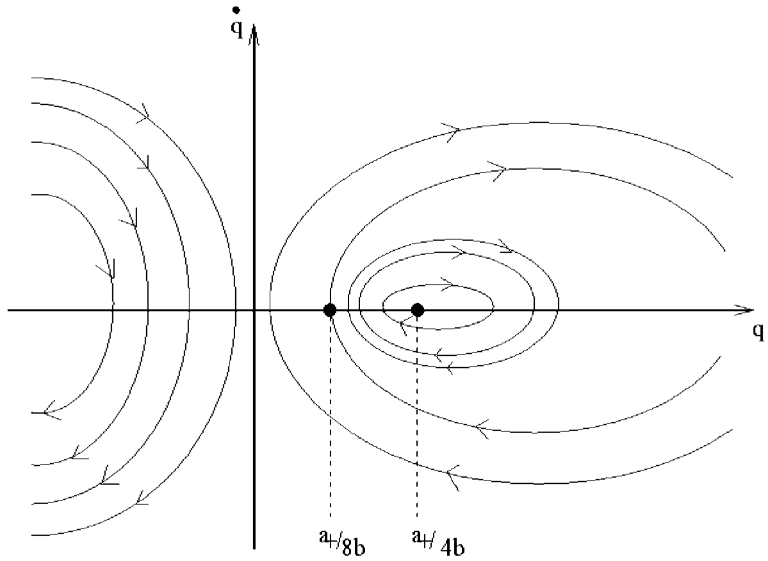

Figure 2.1. Phase portrait of $\ddot{q} = -V_+'(q)$.

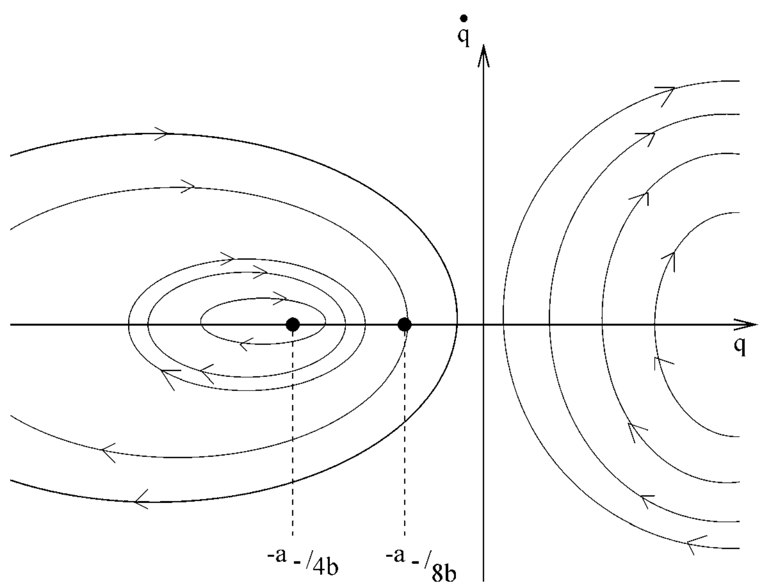

Figure 2.2. Phase portrait of $\ddot{q} = -V_-'(q)$.

To get a clue where to look for periodic solutions of (7), the phase portraits for $\ddot{q} = -V_+'(q)$ resp. for $\ddot{q} = -V_-'(q)$ are given in Figure 2.1 resp. Figure 2.2.

Thus the only possibility to have a periodic solution in $\{q > 0\}$ is to match a periodic orbit from Figure 2.1 to a trajectory from Figure 2.2. The periodic orbits in Figure 2.1 are found to have energies $h_+ \in [-2b^2, 0)$, the value $h_+ = -2b^2$ corresponding to the fixed-point $q = a_+/4b$. The respective periods may then be calculated explicitly as

$$\frac{1}{2}\,T(h_+) = \int\limits_{q_1}^{q_0} \frac{dq}{[2(h_+ - V_+(q))]^{1/2}} = \frac{a_+ b\,\pi}{2\sqrt{2}(-h_+)^{3/2}},$$

where $(q_1, 0)$ and $(q_0, 0)$ with $q_1 \le q_0$ are the intersection points of the orbit of energy h_+ with the axis $\{\dot q = 0\}$.

Let

$$q^* = \begin{cases} a_+/4b & : \quad a_+\pi > 2b^2 L \\ (V_+)^{-1}\Big(-(\sqrt{2}a_+ b\pi/L)^{2/3}\Big) & : \quad a_+\pi \le 2b^2 L \end{cases} \tag{9}$$

with $(V_+)^{-1}\Big(-(\sqrt{2}a_+ b\pi/L)^{2/3}\Big) \in [a_+/4b, \infty)$; observe $V_+ \colon [a_+/4b, \infty) \to [-2b^2, 0)$ is strictly increasing. We define a map $q_0 \mapsto q_1 \mapsto q_2$ as follows.

(1) For given $q_0 \ge q^*$, determine the energy

$$h_+ = V_+(q_0) = \frac{a_+^2}{8q_0^2} - \frac{a_+ b}{q_0} \in [-2b^2, 0). \tag{10}$$

(2) The point $q_1 \le q_0$ then is defined through

$$\frac{L}{4} = \int\limits_{q_1}^{q_0} \frac{dq}{[2(h_+ - V_+(q))]^{1/2}}. \tag{11}$$

(3) Next, $\dot q_1 \ge 0$ is calculated from

$$h_+ = H_+(q_1, \dot q_1) = \frac{1}{2}\,\dot q_1^2 + \frac{a_+^2}{8q_1^2} - \frac{a_+ b}{q_1}. \tag{12}$$

(4) Then we let

$$h_- = H_-(q_1, \dot q_1) = \frac{1}{2}\,\dot q_1^2 + \frac{a_-^2}{8q_1^2} + \frac{a_- b}{q_1} > 0. \tag{13}$$

(5) Finally, $q_2 > 0$ is defined as the unique intersection point of the orbit with energy h_- of $\ddot q = -V_-'(q)$ with the axis $\{\dot q = 0\}$, i.e., the solution of $h_- = H_-(q_2, 0)$.

Remark 2.1 The map $q_0 \mapsto q_1 \mapsto q_2$ is well-defined, since by definition of q^* in (9) we have $\frac{1}{2}T(h_+) \ge \frac{L}{4}$ for $q_0 \ge q^*$ in both cases, and therefore q_1 exists. Note also that all quantities are determined by q_0, or equivalently, by h_+.

Thus the existence of an L-periodic orbit of (7) is equivalent to finding a zero q_0 of the function

$$F(q_0) = \int\limits_{q_2}^{q_1} \frac{dq}{[2(h_- - V_-(q))]^{1/2}} - \frac{L}{4}. \tag{14}$$

Since F is continuous, the existence of a zero will be a consequence of

$$F(q_0) \to -\frac{L}{4} < 0 \qquad \text{as} \quad q_0 \to q^*, \quad \text{and}$$
$$F(q_0) \to \frac{L}{4}\left(\frac{a_+}{a_-} - 1\right) > 0 \quad \text{as} \quad q_0 \to \infty,$$

cf. (8). The following Lemmas 2.1 and 2.2 verify these assertions, completing the proof of Theorem 1.1. Before going on, we will state some identities that will be used frequently throughout. First, from (12) and (13) we infer

$$h_- = h_+ + \frac{a_-^2 - a_+^2}{8q_1^2} + \frac{(a_- + a_+)b}{q_1}. \tag{15}$$

Next, by direct integration of the right-hand side in (11) we obtain

$$\frac{L}{4} = -\frac{\sqrt{X(q_1)}}{4h_+} + \frac{a_+ b}{2\sqrt{2}(-h_+)^{3/2}}\left[\frac{\pi}{2} + \arcsin\left(\frac{2h_+ q_1 + a_+ b}{a_+\sqrt{b^2 + \frac{1}{2}h_+}}\right)\right] \tag{16}$$

with $X(q) = 8h_+ q^2 + 8a_+ bq - a_+^2$; to derive this it is useful to note that $\frac{2h_+ q_0 + a_+ b}{a_+\sqrt{b^2 + \frac{1}{2}h_+}} = -1$ by (10). Similarly, integrating (14) we deduce

$$F(q_0) + \frac{L}{4} = \frac{\sqrt{X(q_1)}}{4h_-} + \frac{a_- b}{2\sqrt{2}h_-^{3/2}} \log\left(\frac{\sqrt{2h_- X(q_1)} + 4h_- q_1 - 2a_- b}{2a_-\sqrt{b^2 + \frac{1}{2}h_-}}\right), \tag{17}$$

utilizing $8h_- q_1^2 - 8a_- bq_1 - a_-^2 = X(q_1)$, cf. (15); the argument of log is ≥ 1, since

$$2a_-\sqrt{b^2 + \frac{1}{2}\,h_-} = 4h_- q_2 - 2a_- b, \tag{18}$$

and $q_1 \geq q_2$.

The right-hand side of (16) contains no q_0, only h_+. It will also be important to have formulae for derivatives w.r.t. h_+. To begin with,

$$\frac{dX(q_1)}{dh_+} = 8q_1^2 + 8(a_+ b + 2h_+ q_1)\left(\frac{dq_1}{dh_+}\right).$$

Through a tedious and lengthy calculation one may then show by differentiating the right-hand side of (16) w.r.t. to h_+ that

$$\frac{dq_1}{dh_+} = \frac{3L}{16}\,\frac{\sqrt{X(q_1)}}{(-h_+ q_1)} - \frac{q_1}{h_+} + \frac{a_+\left(2bq_1 - \frac{1}{2}a_+\right)}{16\left(b^2 + \frac{1}{2}h_+\right)(-h_+ q_1)}; \tag{19}$$

this works by inserting formula (16) after differentiation again for the arcsin$(\ldots)$-term. Additionally, we get from (15)

$$\frac{dh_-}{dh_+} = 1 - \frac{a_-^2 - a_+^2}{4q_1^3}\left(\frac{dq_1}{dh_+}\right) - \frac{(a_- + a_+)b}{q_1^2}\left(\frac{dq_1}{dh_+}\right). \tag{20}$$

After this preparation we can proceed to the proof of Lemma 2.1 and Lemma 2.2.

Lemma 2.1 *As* $q_0 \to q^*$ *we have* $F(q_0) \to -\frac{L}{4}$.

Proof We first consider the case $a_+\pi > 2b^2L$, i.e., $q_0 \to a_+/4b$. By definition, $\bar{q}_0 \le q_1 \le q_0$, with $\bar{q}_0$ and q_0 being the two solutions to $h_+ = H_+(q,0)$. Since $h_+ \to -2b^2$ by (10), it follows that $\bar{q}_0 = (-a_+/2h_+)\Big[b - \sqrt{b^2 + \frac{1}{2}h_+}\Big] \to a_+/4b$, therefore $q_1 \to a_+/4b$, and hence also $X(q_1) \to 0$ as $q_0 \to a_+/4b$. By (15), $h_- \to 2b^2(a_-/a_+)(2 + a_-/a_+)$, and therefore $h_- = H_-(q_2, 0)$ gives $q_2 \to a_+/4b$ as $q_0 \to a_+/4b$. Consequently, $F(q_0) \to -L/4$ as $q_0 \to a_+/4b$ by (17) and (18).

What concerns the second case $a_+\pi \le 2b^2L$ in (9), we then have $T(h_+^*)/2 = L/4$, with $h_+^* = -\big(\sqrt{2}a_+b\pi/L\big)^{2/3}$, by definition of q^*. As $q_0 \to q^*$ therefore q_1 tends to the smaller solution $\bar{q}^*$ of $h_+^* = H(q,0)$, i.e., we have $X(q_1) \to 0$. According to step (3)–(5) in the above construction of the map, q_2 degenerates to $q_2 \to \bar{q}^*$ as $q_0 \to q^*$. Since $h_- \to a_-^2/8(\bar{q}^*)^2 + a_-b/\bar{q}^* > 0$, we may argue as before to conclude $F(q_0) \to -L/4$ as $q_0 \to q^*$.

It remains to analyze the limiting behaviour of $F(q_0)$ as $q_0 \to \infty$.

Lemma 2.2 *As* $q_0 \to \infty$ *we have* $F(q_0) \to \frac{L}{4}\left(\frac{a_+}{a_-} - 1\right)$.

Proof All limits that are taken in this proof are as $q_0 \to \infty$, or, equivalently, as $h_+ \to 0$. Since both terms on the right-hand side of (16) are non-negative and $h_+ \to 0$, we must also have $X(q_1) \to 0$, whence

$$q_1(h_+q_1 + a_+b) \to \frac{a_+^2}{8}, \qquad h_+q_1 \to -a_+b, \tag{21}$$

and therefore $\frac{2h_+q_1 + a_+b}{a_+\sqrt{b^2 + \frac{1}{2}h_+}} \to -1$. By the de L'Hospital rule we are led to check whether

$$\begin{aligned}\Lambda_1 &= -\frac{a_+b}{3\sqrt{2}}\,\frac{\frac{d}{dh_+}\arcsin(\dots)}{(-h_+)^{1/2}} \\ &= -\frac{a_+b}{3\Big(b^2 + \frac{1}{2}h_+\Big)}\left(\frac{2\Big(b^2 + \frac{1}{2}h_+\Big)\Big[h_+\left(\frac{dq_1}{dh_+}\right) + q_1\Big] - \frac{1}{4}(2h_+q_1 + a_+b)}{(-h_+)\sqrt{X(q_1)}}\right)\end{aligned} \tag{22}$$

has a limit as $h_+ \to 0$. Utilizing (19), one arrives after some simplification at

$$\begin{aligned}&2\Big(b^2 + \frac{1}{2}\,h_+\Big)\Big[h_+\left(\frac{dq_1}{dh_+}\right) + q_1\Big] - \frac{1}{4}\,(2h_+q_1 + a_+b) \\ &\qquad = -\frac{3L}{8}\left(b^2 + \frac{1}{2}h_+\right)\frac{\sqrt{X(q_1)}}{q_1} - \frac{X(q_1)}{16q_1}.\end{aligned} \tag{23}$$

Inserting (23) into (22) implies by (21), and since $X(q_1) \to 0$, that $\Lambda_1 \to L/8$. Thus de L'Hospital yields from (16),

$$\frac{a_+b}{2\sqrt{2}(-h_+)^{3/2}}\left[\frac{\pi}{2} + \arcsin\left(\frac{2h_+q_1 + a_+b}{a_+\sqrt{b^2 + \frac{1}{2}h_+}}\right)\right] \to \frac{L}{8}, \qquad -\frac{\sqrt{X(q_1)}}{4h_+} \to \frac{L}{8}. \tag{24}$$

By (15) and (21),

$$\frac{h_-}{h_+} \to -\frac{a_-}{a_+}. \tag{25}$$

Thus as a first step towards deriving the limiting behaviour of $F(q_0)$ we conclude from (24) and (25) that

$$\frac{\sqrt{X(q_1)}}{4h_-} = \left(-\frac{\sqrt{X(q_1)}}{4h_+}\right)\left(-\frac{h_+}{h_-}\right) \to \frac{L}{8}\left(\frac{a_+}{a_-}\right). \tag{26}$$

Next we have to analyze the contribution of the second term on the right-hand side of (17). For this, we proceed as before and consider first the quotient

$$\begin{aligned}\Lambda_2 &= \frac{a_- b}{3\sqrt{2}}\,\frac{\frac{d}{dh_+}\log(\dots)}{h_-^{1/2}\left(\frac{dh_-}{dh_+}\right)}\\ &= \frac{a_- b}{3\sqrt{2}\left(b^2+\frac{1}{2}h_-\right)\left[\sqrt{2h_- X(q_1)}+4h_- q_1 - 2a_- b\right]\left(\frac{dh_-}{dh_+}\right)}\left(\frac{\Lambda_{21}+\Lambda_{22}}{h_-^{1/2}}\right),\end{aligned}$$

where

$$\Lambda_{21} = 4\left(b^2+\frac{1}{2}\,h_-\right)\left[h_-\left(\frac{dq_1}{dh_+}\right)+\left(\frac{dh_-}{dh_+}\right)q_1\right] - \frac{1}{2}\,(2h_- q_1 - a_- b)\left(\frac{dh_-}{dh_+}\right),$$

$$\Lambda_{22} = \frac{\left(b^2+\frac{1}{2}\,h_-\right)}{\sqrt{2h_- X(q_1)}}\left[h_-\left(\frac{dX(q_1)}{dh_+}\right)+\left(\frac{dh_-}{dh_+}\right)X(q_1)\right] - \frac{1}{4}\,\sqrt{2h_- X(q_1)}\left(\frac{dh_-}{dh_+}\right).$$

By (25) we have $\mathcal{O}(h_+) = \mathcal{O}(h_-)$ as $h_\pm \to 0$, whence we can denote such terms simply by $\mathcal{O}(h)$. Because $\frac{1}{q_1^2}\left(\frac{dq_1}{dh_+}\right) \to \frac{1}{a_+ b}$ according to (19), (21) shows $\frac{1}{q_1^3}\,\frac{dq_1}{dh_+} = \mathcal{O}(h)$. In addition, $\sqrt{X(q_1)} = \mathcal{O}(h)$ by (24) and $h_+ q_1 + a_+ b = \mathcal{O}(h)$ by (21). Using this information and the explicit representations (20) of $\frac{dh_-}{dh_+}$ and (19) of $\frac{dq_1}{dh_+}$, it follows after some calculations that

$$\Lambda_{21} = \mathcal{O}(h).$$

Turning our attention to Λ_{22}, we first note $\frac{1}{4}\,\sqrt{2h_- X(q_1)}\left(\frac{dh_-}{dh_+}\right) = \mathcal{O}(h^{3/2})$, since $\frac{dh_-}{dh_+} \to -\frac{a_-}{a_+}$ by (20) and the preceding arguments. Consequently,

$$\begin{aligned}\Lambda_2 = &\frac{a_- b}{6\left[\sqrt{2h_- X(q_1)}+4h_- q_1 - 2a_- b\right]\left(\frac{dh_-}{dh_+}\right)}\\ &\times\ \frac{\left[h_-\left(\frac{dX(q_1)}{dh_+}\right)+\left(\frac{dh_-}{dh_+}\right)X(q_1)\right]}{h_-\sqrt{X(q_1)}} + \mathcal{O}(h^{1/2}).\end{aligned} \tag{27}$$

As before, an elementary but quite lengthy calculation yields

$$h_-\left(\frac{dX(q_1)}{dh_+}\right)+\left(\frac{dh_-}{dh_+}\right)X(q_1) = -\frac{3L}{2}\left[(a_- + 2a_+)b + 2h_+ q_1\right]\frac{\sqrt{X(q_1)}}{q_1} + \mathcal{O}(h^3). \tag{28}$$

As a consequence of $h_-q_1 = (h_+q_1)\left(\frac{h_-}{h_+}\right) \to a_-b$, by inserting (28) into (27) we get $\Lambda_2 \to \frac{L}{8}\left(\frac{a_+}{a_-}\right)$. Thus the rule of de L'Hospital yields

$$\frac{a_-b}{2\sqrt{2}h_-^{3/2}} \log\left(\frac{\sqrt{2h_-X(q_1)} + 4h_-q_1 - 2a_-b}{2a_-\sqrt{b^2 + \frac{1}{2}h_-}}\right) \to \frac{L}{8}\left(\frac{a_+}{a_-}\right). \tag{29}$$

Summarizing (26) and (29), we finally obtain from (17) that $F(q_0) + \frac{L}{4} \to \frac{L}{4}\left(\frac{a_+}{a_-}\right)$.

The method of proof can also be adapted for

$$\ddot{q} = \begin{cases} \frac{a_+^2}{4q^3} - \frac{a_+b}{q^2} & : \quad 0 \le t \le L_1, \;\; L - L_1 \le t \le L \\ \frac{a_-^2}{4q^3} + \frac{a_-b}{q^2} & : \quad L_1 < t < L - L_1 \end{cases}, \tag{30}$$

with $L_1 \in (0, L/2)$, corresponding to the more general dispersion maps (6). We obtain

Theorem 2.1 *For $2L_1\left(1 + \frac{a_+}{a_-}\right) > L$, (30) has an L-periodic solution.*

Proof We can proceed as before, and in particular we find $F(q_0) + (L/2 - L_1) \to L_1(a_+/a_-)$ as $q_0 \to \infty$. The condition $\lim_{q_0\to\infty} F(q_0) > 0$ then means $2L_1(1+a_+/a_-) > L$.

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Hausdorff Dimension Estimates by Use of a Tubular Carathéodory Structure and Their Application to Stability Theory

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Abstract: The paper is concerned with upper bounds for the Hausdorff dimension of flow invariant compact sets on Riemannian manifolds and the application of such bounds to global stability investigations of equilibrium points. The proof of the main theorem uses a special Carathéodory dimension structure in order to get contraction conditions for the considered Carathéodory measures which majorize the Hausdorff measures. The Hausdorff dimension bounds in the general case are formulated in terms of the eigenvalues of the symmetric part of the operator which generates the associated system in normal variations with respect to the direction of the vector field. For sets with an equivariant tangent bundle splitting dimension bounds are derived in terms of uniform Lyapunov exponents. A generalization of the well-known theorems of Hartman-Olech and Borg is given.

Keywords: *Hausdorff dimension; Carathéodory dimension structure; outer measures via tube covers; system in normal variations; global stability; uniform Lyapunov exponents; equivariant tangent bundle splitting; Riemannian manifolds.*

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1 Introduction

The first method of Lyapunov ([9, 36, 47, 49]) traditionally includes all the approaches for the stability investigation of a given solution of an ODE (or an other dynamical

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system) which consider the perturbed solutions by means of various types of linearized or variational equations. In particular this method can be used to construct explicitly (i.e. in the form of a series of known functions and exponential terms including the Lyapunov characteristic exponents) integral manifolds of stationary solutions in order to determine the stability character of these solutions.

As a rule in the given variational equation new coordinates are introduced in order to separate the normal components of the vector fields which act transversally to the flow lines. The main idea of reparametrization and the use of flow information in the transversal to an orbit direction goes back to ([20, 48]). Using these techniques the well-known theorems of Hartman-Olech and Borg ([4, 19, 20]) on global asymptotic stability are derived. For ODE's in $\mathbb{R}^n$ these results were extended and generalized in [29, 32] for other types of stability behavior (stability in the sense of Poincaré and Zhukovskij) including into the consideration Lyapunov functions. Variational systems written in normal coordinates are also used in stability theory to show orbital stability of solutions of a differential equation ([20, 31, 32]). For bounded semi-orbits these methods are extended in [31] to vector fields on Riemannian manifolds. In particular, in this paper sufficient conditions for orbital stability and instability are deduced by estimating the singular values of the fundamental operator of the linearized vector field.

Note that for simple mechanical systems in Lagrange form the physical paths can be interpreted as geodesics on a Riemannian manifold ([17, 23, 24]). A prototype of such systems with instability behavior in the sense of Zhukovskij are geodesic flows on the unit tangent bundle of a manifold with negative curvature ([10, 17, 23, 24, 42]). These systems are characterized by a uniform splitting of the tangent bundle into invariant subbundles (with respect to the linearization) having equal contracting or expanding rates in all points of the bundle. They belong to a special type of (strong) hyperbolic systems. Unfortunately most of the interesting equations are only quasi-hyperbolic ([7, 13, 42, 43]).

Stability investigations of flows are closely connected with global properties of invariant sets or attractors such as dimension (topological, Hausdorff, box-counting etc.) and the topological shape of these sets (connectness, point-like type etc.) ([14, 18]).

The first general results for upper Hausdorff dimension estimates of flow invariant sets in $\mathbb{R}^n$ in terms of singular values of the linearization are given by [6]. This approach was extended in [25, 39] to map-invariant sets on Riemannian manifolds and in [26, 28, 29] by including Lyapunov functions into the contraction conditions for outer Hausdorff measures. In [8, 46] the Douady-Oesterlé results were extended to estimates for evolution systems in general Hilbert spaces. Hausdorff dimension estimates of general flow invariant sets using the eigenvalues of the symmetric part of the operator part of the (standard) equation in variation are deduced in [45] for the $\mathbb{R}^n$ and in [39] for manifolds. Douady-Oesterlé estimates for piecewise smooth maps on manifolds are given in [44]. The hyperbolic or quasi-hyperbolic structure was considered in dimension estimates in [10, 13] where also an entropy term into the estimate was introduced.

Various dimension upper bounds of invariant sets allow conclusions on the dynamical behavior of the system. The key step in the papers [29, 39, 45] is to prove that the Hausdorff dimension for the maximal compact invariant set is less than two. By a result of Smith ([45]) such a set contains no simple closed piecewise smooth invariant curves. In particular the system has no non-constant periodic orbits. On the base of such dimension estimates a generalization of the mentioned global stability results of Hartman-Olech and Borg, but also of other types of classical results from the Bendixson-Poincaré theory were derived in [29, 34, 35].

Parallel to Hausdorff dimension estimates a number of upper bounds for the box dimension of invariant sets were deduced ([3, 21, 22, 30, 38, 46]). The box dimension of a set is always not smaller than the Hausdorff dimension and gives important information about the possibility to use embedding homeomorphisms, which map the given invariant set orthogonal and one-to-one on a hyperplane in standard position ([22, 38]). Recently it was shown that such homeomorphisms can be chosen with Hölder-Lipschitz continuous inverse ([12]) which enables conclusions for dimension estimates.

Hausdorff and box dimension estimates for flow invariant sets show its effectivity if various types of local, global and uniform Lyapunov exponents are introduced ([7, 8, 25, 28, 46]). On the base of such Lyapunov exponents the Lyapunov dimension of a set was defined (Kaplan-Yorke formula [25, 42]) and it was conjectured that in typical cases this dimension coincides with the Hausdorff dimension.

Parallel to the dimension and stability investigation of invariant sets of flows and cascades various types of dimensions of an invariant measure have been developed ([7, 25, 41]). Defining for the invariant ergodic measure of a flow the Lyapunov exponents one can introduce the Lyapunov dimension of this measure which is an upper bound of the Hausdorff dimension of the measure. (The Hausdorff dimension of the measure is the largest lower bound of the Hausdorff dimension of the support of the measure ([25]).) As in the measure free case various stability properties of the underlying flow may be derived from the properties of the Lyapunov exponents of the measure. It is shown in [7] that if the invariant measure is ergodic and all Lyapunov exponents of the measure are negative, the support of this measure is a stable equilibrium point. If exactly one exponent is zero and the remaining ones are negative, the support is an equilibrium point or a stable limit cycle.

An important class of invariant sets of dynamical systems are strange attractors which have locally the structure of the product of a smooth (often one-dimensional) submanifold directed 'along the attractor' and a Cantor-like set 'transversal' to the attractor ([18, 41]). Thus, it is natural to investigate the stability and dimension properties of such attractors considering the intersection of the attractors with surfaces which are locally transversal to the attractor ([20, 26]). The use of transverse intersections (Poincaré sections) is well-known in stability theory investigations of flow orbits: contracting or expanding behavior in sections transverse to the flow line directions is the main reason for properties of stability or instability of the considered orbit ([29, 31, 32]).

The paper is organized as follows. In Section 2 we present a short review of basic facts on Riemannian geometry. We introduce the variational system written in normal variations, transversal to the evolution direction of the flow lines, which is natural to investigate in the case of attractors of differential equations. In Section 3 we give the definition of a special Carathéodory structure adapted for the dimension investigation of flow invariant sets. It is defined via covering elements which are tubular neighborhoods of arcs of smooth curves to approximate the fiber structure of the sets. The main results of the paper are contained in Section 4. For flow negatively invariant sets which do not contain singular points of the vector field an upper bound of the Hausdorff dimension is given. The estimates are derived by means of Carathéodory measures which are contractive under the flow and majorize the Hausdorff measure. These results generalize those from [26, 27] on Riemannian manifolds. The estimates are formulated in terms of the eigenvalues of the symmetric part of the generated operator of the associated system in normal variation. Assuming special properties of the stable and unstable manifolds of equilibrium points the results are generalized for vector fields having a finite number of such equilibrium points in the considered invariant set. The used Carathéodory measures show in many cases a better contracting behavior under the positive semi-flow than the Hausdorff measures do. Section 5 is concerned with Hausdorff dimension estimates of flow invariant sets with an equivariant tangent bundle splitting which are formulated in terms of uniform Lyapunov exponents. In Section 6 we end with a discussion of the effectivity of the obtained Hausdorff dimension estimates. In addition we obtain results about the asymptotic behavior of the dynamical system using the dimension bounds, which are closely related to results in [4, 19, 20].

2 The System in Normal Variation

In this section we introduce a modified variational equation for a vector field f which will be used for modeling the variation of time translated pieces of hypersurfaces orthogonal to a considered orbit. This idea originates from investigations on stability behavior of solutions of a differential equation (see [20, 31, 32]), where together with the movements of phase points along a trajectory one considers their movements in transversal direction. Projecting the covariant derivative of the vector field along a reference orbit into the $(n-1)$-dimensional tangent space lying orthogonal to the vector field in an arbitrary point of the orbit we get a variational equation describing the normal variation. For the first time this type of variational equation has been applied to dimensional estimates in [26, 27].

Let us recall some notation from linear algebra and differential geometry used later. If V and W are m-dimensional Euclidean spaces with scalar products $\langle\cdot,\cdot\rangle_V$ and $\langle\cdot,\cdot\rangle_W$, respectively, and $L\colon V\to W$ is a linear operator, then the adjoint operator $L^*\colon W\to V$ is the linear operator uniquely determined by the relation $\langle L\xi,\eta\rangle_W=\langle\xi,L^*\eta\rangle_V$ for all $\xi\in V$, $\eta\in W$. The *singular values* of the operator L are the eigenvalues of the positive semidefinite operator $(L^*L)^{\frac{1}{2}}\colon V\to V$. We denote them by $\sigma_1(L)\ge\cdots\ge\sigma_m(L)\ge 0$ ordered with respect to size and multiplicity. For $d\in\mathbb{R}$ let $\lfloor d\rfloor$ denote the largest integer less than d. For an arbitrary number $d\in[0,m]$ we define by

$$\omega_d(L)=\begin{cases}1 & \text{for}\quad d=0,\\ \sigma_1(L)\cdot\ldots\cdot\sigma_{\lfloor d\rfloor}(L)\sigma_{\lfloor d\rfloor+1}^{d-\lfloor d\rfloor}(L) & \text{for}\quad d\in(0,m],\end{cases}$$

the *singular value function of order d of L*. Let $\mathcal{E}$ be an ellipsoid in V and let $\sigma_1(\mathcal{E})\ge\ldots\ge\sigma_m(\mathcal{E})\ge 0$ denote the length of its semi-axes. For an arbitrary number $d\in[0,m]$ we introduce the *d-dimensional ellipsoid measure* by

$$\omega_d(\mathcal{E})=\begin{cases}1 & \text{for}\quad d=0,\\ \sigma_1(\mathcal{E})\cdot\ldots\cdot\sigma_{\lfloor d\rfloor}(\mathcal{E})\sigma_{\lfloor d\rfloor+1}^{d-\lfloor d\rfloor}(\mathcal{E}) & \text{for}\quad d\in(0,m].\end{cases}$$

For the linear operator $L\colon V\to W$ and the ball $B(O,r)$ of radius r around the origin O of V the image $LB(O,r)$ is an ellipsoid in W with length of semi-axes $\sigma_i(L)r$. For $d\in[0,m]$ it holds

$$\omega_d(LB(O,r))=\omega_d(L)\,r^d. \tag{2.1}$$

Consider now a Riemannian manifold (M,g) of dimension n $(n\ge 2)$ and, for simplicity, of class C^∞, which we call smooth. Denote by T_pM the tangent space at $p\in M$. The Christoffel symbols of second kind on (M,g) with respect to a chart $x\colon D(x)\to R(x)$

are given by the n^3 smooth functions $\Gamma^k_{ij} = \frac{1}{2} g^{ks}(g_{js,i} + g_{si,j} - g_{ij,s})$ (throughout this paper with summation on repeated indices), where $g_{kl,r} = \frac{\partial g_{kl}}{\partial x^r}$. Here and in the sequel let $f\colon M \to TM$ be a vector field of class C^2 on the n-dimensional Riemannian manifold M $(n \geq 2)$ and let us consider the corresponding differential equation

$$\dot{u} = f(u). \tag{2.2}$$

For simplicity we assume that the global flow $\varphi\colon \mathbb{R} \times M \to M$ of (2.2) exists. This flow φ can also be written as one-parameter family of C^2-diffeomorphisms $\{\varphi^t\}_{t \in \mathbb{R}}$ with $\varphi^t(\cdot) = \varphi(t, \cdot)$. In a chart x around p let $\{\partial_i(p)\}$ be the canonical basis of T_pM and $f(p) = f^i \partial_i(p)$ the representation of the vector field (2.2). The *covariant derivative* of f in p is the linear operator $\nabla f(p)\colon T_pM \to T_pM$ defined by $\nabla f(p)v = \nabla_i f^k v^i \partial_k(p) = \Big(\frac{\partial f^k}{\partial x^i} v^i + \Gamma^k_{ij} f^j v^i\Big)\partial_k(p)$ for all $v = v^i \partial_i(p) \in T_pM$. For the linear operator $\nabla f(p)\colon T_pM \to T_pM$ in the Euclidean space $(T_pM, \langle\cdot,\cdot\rangle_{T_pM})$ we denote by $\nabla f(p)^*$ the adjoint operator and by $S\nabla f(p) := \frac{1}{2}[\nabla f(p) + \nabla f(p)^*]$ the symmetric part of $\nabla f(p)$.

Let $c\colon [a,b] \to M$ be a piecewise smooth curve such that the restrictions $c|_{[t_j, t_{j+1}]}$ are smooth for any $j = 1, \dots, m-1$. Recall that the length $l(c)$ of c is defined as $l(c) = \sum\limits_{j=1}^{m-1} \int\limits_{t_j}^{t_{j+1}} \|\dot{c}(t)\|\, dt$. For a C^1-curve $c\colon [a,b] \to M$ let $x^i(t)$ be the local coordinates of $c(t)$ in the chart x. Let $F(t)$ be a vector field along c, i.e., $F(t) \in T_{c(t)}M$ for all $t \in [a,b]$. The absolute derivative $\frac{DF(t)}{dt} \in T_{c(t)}M$ of F along c is defined in the chart x by

$$\frac{DF(t)}{dt} \equiv \nabla_{\dot{c}} F(t) := \Big(\frac{dF^k}{dt} + \Gamma^k_{ij} F^j \dot{c}^i\Big)\partial_k(c(t)).$$

For a given C^1-curve $c\colon [a,b] \to M$ and $v \in T_{c(t_0)}M$ $(t_0 \in [a,b])$ there exists a unique vector field F_v along c such that F_v is parallel along c, i.e., $\nabla_{\dot{c}} F_v \equiv 0$ and $F_v(t_0) = v$. This defines for any $s, t \in [a,b]$ with $s < t$ the parallel transport $\tau^{c(t)}_{c(s)}\colon T_{c(s)}M \to T_{c(t)}M$ along c from $c(s)$ to $c(t)$ which relates to any $v \in T_{c(s)}M$ the vector $F_v(t) \in T_{c(t)}M$.

Recall that a geodesic on (M,g) is a smooth curve $c\colon [a,b] \to M$ satisfying $\frac{D\dot{c}(t)}{dt} \equiv 0$. For any $p \in M$ and $v \in T_pM$ we denote the maximal geodesic with $\dot{c}(0) = v$ and $c(0) = p$ by $c_{p,v}$. Let $\mathcal{D}^1 \subset TM$ be the set of pairs $\{(p,v)\}$ with $p \in M$ and $v \in T_pM$ such that $c_{p,v}(1)$ exists. Then the *exponential map* $\exp\colon \mathcal{D}^1 \to M$ on (M,g) is given by $\exp((p,v)) = c_{p,v}(1)$ for all $(p,v) \in \mathcal{D}^1$ and $\exp_p$ is the restriction $\exp|_{T_pM \cap \mathcal{D}^1}$. It is well-known (see [24]) that $\mathcal{D}^1$ is open in TM, that $\exp\colon \mathcal{D}^1 \to M$ is smooth, and for any $p \in M$ there exists an open set $\mathcal{D}^1_p \subset T_pM$ such that $\exp_p$ is a diffeomorphism on $\mathcal{D}^1_p$ and $\|d_{O_p} \exp_p\| = 1$.

The behavior of system (2.2) near a given solution $\varphi^{(\cdot)}(p)$ is described by the *variational equation*

$$\frac{Dy}{dt} = \nabla f(\varphi^t(p))y \tag{2.3}$$

(see [31, 39]). In local coordinates of a chart x around $\varphi^t(p)$ system (2.3) takes the form

$$\frac{Dy^k}{dt} = \frac{\partial f^k}{\partial x^i} y^i + \Gamma^k_{ij} f^j y^i = \nabla_i f^k y^i, \qquad k = 1, \dots, n.$$

For any $p \in M$ the differential $Y(t,p) = d_p\varphi^t$ is the operator solution of (2.3) with initial condition $Y(0,p) = \mathrm{id}_{T_pM}$.

All points $p \in M$ with $f(p) \neq O_p$ $(f(p) = O_p)$, where O_p denotes the origin of the tangent space T_pM, we call *regular* (*singular*) points of the vector field f. If p is a regular point we may consider the *system in normal variations* with respect to the solution $\varphi^{(\cdot)}(p)$ of (2.2) ([31])

$$\frac{Dz}{dt} = A(\varphi^t(p))z, \tag{2.4}$$

where the linear operator $A(p)\colon T_pM \to T_pM$ is given by

$$\begin{aligned} &A(p) = \nabla f(p) - B(p), \quad \text{where} \\ &B(p)v = 2\,\frac{f(p)}{\|f(p)\|^2}\,\langle f(p), S\nabla f(p)v\rangle \quad \text{for all} \quad v \in T_pM. \end{aligned} \tag{2.5}$$

The scalar product $\langle\cdot,\cdot\rangle$ and the associated norm $\|\cdot\|$ are taken in the tangent space T_pM. In coordinates of an arbitrary chart $x\colon D(x) \to R(x)$ around the regular point p the linear operator $A(p)$ is given by

$$A_i^k = \nabla_i f^k - \frac{2}{g_{mn}f^m f^n}\, f^k g_{jl} f^l S_i^j, \qquad k,i = 1,\ldots,n,$$

where f^k and g_{jl} are the coordinates of the vector field f and the Riemannian metric tensor g in the chart x, respectively, and $S_i^j = \frac{1}{2}\big[g^{jk}\nabla_k f^p g_{pi} + \nabla_i f^j\big]$ is the representation in coordinates of the symmetric part $S\nabla f(p)$ of the covariant derivative of the vector field f in this chart. Note that for ODE's in $\mathbb{R}^n$ with standard metric the system in normal variations (2.4) coincides with the system in modified variations in [28, 29, 32]. Suppose that $p \in M$ is a regular point of f and $y(\cdot)$ is a solution of (2.3) along $\varphi^{(\cdot)}(p)$. This solution can be splitted for any $t \in \mathbb{R}$ into two orthogonal components as

$$y(t) = z(t) + \mu(t) f(\varphi^t(p)), \tag{2.6}$$

where $z(\cdot)$ is the solution of (2.4) with respect to $\varphi^{(\cdot)}(p)$ with initial condition $z(0) = y(0)$ and $\mu(\cdot)$ is a scalar valued C^1-function given by $\mu(t) = \langle y(t), f(\varphi^t(p))\rangle / \|f(\varphi^t(p))\|^2$.

For every regular point $p \in M$ of f we introduce the $(n-1)$-dimensional linear subspace

$$T^\perp(p) = \big\{v \in T_pM\colon \langle v, f(p)\rangle = 0\big\}$$

of the tangent space T_pM. Denote by $SA(p) := \frac{1}{2}[A(p) + A(p)^*]$ the symmetric part of the operator $A(p)$. A straight forward calculation shows that for all $v \in T^\perp(p)$ the following two relations

$$\langle f(p), SA(p)v\rangle = 0 \quad \text{and} \quad \langle v, A(p)v\rangle = \langle v, \nabla f(p)v\rangle \tag{2.7}$$

are satisfied. Hence, we have $SA(p)\colon T^\perp(p) \to T^\perp(p)$. Using this fact one can easily prove the first part of the following lemma.

Lemma 2.1 *For an arbitrary regular point $p \in M$ of the vector field (2.2) the eigenvalues of the operator $SA(p)\colon T_pM \to T_pM$ are the eigenvalues of the operator $SA(p)$*

which is restricted to the linear subspace $T^{\perp}(p)$ *and the value* $-\langle \nabla f(p) f(p), f(p)\rangle \backslash \|f(p)\|^2$. $\blacksquare$
Further we have

$$S\nabla f(p)z - \frac{f(p)}{\|f(p)\|^2}\langle f(p), S\nabla f(p)z\rangle = SA(p)z \quad \textit{for all} \quad z \in T^{\perp}(p).$$

In the following we denote at any regular point p of (2.2) the eigenvalues of the operator $SA(p)$ restricted to the subspace $T^{\perp}(p)$ by $\beta_1(p) \geq \dots \geq \beta_{n-1}(p)$, which are ordered with respect to size and multiplicity. By $Z(t,p)$ we denote the operator solution of (2.4) with initial condition $Z(0,p) = \mathrm{id}_{T^{\perp}(p)}$. For every $t \in \mathbb{R}$ the linear operator $Z(t,p)\colon T^{\perp}(p) \to T^{\perp}(\varphi^t(p))$ maps between the subspaces $T^{\perp}(p)$ and $T^{\perp}(\varphi^t(p))$ being orthogonal to the vector field in p and $\varphi^t(p)$, respectively. The next lemma will be needed in the sequel and can be proved analogously to [39].

Lemma 2.2 *Suppose that* $p \in M$ *is a regular point of the vector field (2.2) and* $Z(\cdot,p)$ *is the operator solution of (2.4). Let* $d \in (0, n-1]$. *Then for all* $t \geq 0$ *it holds*

$$\omega_d(Z(t,p)) \leq \exp\Bigg\{ \int_0^t \big[\beta_1(\varphi^\tau(p)) + \ldots + \beta_{\lfloor d\rfloor}(\varphi^\tau(p)) + (d - \lfloor d\rfloor)\beta_{\lfloor d\rfloor+1}(\varphi^\tau(p))\big]\, d\tau \Bigg\}.$$

Let $B(O_p, r)$ denote the ball of radius r around the origin O_p of T_pM. For a regular point $p \in M$ of f let $B^{\perp}(O_p, r) = B(O_p, r) \cap T^{\perp}(p)$ be the ball in the subspace $T^{\perp}(p)$ centered in the origin O_p of T_pM with radius r. Fix p and r and consider for any $t \geq 0$ the ellipsoid $\mathcal{E}(t) = Z(t,p)B^{\perp}(O_p, r)$ in the subspace $T^{\perp}(\varphi^t(p))$. If $\sigma_1(\mathcal{E}(t)) \geq \dots \geq \sigma_{n-1}(\mathcal{E}(t))$ are the lengths of the semi-axes of $\mathcal{E}(t)$ and if d is an arbitrary number in $(0, n-1]$ we have by (2.1)

$$\omega_d(\mathcal{E}(t)) = \omega_d(Z(t,p))r^d. \tag{2.8}$$

Our aim is to describe the variation of time translated pieces of hypersurfaces, i.e., $(n-1)$-dimensional submanifolds, orthogonal to a considered orbit of (2.2). For this purpose we will use methods from [31, 32] developed there for stability investigations of flows on manifolds, in order to get information for the Hausdorff dimension of underlying flow invariant sets. Considering a non-equilibrium solution $\varphi^{(\cdot)}(p)$ of (2.2) with $p \in M$ the local transformation of small pieces of a hypersurface can be described by a *reparametrized local flow*. For $\delta > 0$ so small that $\exp_p$ is defined on $B(O_p, \delta)$ we consider the $(n-1)$-dimensional submanifold

$$B^{\perp}(p, \delta) := \exp_p(B^{\perp}(O_p, \delta))$$

of M through p which is local transversal at the point p to the trajectory of the vector field passing through the point p. Every point $u \in B^{\perp}(p, \delta)$ can be uniquely written in the form $u = \exp_p(rv)$, where $v \in T^{\perp}(p)$ is a vector of length $\|v\| = 1$ and $r \in [0, \delta)$ measures the arc length of the geodesic $c_{p,v}$ connecting p and u. This defines us a unique representation $u = u(r, v)$ of a point $u \in B^{\perp}(p, \delta)$.

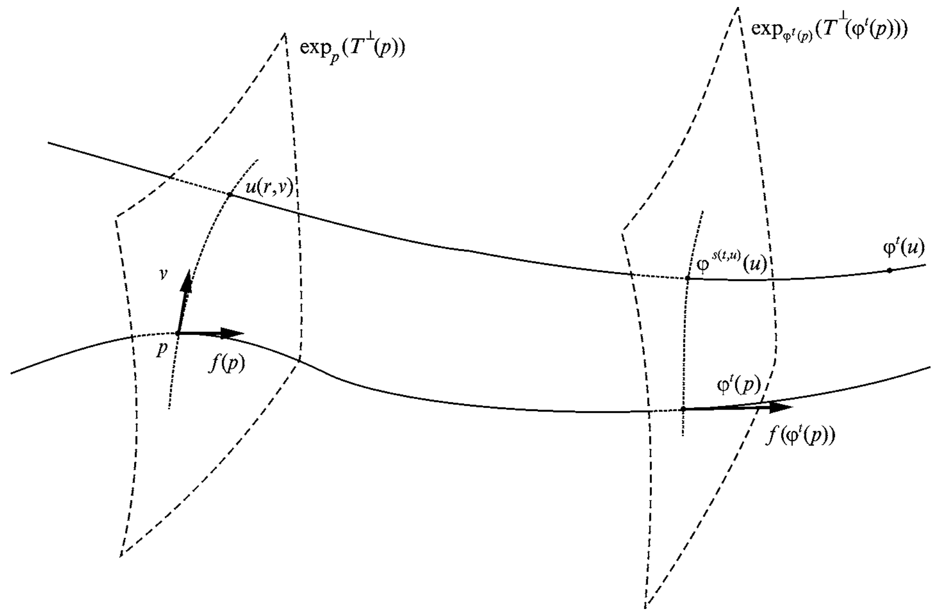

Figure 2.1. Reparametrization of the flow.

The main properties of the described reparametrization are summarized in the following two lemmata which proofs are similar to [20, 31], where a slightly different reparametrization is considered. Results on reparameterization for flows in $\mathbb{R}^n$ are given in [28,29,32]. ∎

Lemma 2.3 *Suppose that $\varphi^{(\cdot)}(p)$ is a non-equilibrium solution of the C^2-vector field (2.2). Then for any finite number $T_0 > 0$ there exists a number $\varepsilon_1 > 0$ such that for every $u \in B^{\perp}(p, \varepsilon_1)$ there is a monotonously increasing differentiable function $s(\cdot, u)\colon \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $s(\cdot, p) = \mathrm{id}\,|_{[0,T_0]}$ and*

$$\left\langle \exp^{-1}_{\varphi^t(p)} \left(\varphi^{s(t,u)}(u)\right), f(\varphi^t(p))\right\rangle = 0 \quad \text{for all} \quad t \in [0, T_0]. \tag{2.9}$$

The next lemma states that for any regular point $p \in M$ of f for the locally defined reparametrized flow $\phi^t(\cdot) \equiv \phi(t, \cdot) := \varphi(s(t, \cdot), \cdot)$ the differential $d_p\phi^t$ of ϕ^t restricted to $T^{\perp}(p)$ satisfies (2.4). This provides the desired description of the variation of time translated pieces of hypersurfaces orthogonal to the considered orbit. For the proof again we refer to the method of [31].

Lemma 2.4 *Suppose that $\varphi^{(\cdot)}(p)$ is a non-equilibrium solution of (2.2) and the function $s(\cdot, \cdot)\colon [0, T_0] \times B^{\perp}(p, \varepsilon_1) \to \mathbb{R}_+$ as given in Lemma 2.3 defines a reparametrized local flow $\phi^t(u) := \varphi^{s(t,u)}(u)$. Then for all $t \in [0, T_0]$ there holds*

$$d_p\phi^t|_{T^{\perp}(p)} = Z(t, p),$$

where $Z(t, p)$ denotes the operator solution of (2.4) with $Z(0, p) = id_{T^{\perp}(p)}$.

We return to the Lemmata 2.3 and 2.4 in Section 4 where they are needed in the proof of Theorem 4.1.

3 Tubular Carathéodory Structure

In this section we define a special Carathéodory structure for flow negatively invariant sets on Riemannian manifolds. The outer measures which arise from this structure will majorize the Hausdorff measures and will be applied to obtain Hausdorff dimension estimates of flow-invariant sets on the manifold.

Carathéodory dimension structures were introduced by Pesin [41] (see also [42]) in order to give a general concept for most of the dimension-like characteristics of sets and measures. Such structures may be considered as a generalization of a well-known measure-theoretic construction of Carathéodory [5, 11]. The essential parts of such a structure are the following ([15]).

Let X be an arbitrary set, $\mathcal{F}$ be a family of subsets of X, $\mathbb{P} = [d^*, +\infty)$ for finite d^* or $\mathbb{P} = \mathbb{R}$ be a parameter set, and let $\xi\colon \mathcal{F} \times \mathbb{P} \to [0, \infty)$, $\eta\colon \mathcal{F} \times \mathbb{R} \to [0, \infty)$, and $\psi\colon \mathcal{F} \to [0, \infty)$ be functions. A sub-family $\mathcal{G} \subset \mathcal{F}$ is said to be an *ε-cover* of a set $Y \subset X$ if $Y \subset \bigcup_{u \in \mathcal{G}} U$ and $\psi(\mathcal{G}) := \sup\{\psi(U) \mid U \in \mathcal{G}\} \leq \varepsilon$ hold. The following conditions are assumed to be satisfied:

(A1) $\emptyset \in \mathcal{F}$, $\psi(\emptyset) = 0$, and $\xi(\emptyset, d) = 0$ for all $d \in \mathbb{P}$.

(A2) $\xi(U, s) = \eta(U, s - d)\xi(U, d)$ for all d, $s \in \mathbb{P}$ and all $U \in \mathcal{F}$.

(A3) For any $\Delta > 0$ there exists $\varepsilon > 0$ such that for all $U \in \mathcal{F}\backslash\{\emptyset\}$ with $\psi(U) \leq \varepsilon$ we have $\eta(U, d) \leq \Delta$ if $d > 0$ and $\eta(U, d) \geq \Delta^{-1}$ if $d < 0$.

(A4) For any subset $Y \subset X$ and for arbitrary $\varepsilon > 0$ there exists a countable ε-cover of Y.

In analogy to [42] we call such a collection $(\mathcal{F}, \mathbb{P}, \xi, \eta, \psi)$ which satisfies (A1)–(A4) a *Carathéodory (dimension) structure* on X. For a given Carathéodory structure $(\mathcal{F}, \mathbb{P}, \xi, \eta, \psi)$, an arbitrary set $Y \subset X$, $d \in \mathbb{P}$, and $\varepsilon > 0$ we define the *Carathéodory d-measure at level ε of Y with respect to* $(\mathcal{F}, \mathbb{P}, \xi, \eta, \psi)$ by

$$\mu_{\mathcal{C}}(Y, d, \varepsilon) = \inf_{\mathcal{G}} \sum_{U \in \mathcal{G}} \xi(U, d),$$

where the infimum is taken over all countable sub-collections $\mathcal{G} \subset \mathcal{F}$ being ε-covers of the set Y. For fixed Y and d the function $\mu_{\mathcal{C}}(Y, d, \varepsilon)$ is non-increasing with respect to ε. Therefore, there exists the limit

$$\mu_{\mathcal{C}}(Y, d) = \lim_{\varepsilon \to 0+0} \mu_{\mathcal{C}}(Y, d, \varepsilon)$$

which is called the *Carathéodory d-measure of Y with respect to* $(\mathcal{F}, \mathbb{P}, \xi, \eta, \psi)$. For arbitrary $d \in \mathbb{P}$ and arbitrary $\varepsilon > 0$ the functions $\mu_{\mathcal{C}}(\cdot, d, \varepsilon)$ and $\mu_{\mathcal{C}}(\cdot, d)$ are outer measures on X. It turns out that for any set $Y \subset X$ there exists a unique number $d_{\mathrm{cr}}(Y) \in \overline{\mathbb{P}}$ having the property that

$$\mu_{\mathcal{C}}(Y, d) = \begin{cases} 0 & \text{for} \quad d > d_{\mathrm{cr}}(Y) \\ +\infty & \text{for} \quad d < d_{\mathrm{cr}}(Y) \end{cases}$$

holds for $d \in \mathbb{P}$. This critical value $d_{\mathrm{cr}}(Y)$ is called *Carathéodory dimension* $\dim_{\mathcal{C}} Y$ *of Y with respect to the structure* $(\mathcal{F}, \mathbb{P}, \xi, \eta, \psi)$.

Note that our system of conditions (A1)–(A4) which leads to a Carathéodory structure is slightly different from the system in [41, 42]. In contrast to these works we assume that our family of objects in the Carathéodory construction depends on parameters which come from a (possibly proper) subset of $\mathbb{R}$.

For a standard Carathéodory structure let X be a separable metric space, $\mathcal{F}$ the family consisting of open balls $B(u,r)$ in X with center u and radius r and the empty set, $\mathbb{P}=\mathbb{R}_+$, $\xi(B(u,r),d)=r^d$, $\eta(B(u,r),s)=r^s$, $\psi(B(u,r))=r$, $\xi(\emptyset,d)=\psi(\emptyset)=0$, and $\eta(\emptyset,s)=1$ for each $u\in X$, $r>0$ and each $d\geq 0$, $s\in\mathbb{R}$. It is easy to see that such a system $(\mathcal{F},\mathbb{P},\xi,\eta,\psi)$ defines a Carathéodory structure on X. We denote by $\mu_H(\cdot,d,r)$, $\mu_H(\cdot,d)$ and $\dim_H$ the resulting Carathéodory measures and Carathéodory dimension which are in fact the Hausdorff d-measure at level r, the Hausdorff d-measure and the Hausdorff dimension, respectively. The concept of the Carathéodory dimension covers not only several dimension type characteristics of sets but also characteristics of dynamical systems such as topological pressure and topological entropy (see [41, 42]) or a dimension introduced for Poincaré recurrences ([1]).

Let (M,g) be a smooth n-dimensional Riemannian manifold and ρ the metric induced by g. For a piecewise smooth curve $\gamma\colon I\to M$ $(I\subset\mathbb{R}$ an interval) of finite length and arbitrary $\varepsilon>0$ we define the *ε-tubular neighborhood* $\Omega(\gamma,\varepsilon)$ of γ by

$$\Omega(\gamma,\varepsilon)=\bigcup_{u\in\gamma(I)}B(u,\varepsilon),$$

where $B(u,\varepsilon)=\{p\in M\mid \rho(u,p)<\varepsilon\}$ is again a metric ε-ball on M centered in the point u. For simplicity we call the ε-tubular neighborhood $\Omega(\gamma,\varepsilon)$ around the curve γ of length l shortly *tube of length l*.

For a given compact set $K\subset M$ and a given number $l_0>0$ we denote by $\Gamma=\{\gamma\}$ a family of piecewise smooth curves of a finite length $l(\gamma)=l_0$ such that for any $\varepsilon>0$ the following condition is satisfied:

(A) K is contained in the union of ε-tubular neighborhoods $\Omega(\gamma,\varepsilon)$ with $\gamma\in\Gamma$.

Condition (A) guarantees the existence of arbitrarily fine covers of the set K which are generated by the family Γ. For a family Γ satisfying (A) we define a family of subsets $\mathcal{F}$, a parameter set $\mathbb{P}$, and the functions $\xi\colon\mathcal{F}\times\mathbb{P}\to[0,\infty)$, $\eta\colon\mathcal{F}\times\mathbb{R}\to[0,\infty)$, and $\psi\colon\mathcal{F}\to[0,\infty)$ by

$$\begin{gathered}\mathcal{F}=\{\Omega(\gamma,\varepsilon)\cap K\mid\gamma\in\Gamma,\ \varepsilon>0\}\cup\{\emptyset\},\qquad\mathbb{P}=[1,+\infty),\\ \xi(\Omega(\gamma,\varepsilon)\cap K,d)=\varepsilon^{d-1},\quad\eta(\Omega(\gamma,\varepsilon)\cap K,s)=\varepsilon^s,\\ \psi(\Omega(\gamma,\varepsilon)\cap K)=\varepsilon\end{gathered}\tag{3.1}$$

for $\gamma\in\Gamma$, $\varepsilon>0$ with $\Omega(\gamma,\varepsilon)\cap K\neq\emptyset$, $\xi(\emptyset,d)=\psi(\emptyset)=0$, and $\eta(\emptyset,s)=1$ for all $d\in\mathbb{P}$, $s\in\mathbb{R}$.

Straight forward, one can verify that the collection $(\mathcal{F},\mathbb{P},\xi,\eta,\psi)$ defined via (3.1) with Γ satisfying (A) is a Carathéodory structure on K in the sense as considered above. In the sequel we will call such a structure simply a *Carathéodory structure with tubes of length l_0 on K* or *tubular Carathéodory structure on K*, if the underlying set K and the family Γ are clear from the context. The next proposition shows the relations between the Carathéodory measures and the Hausdorff measures, as well as between the Carathéodory dimension and the Hausdorff dimension, generated by this structure. For the proof we refer to [15, 16] and for the $\mathbb{R}^n$-case to [27].

Proposition 3.1 *Suppose that K is a compact set on the smooth n-dimensional Riemannian manifold (M,g). Suppose that $(\mathcal{F},\mathbb{P},\xi,\eta,\psi)$ is a tubular Carathéodory structure on K with tubes of length l_0 defined by (3.1) and with respect to this structure let be $\mu_{\mathcal{C}}(\cdot,d,\varepsilon)$, $\mu_{\mathcal{C}}(\cdot,d)$, and $\dim_{\mathcal{C}}$ the Carathéodory d-measure at level ε, the Carathéodory d-measure, and the Carathéodory dimension, respectively. Then there exist two numbers $k>0$ and $\varepsilon_0>0$ depending only on K such that for any set $Y\subset K$ and any $d\ge 1$ the inequality*

$$\mu_H(Y,d,\varepsilon)\le l_0 k\mu_{\mathcal{C}}(Y,d,\varepsilon) \tag{3.2}$$

holds for all $\varepsilon\in(0,\varepsilon_0]$. Therefore, we have

$$\mu_H(Y,d)\le l_0k\mu_{\mathcal{C}}(Y,d) \quad \textit{and thus} \quad \dim_H Y\le \dim_{\mathcal{C}} Y.$$

Now we specify the family Γ of curves which will be used further for the considerations of sets being negatively invariant with respect to a flow. As in the previous section we consider the complete C^2-vector field $f\colon M\to TM$ on a smooth n-dimensional Riemannian manifold and the corresponding differential equation (2.2) with global flow $\{\varphi^t\}_{t\in\mathbb{R}}$. Let K and $\widetilde{K}$ be two compact sets in M satisfying

$$K\subset\varphi^t(K)\subset\widetilde{K} \quad \text{for all} \quad t\ge 0. \tag{3.3}$$

(A set K satisfying $K\subset\varphi^t(K)$ for all $t\ge0$ is usually called *negatively invariant* with respect to the flow.) At first we suppose that the set K does not contain equilibrium points of (2.2).

To construct the family Γ we denote by Λ the set of all equilibrium points of (2.2) in $\widetilde{K}$ and set $e_1=\frac{1}{2}\operatorname{dist}(\Lambda,K)$, where $\operatorname{dist}(\Lambda,K)=\inf\limits_{u\in\Lambda,\,p\in K}\rho(u,p)$ is the usual metric distance between two sets in M, and define

$$\Phi:=\widetilde{K}\cap\bigcup_{p\in K}B(p,e_1). \tag{3.4}$$

With respect to the vector field f, the compact set $\widetilde{K}$ from (3.3), and the set Φ from (3.4) define the following coefficient

$$V(f,\widetilde{K},\Phi):=\frac{\max\limits_{u\in\widetilde{K}}\|f(u)\|_{T_uM}}{\min\limits_{u\in\Phi}\|f(u)\|_{T_uM}}, \tag{3.5}$$

which will be important for the proofs in Section 4. For any $p\in K$ we take a time $b_p>0$ such that $\varphi^t(p)\in\Phi$ for all $t\in[0,b_p]$. Further, since $d_p\varphi^t|_{t=0}=\operatorname{id}_{T_pM}$ we can suppose that $\|d_p\varphi^t\|\le 2$ holds for all $t\in[0,b_p]$. Since K is compact and contains no equilibrium points of f there exists a number $e_2>0$ such that for the length of the integral curve pieces it holds $l(\varphi(\cdot,p)|_{[0,b_p]})\ge e_2$ for any $p\in K$. We set

$$l_0:=\frac{1}{2}\min\{e_1,e_2\},$$

introduce for any $q\in K$ the number $\tau(q)>0$ satisfying $l(\varphi(\cdot,q)|_{[0,\tau(q)]})=l_0$, and define the set

$$\Gamma:=\{\varphi(\cdot,q)|_{[0,\tau(q)]}\mid q\in K\}. \tag{3.6}$$

Obviously this family Γ satisfies condition (A) and $(\mathcal{F}, \mathbb{P}, \xi, \eta, \psi)$ defined by (3.1) on the base of this family is a Carathéodory structure on K – a Carathéodory structure with tubes of length l_0 – which will be used in Section 4.

4 Dimension Estimates of Flow Negatively Invariant Sets

In the present section we derive upper bounds for the Hausdorff dimension of compact sets being negatively invariant with respect to the flow of the differential equation (2.2). Investigating the deformation of such a set under shift maps generated by the flow the deformation transversal to the flow lines is of great importance.

Our main result is the following theorem which generalizes the results of [26, 27] to vector fields on manifolds. Recall that for $d \in \mathbb{R}$ we denote by $\lfloor d \rfloor$ the largest integer less than d.

Theorem 4.1 *Let $f\colon M \to TM$ be the C^2-vector field (2.2) on the smooth n-dimensional $(n \geq 2)$ Riemannian manifold (M, g) satisfying the following conditions:*

(a) *The flow $\{\varphi^t\}_{t\in\mathbb{R}}$ of (2.2) satisfies (3.3) with respect to the compact sets K and $\widetilde{K}$ in M, where K does not contain equilibrium points of (2.2).*

(b) *For a regular point $p \in \widetilde{K}$ let $\beta_1(p) \geq \cdots \geq \beta_{n-1}(p)$ be the eigenvalues of the symmetric part $SA(p) = \frac{1}{2}[A(p) + A(p)^*]$ restricted to the subspace $T^\perp(p)$, where $A(p)$ is the operator from (2.5). There exist a number $d \in (0, n-1]$, a number $\Theta > 0$, and a time $T_0 > 0$ such that*

$$\int_0^{T_0} \left[\beta_1(\varphi^\tau(p)) + \cdots + \beta_{\lfloor d \rfloor}(\varphi^\tau(p)) + (d - \lfloor d \rfloor)\beta_{\lfloor d \rfloor + 1}(\varphi^\tau(p))\right] d\tau \leq -\Theta \qquad (4.1)$$

is satisfied for all regular points $p \in \widetilde{K}$.

Then it holds $\dim_H K < d + 1$. If $d = 1$ we have $\dim_H K \leq 1$.

Before proving Theorem 4.1 we formulate some lemmata. The special flow line structure of sets which are flow negatively invariant allows us to obtain the dimension estimate. In order to describe the deformation under the map φ^t of tubular neighborhoods around an arc of a trajectory we investigate the evolution of time translated pieces of hypersurfaces lying transversal to the considered trajectory. In the next lemma we consider the influence of φ^t on arcs of a trajectory.

For an arbitrary piecewise smooth curve $c\colon [t_1, t_2] \to M$ we denote its length by $l(c)$.

Lemma 4.1 *Suppose that $\{\varphi^t\}_{t\in\mathbb{R}}$ is the flow of (2.2), Φ and $\widetilde{K}$ are compact sets in M, Φ does not contain any equilibrium points of (2.2), and $V(f, \widetilde{K}, \Phi)$ is the coefficient from (3.5). Let $p \in \Phi$ and let $c^t\colon [t_1, t_2] \to M$ be a restriction of the integral curve of (2.2) through p given by $c^t(\cdot) = \varphi(t + \cdot, p)|_{[t_1,t_2]}$ and satisfying $c^0([t_1, t_2]) \subset \Phi$ and $c^t([t_1, t_2]) \subset \widetilde{K}$ for all $t > 0$. Then the length $l(c^t)$ of such a restriction satisfies $l(c^t) \leq V(f, \widetilde{K}, \Phi)l(c^0)$ for all $t \geq 0$.*

Proof The statement follows immediately from

$$l(c^t) = \int_{t_1}^{t_2} \|\dot{\varphi}(\tau, \varphi^t(p))\|\, d\tau = \int_{t_1}^{t_2} \frac{\|\dot{\varphi}(\tau + t, p)\|}{\|\dot{\varphi}(\tau, p)\|} \|\dot{\varphi}(\tau, p)\|\, d\tau$$

$$\leq V(f, \widetilde{K}, \Phi)l(c^0).$$

We consider now the family Γ of curves of length l_0 from (3.6) and the chosen Carathéodory structure $(\mathcal{F}, \mathbb{P}, \xi, \eta, \psi)$ on the compact set K with tubes of length l_0 from (3.1). The next lemma estimates the tubular measures $\mu_{\mathcal{C}}(\cdot, d, \varepsilon)$, generated with respect to $(\mathcal{F}, \mathbb{P}, \xi, \eta, \psi)$, of the flow-transformed set K. Its proof is based on the consideration of the deformation of tubular neighborhoods around trajectory pieces.

Lemma 4.2 *Suppose that $\{\varphi^t\}_{t\in\mathbb{R}}$ is the flow of (2.2) satisfying (3.3) with respect to the compact sets K and $\widetilde{K}$ in M, where K does not contain equilibrium points of (2.2). Suppose also that Φ, $V(f, \widetilde{K}, \Phi)$ and l_0 are given by (3.4), (3.5), and (3.6), respectively. For $p \in \widetilde{K}$ let $\alpha_1(p)$ be the largest eigenvalue of $S\nabla f(p)$, and for a regular point $p \in \widetilde{K}$ let $\beta_1(p) \geq \cdots \geq \beta_{n-1}(p)$ be the eigenvalues of $SA(p)|_{T^\perp(p)}$, where $A(p)$ is the operator from (2.5). Define for a number $d \in (0, n-1]$ and a time $T_0 > 0$ the values*

$$
\begin{aligned}
k &:= \max_{p\in K} \exp\left\{ \int_0^{T_0} \big[\beta_1(\varphi^\tau(p)) + \cdots + \beta_{\lfloor d\rfloor}(\varphi^\tau(p)) \right. \\
&\qquad \left. + (d - \lfloor d\rfloor)\beta_{\lfloor d\rfloor+1}(\varphi^\tau(p))\big]\, d\tau \right\}, \\
a &:= \exp\left[3l_0 \max_{p\in\widetilde{K}} \alpha_1(p) \frac{V(f, \widetilde{K}, \Phi)}{\min\limits_{p\in\Phi} \|f(p)\|_{T_pM}} \right], \\
\lambda &:= 2^6 \sqrt{\lfloor d\rfloor + 1}\, a, \quad \text{and} \quad C := \big(3V(f, \widetilde{K}, \Phi) + 1\big) 2^{\lfloor d\rfloor} \lambda^d.
\end{aligned}
\tag{4.2}
$$

Then for any $l > k$ there exists an $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$ the Carathéodory $(d+1)$-measure $\mu_{\mathcal{C}}(\cdot, d+1, \varepsilon)$ at level ε, generated with respect to the Carathéodory structure (3.1) with tubes of length l_0, satisfies the inequality

$$
\mu_{\mathcal{C}}(\varphi^{T_0}(K) \cap K, d+1, \lambda l^{1/d} \varepsilon) \leq C l \mu_{\mathcal{C}}(K, d+1, \varepsilon). \tag{4.3}
$$

Proof Fix some $\gamma \in \Gamma$. For arbitrary $l > k$ we can choose an $\varepsilon_1 > 0$ such that the set $V := \bigcup\limits_{p\in K} B(p, \varepsilon_1)$ contains no equilibrium points of (2.2) and that the inequality

$$
\begin{aligned}
k' &:= \max_{u\in\overline{V}} \exp\left\{ \int_0^{T_0} \big[\beta_1(\varphi^\tau(u)) + \cdots + \beta_{\lfloor d\rfloor}(\varphi^\tau(u)) \right. \\
&\qquad \left. + (d - \lfloor d\rfloor)\beta_{\lfloor d\rfloor+1}(\varphi^\tau(u))\big]\, d\tau \right\} < l
\end{aligned}
\tag{4.4}
$$

is satisfied. We set

$$
\sigma := \max_{p\in\overline{V}} \exp\left\{ \int_0^{T_0} \beta_1(\varphi^\tau(p))\, d\tau \right\} \tag{4.5}
$$

and take a number $m > 0$ such that $k' < m^d$ and $\sigma \leq m$ are satisfied. Since $l > k'$ the equation

$$
\left[1 + \left(\frac{m^{\lfloor d\rfloor}}{k'} \right)^{1/(1-\lfloor d\rfloor)} \eta \right]^d k' = l
$$

uniquely defines a number $\eta > 0$.

Choose $\delta > 0$ such that for any $u \in \widetilde{K}$ the map $\exp_u$ maps the ball $B(O_u, \delta) \subset T_u M$ diffeomorphically onto the geodesic ball $B(u, \delta) \subset M$. Further with $\|d_{O_u} \exp_u\| = 1$ we can suppose that $\|d_v \exp_u\| \leq 2$ and therefore

$$\rho(\exp_u v_1, \exp_u v_2) \leq 2\rho(v_1, v_2)$$

holds for all $v,\, v_1,\, v_2 \in B(O_u, \delta)$.

To simplify the use of the reparametrized local flow we cover $\Omega(\gamma, r)$ by a set $S(\gamma_p, r)$ as follows. Let for some $p \in K$ and the associated time $t(p) > 0$ be $\gamma_p(\cdot) = \varphi(\cdot, p)|_{[0,t(p)]}$ the integral curve of lenght $2l_0$ such that $\gamma_p \supset \gamma$ and for any $r \in (0, l_0]$ the inclusion $\Omega(\gamma, r) \subset S(\gamma_p, r)$ holds, where

$$S(\gamma_p, r) := \bigcup_{u \in \gamma_p} B^{\perp}(u, r).$$

Let p and $t(p)$ be fixed in the sequel. We take now

$$\varepsilon_0(\gamma) < \frac{1}{4} \min\{\varepsilon_1, \delta, \operatorname{dist}(K, M \backslash V), l_0\}$$

small enough such that the following conditions are satisfied:

(1) The function $s\colon [0, \max\{T_0, t(p)\}] \times B^{\perp}(p, 4\varepsilon_0(\gamma)) \to \mathbb{R}_+$ as characterized in the Lemma 2.3 defines a local reparametrization of the flow φ by $\phi\colon [0, \max\{T_0, t(p)\}] \times B^{\perp}(p, 4\varepsilon_0(\gamma)) \to M$ with $\phi(t, \cdot) \equiv \phi^t(\cdot) := \varphi^{s(t,\cdot)}(\cdot)$ for $t \in [0, \max\{T_0, t(p)\}]$.

(2) $\phi^{T_0}(B^{\perp}(p, 4\varepsilon_0(\gamma))) \subset B(\varphi^{T_0}(p), \delta)$.

(3) The distance between the points $\phi^t(u)$ on an integral curve starting in $u = \exp_p(rv) \in B^{\perp}(p, \varepsilon_0(\gamma))$ and the reference orbit through p for a fixed $t \in [0, t(p)]$ is of the size

$$\rho(\varphi^t(p), \phi^t(u)) = \|d_p \phi^t\| \cdot r(1 + O(r))$$

as $r \to 0$. It holds $\|d_p \phi^t\| \leq \|d_p \varphi^t\|$ and $\|d_p \varphi^t\| \leq 2$ for any $t > 0$ such that $l(\varphi([0,t],p)) \leq 2l_0$. Thus, for any $u \in B^{\perp}(p, \varepsilon_0(\gamma))$ it is $\rho(\varphi^t(p), \phi^t(u)) \leq 4\rho(p, u)$ for any such t. We can assume analogous assumptions for the flow in reverse time-direction. Let for $\varepsilon_0(\gamma) > 0$ the following be satisfied: If $\gamma' = \phi([0, t(p)], u)$ is some arc of trajectory intersecting $S(\gamma_p, \varepsilon_0(\gamma))$ then γ' is completely contained in $S(\gamma_p, 4\varepsilon_0(\gamma))$ and satisfies $l(\gamma') \leq 3l_0$.

(4) For any $u \in \widetilde{K}$ and for any time $\tau > 0$ such that the integral curve $\varphi([0, \tau], u)$ is of maximal lenght $3l_0 V(f, \widetilde{K}, \Phi)$ it holds

$$\sup_{q \in B(u, 16\sigma\varepsilon_0(\gamma))} \left\| \tau_{\varphi^t(q)}^{\varphi^t(u)} d_q \varphi^t \tau_u^q - d_u \varphi^t \right\| \leq a \quad \text{for all} \quad t \in (0, \tau). \tag{4.6}$$

Suppose that it holds

$$\sup_{q \in B^{\perp}(p, 4\varepsilon_0(\gamma))} \left\| \tau_{\phi^{T_0}(q)}^{\phi^{T_0}(p)} d_q \phi^{T_0} \tau_p^q - d_p \phi^{T_0} \right\| \leq \eta. \tag{4.7}$$

(5) For any $u = u(r,v) \in B^{\perp}(p, 4\varepsilon_0(\gamma))$ the deviation arising from the local reparametrization of the flow is of the form $s(T_0, u(r,v)) - T_0 = O(r)$ as $r \to 0$ which gives for the point $\phi^{T_0}(u) = \varphi^{s(T_0,u)-T_0}(\varphi^{T_0}(u))$ the representation

$$\exp^{-1}_{\varphi^{T_0}(u)}(\phi^{T_0}(u)) = O_{\varphi^{T_0}(u)} + f(\varphi^{T_0}(u))O(r) + o(r)$$

as $r \to 0$. The vector field C^2-varies on M. So we can suppose that for any point $u \in B^{\perp}(\varphi^{T_0}(p), \delta)$ for $\nu < 2^4\sqrt{\lfloor d \rfloor + 1}\,\sigma\varepsilon_0(\gamma)$ any set $(\varphi^{T_0} \circ \phi^{-T_0})B(u,\nu)$ is contained in a 2ν-tubular neighborhood of a curve $\varphi(\cdot, (\varphi^{T_0} \circ \phi^{-T_0})(u))|_{(-\tau,\tau)}$ of some finite length, say of length l_0.

Now let $r \le \varepsilon_0(\gamma)$. Suppose $\varphi^{T_0}(\Omega(\gamma, r)) \cap K \neq \emptyset$. The set $B(p, 4r)$ is contained in the open set V. Taylor's formula for the differentiable map ϕ^{T_0} provides ([39]) that for every $u \in B^{\perp}(p, 4r)$

$$\begin{aligned} &\left\| \exp^{-1}_{\phi^{T_0}(p)} \phi^{T_0}(u) - d_p\phi^{T_0}(\exp_p^{-1}(u)) \right\| \\ &\le \sup_{q \in B(p,4r)} \left\| \tau^{\phi^{T_0}(p)}_{\phi^{T_0}(q)} d_q\phi^{T_0}\tau_p^q - d_p\phi^{T_0} \right\| \cdot \left\| \exp_p^{-1}(q) \right\| \end{aligned} \tag{4.8}$$

holds. Considering the image of $B^{\perp}(p, 4r)$ under ϕ^{T_0} with (4.7) we obtain the inclusion

$$\exp^{-1}_{\phi^{T_0}(p)}\left(\phi^{T_0}\left(B^{\perp}(p,4r)\right)\right) \subset d_p\phi^{T_0}\left(B^{\perp}(O_p, 4r)\right) + B^{\perp}\left(O_{\varphi^{T_0}(p)}, \eta 4r\right).$$

The set $d_p\phi^{T_0}\left(B^{\perp}(O_p, 4r)\right)$ is an ellipsoid with half-axes of length $4r\sigma_k(p)$, where $\sigma_k(p)$ $(k = 1, \dots, n-1)$ denote the singular values of the linear operator $d_p\phi^{T_0}\colon T^{\perp}(p) \to T^{\perp}(\varphi^{T_0}(p))$. Using the definition of k', Lemma 2.2 and (2.8) we conclude

$$\omega_d\left(d_p\phi^{T_0}(B^{\perp}(O_p, 4r))\right) \le (4r)^d k'. \tag{4.9}$$

By standard covering results (see e.g. [39]) an ellipsoid $\mathcal{E} \subset T^{\perp}(\varphi^{T_0}(p))$ can be found containing $d_p\phi^{T_0}\left(B^{\perp}(O_p, 4r)\right) + B\left(O_{\varphi^{T_0}(p)}, \eta 4r\right)$ and satisfying $\omega_d(\mathcal{E}) \le l(4r)^d$. Any set $\mathcal{E}$ can be covered by N balls of radius $R = \sqrt{\lfloor d \rfloor + 1}\,\sigma_{\lfloor d \rfloor + 1}(\mathcal{E})$. The number N can be estimated from above by

$$N \le \frac{2^{\lfloor d \rfloor}\omega_d(\mathcal{E})}{\sigma_{\lfloor d \rfloor + 1}(\mathcal{E})^d}\,.$$

Thus, any set $\exp_{\varphi^{T_0}(p)}(\mathcal{E})$ and therefore $\phi^{T_0}(B^{\perp}(p, 4r))$ can be covered by N geodesic balls in M of radius $2R$. Fixing such a cover $\{B(\widetilde{u}_j, 2R)\}_{j \ge 1}$, where $\widetilde{u}_j \in M$ $(j \ge 1)$, we choose in every set

$$K \cap B(\widetilde{u}_j, 2R) \cap B^{\perp}(\varphi^{T_0}(p), \delta)$$

a point u_j and obtain the cover $\{B_j\}_{j \ge 1}$ of the set $\phi^{T_0}(B^{\perp}(p, 4r)) \cap K$ with $B_j = B(u_j, 4R) \cap B^{\perp}(\varphi^{T_0}(p), \delta)$.

Now we consider the deviation arising from the reparametrization. By the property (5) any set $(\varphi^{T_0} \circ \phi^{-T_0})(B_j)$ is with precision $o(r)$ $(r \le \varepsilon_0(\gamma))$ contained in a

$4R$-neighborhood of the orbit trough u_j, or more precise, in an $8R$-neighborhood of a trajectory piece $\varphi(\cdot,(\varphi^{T_0}\circ\phi^{-T_0})(u_j))|_{(-\tau,\tau)}$ of length l_0.

By the choise of $\varepsilon_0(\gamma)$ any trajectory piece in $S(\gamma_p,4r)$ which intersects $S(\gamma_p,r)$ is of maximal length $3l_0$. We shift the balls $B((\varphi^{T_0}\circ\phi^{-T_0})(u_j),8R)$ along the flow lines. Thus, with the above and (4.6) the set $\varphi^{T_0}(S(\gamma_p,r))$ can be covered by N tubes of length $3l_0V(f,\widetilde{K},\Phi)+l_0$ and diameter $2a\cdot 8R$.

Covering each curve arc by curve arcs of length l_0 we conclude

$$\begin{aligned}&\mu_{\mathcal{C}}\Big(\varphi^{T_0}(\Omega(\gamma,r))\cap K,\ d+1,2^6\sqrt{\lfloor d\rfloor+1}\,l^{1/d}ar\Big)\\&\le N(3V(f,\widetilde{K},\Phi)+1)\Big(2^6a\sqrt{\lfloor d\rfloor+1}\,\sigma_{\lfloor d\rfloor+1}(\mathcal{E})\Big)^d\le Clr^d.\end{aligned}\tag{4.10}$$

Since Γ is the set of trajectory pieces starting in a point p in the compact set K we can pass to $\varepsilon_0:=\inf_{\gamma\in\Gamma}\varepsilon_0(\gamma)>0$ such that the (4.10) holds for any $\Omega(\gamma,r)$ with $\gamma\in\Gamma$ and $r\le\varepsilon_0$. Let $\varepsilon\le\varepsilon_0$. For any $\nu>0$ there exists a finite family $\{\Omega(\gamma_i,r_i)\}_{i\ge1}$ with $\gamma_i\in\Gamma,\ r_i\le\varepsilon$ having the property that $\bigcup_i\Omega(\gamma_i,r_i)\supset K$ and $\sum_i r_i^d\le\mu_{\mathcal{C}}(K,d+1,\varepsilon)+\nu$. We obtain $\mu_{\mathcal{C}}(\varphi^{T_0}(K)\cap K,d+1,\lambda l^{1/d}\varepsilon)\le\sum_i\mu_{\mathcal{C}}(\varphi^{T_0}(\Omega(\gamma_i,r_i))\cap K,d+1,\lambda l^{1/d}\varepsilon)\le Cl\sum_i r_i^d\le Cl(\mu_{\mathcal{C}}(K,d+1,\varepsilon)+\nu)$, where λ and C are defined by (4.2). Since ν has been chosen arbitrarily we obtain that (4.3) holds for any $\varepsilon\in(0,\varepsilon_0]$.

Although we are mainly interested in upper estimates of the Hausdorff dimension of flow negatively invariant sets we can deduce upper bounds of its Carathéodory dimension with respect to the chosen tubular Carathéodory structure.

Proposition 4.1 *Let the differential equation (2.2) satisfy the conditions of Theorem 4.1 with the number $d\in(0,n-1]$ in (4.1) and the negatively invariant set K. Then the Carathéodory dimension of K, determined with respect to the Carathéodory structure (3.1) on K consisting of tubes with length l_0 determined in (3.6), satisfies*

$$\dim_{\mathcal{C}}K<d+1.$$

Proof It follows from (4.1) that for an arbitrarily small number $\varkappa\in(0,1)$ there exists some number $m=m(\varkappa)>0$ such that

$$\begin{aligned}k:=\sup_{p\in K}\exp\Bigg\{&\int_0^{mT_0}\big[\beta_1(\varphi^\tau(p))+\cdots+\beta_{\lfloor d\rfloor}(\varphi^\tau(p))\\&+(d-\lfloor d\rfloor)\beta_{\lfloor d\rfloor+1}(\varphi^\tau(p))\big]\,d\tau\Bigg\}\le\exp(-m\Theta)<\varkappa.\end{aligned}\tag{4.11}$$

Without loss of generality we can assume that this number k satisfies $\lambda k^{1/d}<1$ and $Ck<1$, where λ and C are the constants given in (4.2). We choose $l>k$ with $\lambda l^{1/d}<1$ and $Cl<1$. Lemma 4.2, applied to the map φ^{mT_0}, guarantees that for the chosen number l there exists a number $\varepsilon_0>0$ such that for all $\varepsilon\in(0,\varepsilon_0]$ the inequality

$$\mu_{\mathcal{C}}\big(\varphi^{mT_0}(K)\cap K,d+1,\lambda l^{1/d}\varepsilon\big)\le Cl\mu_{\mathcal{C}}(K,d+1,\varepsilon)\tag{4.12}$$

holds. Let $\varepsilon \in (0, \varepsilon_0]$ be arbitrarily small. Since K is compact the value $\mu_{\mathcal{C}}(K, d+1, \varepsilon)$ is finite. Since K is negatively invariant with respect to φ^{mT_0} we have $K = \varphi^{mT_0}(K) \cap K$. Using inequality $\lambda l^{1/d} < 1$ we conclude $\mu_{\mathcal{C}}(K, d+1, \varepsilon) < CL\mu_{\mathcal{C}}(K, d+1, \varepsilon)$. From this we follow that the equality $\mu_{\mathcal{C}}(K, d+1, \varepsilon) = 0$ holds for every $\varepsilon \in (0, \varepsilon_0]$. We see that $\mu_{\mathcal{C}}(K, d+1) = 0$. This implies $\dim_{\mathcal{C}} K \le d+1$. Since (4.11) holds true if we slightly reduce d we conclude $\dim_{\mathcal{C}} K < d+1$.

Proof of Theorem 4.1 Applying Proposition 4.1 and Proposition 3.1 we obtain $\dim_H K < d+1$. If condition (4.1) is also satisfied for $d = 1$ it is satisfied for all $d \in (0, n-1]$. Thus, $\dim_H K < d+1$ for all $d \in (0, n-1]$ and we obtain $\dim_H K \le 1$. This proves the Theorem.

Let us again consider compact sets K and $\widetilde{K}$ in M satisfying (3.3) with respect to the flow of (2.2). We may now assume that the set K possesses equilibrium points and satisfies the following condition:

(S) The set K contains at most a finite number of equilibrium points of (2.2). Every such equilibrium point possesses a local stable manifold with dimension at least $n-1$. Trajectories starting in local unstable manifolds or local center manifolds of such an equilibrium point in K converge for $t \to +\infty$ to an asymptotically stable equilibrium point of (2.2) in $\widetilde{K}$.

The special structure of equilibrium points satisfying (S) allows us to obtain the following theorem. The reason for this is that in some sense in open and flow positively invariant neighborhoods of these points the flow preserves its contracting property with respect to the Hausdorff measure ([16]).

Theorem 4.2 *Let $f\colon M \to TM$ be a C^2-vector field (2.2) on the smooth n-dimensional Riemannian manifold (M, g). Suppose that the flow $\{\varphi^t\}_{t\in\mathbb{R}}$ of (2.2) satisfies (3.3) and condition* (S) *with respect to compact sets K and $\widetilde{K}$ in M. Suppose also that condition (b) of Theorem 4.1 is satisfied. Then the conclusion of Theorem 4.1 holds.*

In the following statement we denote for a differentiable function $v\colon U \subset M \to \mathbb{R}$, U an open set, by $L_f v(p)$ the Lie derivative of v in p in direction of the vector field f.

Corollary 4.1 *Suppose that the flow $\{\varphi^t\}_{t\in\mathbb{R}}$ of (2.2) satisfies (3.3) and condition* (S) *with respect to compact sets K and $\widetilde{K}$ in M.*

Denote by Λ the set of equilibrium points of (2.2) in M. For $p \in M\backslash\Lambda$ let $\beta_1(p) \ge \ldots \ge \beta_{n-1}(p)$ be the eigenvalues of the symmetric part $SA(p)$ restricted to the subspace $T^{\perp}(p)$, where $A(p)$ is the operator from (2.5), and let $v\colon M\backslash\Lambda \to \mathbb{R}$ be a C^1-function. Suppose also that for a number $d \in (0, n-1]$ there exist a number $\Theta > 0$ and a time $T_0 > 0$ such that

$$\int_0^{T_0} \big[\beta_1(\varphi^\tau(p)) + \cdots + \beta_{\lfloor d\rfloor}(\varphi^\tau(p)) + (d - \lfloor d\rfloor)\beta_{\lfloor d\rfloor+1}(\varphi^\tau(p)) + L_f v(\varphi^\tau(p))\big]\, d\tau \le -\Theta \tag{4.13}$$

holds for all regular points $p \in \widetilde{K}$. Then the conclusion of Theorem 4.1 holds.

Proof As mentioned above, on open and flow positively invariant neighborhoods of equilibrium points of (2.2) which satisfy (S) the flow preserves its contracting property with respect to the Hausdorff measure. So it remains to show that for any compact, flow negatively invariant set $K_1 \subset K$ which does not contain equilibrium points of (2.2) it holds $\dim_H K_1 < d+1$. On $M\backslash\Lambda$ we introduce a new metric tensor by $\hat{g}(p) := \exp\left(\frac{2v(p)}{d}\right) g(p)$ for $p \in M\backslash\Lambda$. On K_1 the Riemannian metric $\hat{g}$ is equivalent to g. Changing to the metric $\hat{g}$ does not alter the Hausdorff dimension of the compact set K_1. Consider the operator $\hat{A}(p)$ from (2.5), the symmetric part $S\hat{A}(p)$ of $\hat{A}(p)$, the operator $\hat{\nabla} f(p)$, and $S\hat{\nabla} f(p)$, which are defined regarding to the scalar product in T_pM induced by the metric $\hat{g}$. As in [39] one shows that $S\hat{\nabla} f(p) = S\nabla f(p) + \frac{L_f v(p)}{d}\,\mathrm{id}_{T_pM}$. Using (2.7) we obtain that for a regular point $p \in M$ the eigenvalues $\hat{\beta}_i(p)$ of the operator $S\hat{A}(p)|_{T^\perp(p)}$ are related to the eigenvalues $\beta_i(p)$ $(i = 1, \ldots, n-1)$ with respect to the original metric g by $\hat{\beta}_i(p) = \beta_i(p) + \frac{L_f v(p)}{d}$. Therefore,

$$\begin{aligned} &\hat{\beta}_1(p) + \cdots + \hat{\beta}_{\lfloor d\rfloor}(p) + (d - \lfloor d\rfloor)\hat{\beta}_{\lfloor d\rfloor+1}(p) \\ &= \beta_1(p) + \cdots + \beta_{\lfloor d\rfloor} + (d - \lfloor d\rfloor)\beta_{\lfloor d\rfloor+1}(p) + L_f v(p) \end{aligned}$$

guarantees (4.13) and thus (4.1) of Theorem 4.1. Hence $\dim_H K_1 < d+1$.

Corollary 4.2 *Consider a 2-dimensional Riemannian manifold M. Suppose that the flow $\{\varphi^t\}_{t\in\mathbb{R}}$ of (2.2) satisfies (3.3) and condition* (S) *with respect to compact sets K and $\widetilde{K}$ in M. If* div $f(p) < 0$ *holds for any regular points $p \in \widetilde{K}$ then* $\dim_H K \le 1$.

Proof For the operator $A(p)$ from (2.5) it holds $\mathrm{tr}\,(SA(p)|_{T^\perp(p)}) = \mathrm{tr}\,\nabla f(p) - \langle \nabla f(p) f(p), f(p)\rangle / \|f(p)\|^2$. We define the C^1-function v on the set of all regular points p in M by $v(p) = \frac{1}{2}\ln\|f(p)\|^2$. The statement follows with Corollary 4.1.

5 Flow Invariant Sets with an Equivariant Tangent Bundle Splitting

The considered outer measures defined via tube covers show in many cases a better contraction behavior under the flow operator of a vector field in positive time direction than conventional outer measures defined via a covering of balls do. Using such an approach for a class of generalized hyperbolic flows on n-dimensional Riemannian manifolds we may improve upper Hausdorff dimension estimates which are obtained with methods from [39] (or from [45] for the $\mathbb{R}^n$).

Consider again the vector field $f\colon M \to TM$ from (2.2) on the smooth n-dimensional Riemannian manifold (M, g). Let us introduce a property of flow-invariant sets which may be considered as a generalized hyperbolic structure. We say that a flow-invariant compact set $K \subset M$ possesses an *equivariant tangent bundle splitting* (which for simplicity consists of only two components) $T_K M = E^1 \oplus E^2$ with respect to the flow $\{\varphi^t\}_{t\in\mathbb{R}}$ if for any $p \in K$ and $i = 1, 2$ the space $E^i_p = E^i \cap T_pM$ is an n_i-dimensional subspace of T_pM such that $n_1 + n_2 = n$ and $d_p\varphi^t(E^i_p) = E^i_{\varphi^t(p)}$ hold for any $p \in K$ and $t \in \mathbb{R}$. Recall that an *Anosov flow* on K is a flow without equilibria for which among other properties there exists an equivariant tangent bundle splitting $T_K M = E^1 \oplus E^2$, where $E^2_p = \mathrm{span}\,\{f(p)\}$ for each $p \in K$. For $d \in (0, n - n_2]$ and $t \in \mathbb{R}$ we introduce the *singular value function of order d of φ^t on K with respect to the splitting* $E^1 \oplus E^2$ which is defined by

$$\omega^{E^1,E^2}_{d,K}(\varphi^t) := \sup_{p\in K} \omega_d\big(d_p\varphi^t|_{E^1(p)}\big).$$

Since $\omega_{d,K}^{E^1,E^2}(\varphi^t)$ is a sub-exponential function the limit

$$\nu_d := \lim_{t\to\infty} \frac{1}{t} \ln \omega_{d,K}^{E^1,E^2}(\varphi^t)$$

exists for any $d \in (0, n-n_2]$ ([46]). We call the numbers

$$\nu_1^u := \nu_1, \quad \nu_i^u := \nu_i - \nu_{i-1} \quad \text{for} \quad i = 1, \dots, n-n_2$$

the *uniform Lyapunov exponents of* $\{\varphi^t\}$ *with respect to the splitting* $E^1 \oplus E^2$. Let us investigate the splitting $T_K M = E^2 \oplus E^2$ such that $E^1 = T^\perp$ with $E_p^1 = T^\perp(p)$ and $E^2 = T^\parallel$ with $E_p^2 = T^\parallel(p) = \operatorname{span}\{f(p)\}$.

With the help of Lemma 2.1 one shows that for any regular point $p \in M$ satisfying

$$\langle S\nabla f(p)z, f(p)\rangle = 0 \quad \text{for all} \quad z \in T^\perp(p) \tag{5.1}$$

the $n-1$ eigenvalues $\beta_1(p), \dots, \beta_{n-1}(p)$ of $SA(p)|_{T^\perp(p)}$, with the operator $A(p)$ from (2.5), coincide with $n-1$ eigenvalues of $S\nabla f(p)$. The subspace $T^\parallel(p)$ is the eigenspace of the remaining nth eigenvalue $\overline{\alpha}(p) = \langle \nabla f(p)f(p), f(p)\rangle / \|f(p)\|^2$ of $S\nabla f(p)$.

We consider now two compact sets K and $\widetilde{K}$ in M without equilibrium points of (2.2) satisfying (3.3) and suppose that (5.1) is satisfied for any $p \in \widetilde{K}$. By $\alpha_1(p) \ge \dots \ge \alpha_n(p)$ denote the eigenvalues of $S\nabla f(p)$. For that case Theorem 3.1 from [39] states that if for some $d \in (0, n]$ the inequality

$$\alpha_1(p) + \ldots + \alpha_{\lfloor d\rfloor}(p) + (d - \lfloor d\rfloor)\alpha_{\lfloor d\rfloor+1}(p) < 0$$

holds for all $p \in \widetilde{K}$, the estimate $\dim_H K < d$ is true. For the C^1-function $v\colon \widetilde{K} \to \mathbb{R}$ given by $v(p) = \frac{1}{2}\ln\|f(p)\|^2$ we have $L_f v(p) = \langle \nabla f(p)f(p), f(p)\rangle / \|f(p)\|^2 = \overline{\alpha}(p)$ for each $p \in \widetilde{K}$. If $\overline{\alpha}(p) \ge 0$ holds for all $p \in \widetilde{K}$ then

$$\begin{aligned} &\alpha_1(p) + \dots + \alpha_{\lfloor d\rfloor}(p) + (d - \lfloor d\rfloor)\alpha_{\lfloor d\rfloor+1}(p) \\ &= \beta_1(p) + \dots + \beta_{\lfloor d\rfloor-1}(p) + (d - \lfloor d\rfloor)\beta_{\lfloor d\rfloor}(p) + L_f v(p). \end{aligned}$$

With this Corollary 4.1 gives an upper bound of $\dim_H K$ which is less than or equal to the upper bound we would get applying Theorem 3.1 from [39]. If $d = 2$ then Corollary 4.1 gives the better estimate $\dim_H K \le 1$.

One easily shows that a compact, flow-invariant set K without equilibrium points possesses an equivariant tangent bundle splitting $T^\perp \oplus T^\parallel$ if and only if (5.1) holds for any $p \in K$. Obviously the flow $\{\varphi^t\}_{t\in\mathbb{R}}$ on K then is already reparametrized globally if one considers the reparametrization described in Lemma 2.3. For that case the assumptions of Theorem 4.1 can be weaken if we consider the long-time behavior.

Proposition 5.1 *Let* $f\colon M \to TM$ *be the* C^2*-vector field from (2.2) on the* n*-dimensional Riemannian manifold* (M, g)*. Suppose that* $K \subset M$ *is a compact and flow-invariant set without equilibrium points of (2.2) and that* K *possesses an equivariant tangent bundle splitting* $T_K M = T^\perp \oplus T^\parallel$ *with respect to the flow. Let* $D \in \{0, \dots, n-1\}$ *be the smallest number such that* $\nu_1^u + \dots + \nu_D^u + \nu_{D+1}^u < 0$*. Then it holds*

$$\dim_H K \le D + \frac{\nu_1^u + \dots + \nu_D^u}{|\nu_{D+1}^u|} + 1.$$

Proof Take an arbitrary number $d \in \left(D + \frac{\nu_1^u + \cdots + \nu_D^u}{|\nu_{D+1}^u|}, n-1\right]$. Then it holds $\nu_d = \nu_1^u + \cdots + \nu_{\lfloor d \rfloor}^u + (d - \lfloor d \rfloor)\nu_{\lfloor d \rfloor + 1}^u < 0$. Fix some $\varepsilon \in (0, \nu_d)$. By definition of ν_d there is a finite number $T_0 > 0$ such that $\frac{1}{T_0} \ln \omega_{d,K}^{T^\perp, T^\parallel}(\varphi^{T_0}) < \nu_d + \varepsilon$, i.e., $\omega_{d,K}^{T^\perp, T^\parallel}(\varphi^{T_0}) < \exp(T_0(\nu_d + \varepsilon)) < 1$. Theorem 4.1 basically uses properties of the singular value function which is estimated from above applying Lemma 2.2. Thus, the proposition can be proved applying analogous arguments and using $\omega_{d,K}^{T^\perp, T^\parallel}(\varphi^{T_0}) = \sup_{p \in K} \omega_d\big(d_p \varphi^{T_0}|_{T^\perp(p)}\big)$.

Example 5.1 Consider the vector field in $\mathbb{R}^2$ given by

$$\dot{\theta} = a \sin\theta, \qquad \dot{x} = -x + b \tag{5.2}$$

(with parameters $a \geq 1$, $b \neq 0$), being in the first coordinate periodic with period 2π. The arising dynamical system can be interpreted as a dynamical system on the flat cylinder Z of all equivalence classes $[u]$, $u \in \mathbb{R}^2$, being a smooth 2-dimensional Riemannian manifold with the standard metric for factor manifolds. Every solution of (5.2) is bounded in the second coordinate. Obviously, the set $K = \{z \in Z | z = [u], u = (\theta, 0), \theta \in \mathbb{R}\}$ is compact and flow-invariant with respect to (5.2). The variational system (2.3) and the system in normal variations (2.4) with respect to any solution $(\theta(t), 0)$ in K are given by

$$\dot{y} = \begin{pmatrix} a\cos\theta(t) & 0 \\ 0 & -1 \end{pmatrix} y \quad \text{and} \quad \dot{z} = \begin{pmatrix} -a\cos\theta(t) & 0 \\ 0 & -1 \end{pmatrix} z,$$

respectively. Thus, $\beta_1(z) = -1$ for any $z \in K$ and condition (4.1) is satisfied with $d = 1$ and $\Theta = T = 1$. By Theorem 4.1 we conclude that $\dim_H K \leq 1$. Note that in the present situation other available theorems [39, 45] are not applicable since the divergence of the right-hand side of (5.2) gives the expression $a\cos\theta - 1$ which is, in contrast to the assumptions of Theorem 3.1 from [39], not always negative.

6 Generalizations of the Theorems of Hartman-Olech and Borg

In this section we show that for certain vector fields in $\mathbb{R}^3$ the methods of the present paper provide always more effective conditions for upper Hausdorff bounds than those which work without projection onto transversal submanifolds (e.g. [39, 45]). In addition to this we improve for these systems results about the structure of ω-limit sets, which are closely related to results in [4, 19, 20].

Consider an arbitrary C^2-vector field f in $\mathbb{R}^3$ with the standard Euclidean metric, i.e., the differential equation

$$\dot{x} = f(x). \tag{6.1}$$

Suppose that for (6.1) the global flow $\{\varphi^t\}_{t\in\mathbb{R}}$ exists. Let K and $\widetilde{K}$ be two compact sets in $\mathbb{R}^3$ satisfying $K \subset \varphi^t(p) \subset \widetilde{K}$ for all $t \geq 0$. For that case for any $x \in \mathbb{R}^3$ the covariant derivative $\nabla f(x)$ can be identified with the Jacobi matrix $Df(x)$ of f in x. Suppose that f possesses in $\widetilde{K}$ a finite number of equilibrium points and that for any such equilibrium point all eigenvalues of $Df(x)$ have negative real part.

Consider the symmetric part $SDf(x) = \frac{1}{2}\big(Df(x) + Df(x)^*\big)$ of $Df(x)$. As in the previous sections for any regular p of f define the hyperplanes $T^{\perp}(x) = \{z \in \mathbb{R}^3 \mid f(x)^* z = 0\}$, where $f(x)^*$ denotes the transposed vector. Let the linear operator $SA(x)\colon T^{\perp}(x) \to T^{\perp}(x)$ be given by

$$SA(x) = SDf(x) - \frac{f(x)f(x)^*}{\|f(x)\|^2}\, SDf(x) = \left(I - \frac{f(x)f(x)^*}{\|f(x)\|^2}\right) Df(x)$$

(compare with Lemma 2.1). Denote the eigenvalues of $SDf(x)$, ordered with respect to size and multiplicity, by $\alpha_1(x) \ge \alpha_2(x) \ge \alpha_3(x)$. Suppose that $\beta_1(x) \ge \beta_2(x)$ are the eigenvalues of $SA(x)$ restricted to the subspace $T^{\perp}(x)$ and suppose further that $\beta_1(x)$ and $\beta_2(x)$ are not eigenvalues of $S\nabla f(x)$. It is easy to see that $\beta_1(x)$ and $\beta_2(x)$ are the zeros of the equation $f(x)^*\big(\beta_i(x)I - SDf(x)\big)^{-1} f(x) = 0$. We introduce the polynomial

$$\det(\beta I - Df(x)) \equiv \beta^3 + \delta_2(x)\beta^2 + \delta_1(x)\beta + \delta_0(x). \tag{6.2}$$

Let $x \in \widetilde{K}$. Note that we have $\delta_2(x) = -(\alpha_1(x) + \alpha_2(x) + \alpha_3(x))$, $\delta_1(x) = \alpha_1(x)\alpha_2(x) + \alpha_2(x)\alpha_3(x) + \alpha_1(x)\alpha_3(x)$ and $\delta_0(x) = -\alpha_1(x)\alpha_2(x)\alpha_3(x)$. From this with elementary calculations (see [16]) it follows that the eigenvalues $\beta_i(x)$ $(i = 1, 2)$ of $SA(x)$ are the zeros of the polynomial

$$\beta^2 + \big[\delta_2(x) + \Delta_1(x)\big]\beta + \big[\delta_1(x) + \delta_2(x)\Delta_1(x) + \Delta_2(x)\big],$$

where

$$\begin{aligned} \Delta_1(x) &= \frac{1}{\|f(x)\|^2}\, f(x)^* Df(x) f(x) \quad \text{and} \\ \Delta_2(x) &= \frac{1}{\|f(x)\|^2}\, f(x)^* Df(x)^2 f(x). \end{aligned} \tag{6.3}$$

Using this fact one sees immediately that the assumptions of Corollary 4.1 are satisfied for (6.1) if we suppose for the auxiliary function $v(x) = \frac{1}{2}\ln\|f(x)\|^2$, defined on the set of all regular points of $\mathbb{R}^3$, the following conditions: There exists a continuous function $s\colon \widetilde{K} \to [0, d_1]$ with $d_1 \in (0, 1]$ such that for any regular point $x \in \widetilde{K}$ with $h(x) := \frac{1-s(x)}{1+s(x)}$ the inequalities

$$\begin{gathered} \delta_2(x) - h(x)\Delta_1(x) > 0 \quad \text{and} \\ \frac{1}{4h(x)^2}\big(\delta_2(x) - h(x)\Delta_1(x)\big)^2 > \frac{1}{4}\big(\delta_2(x) - \Delta_1(x)\big)^2 - \delta_1(x) - \Delta_2(x) \end{gathered}$$

hold. As a corollary we get that if the inequalities

$$\begin{aligned} &\delta_2(x) - \Delta_1(x) > 0 \quad \text{and} \\ &\delta_1(x) + \Delta_2(x) > 0 \end{aligned} \tag{6.4}$$

are satisfied for all regular points x of f on $\widetilde{K}$ then by Corollary 4.1 it holds that $\dim_H K \le 1$. Further, the set K consists of a finite number of equilibrium points and closed trajectories of (6.1). This can be easily shown using coverings of appropriated tubular neighborhoods. Note that the last result is closely related to results in [4, 19, 20]. If in addition to this the set $\widetilde{K}$ is positively invariant with respect to the flow of (6.1), connected, and if $\widetilde{K}$ contains exactly one equilibrium point being asymptotically stable, then $\widetilde{K}$ is contained in the basin of attraction of this equilibrium point.

The Hartman-Olech condition ([20]) requires that $\alpha_1(x) + \alpha_2(x) < 0$ for all regular points $x \in \widetilde{K}$. This is one of the most effective sufficient condition which guarantees that in the present situation the set $\widetilde{K}$ is contained in the basin of attraction of an equilibrium. Note that this is always sufficient for the condition (6.4).

Let us formulate a further corollary from Theorem 4.2 for the case $M = \mathbb{R}^3$. Suppose now that $\delta_2(x) > 0$ for all regular points $x \in \widetilde{K}$ and that there exists a continuous function $s\colon \widetilde{K} \to [0, d_1)$ with $d_1 \in (0, 1]$ such that the inequalities

$$\begin{gathered}\frac{1+s(x)}{1-s(x)}\,\delta_2(x) - \Delta_1(x) \geq 0 \quad \text{and} \\ \frac{s(x)}{(1-s(x))^2}\,\delta_2(x)^2 - \frac{s(x)}{1-s(x)}\,\delta_2(x)\Delta_1(x) + \delta_1(x) + \Delta_2(x) \geq 0\end{gathered} \tag{6.5}$$

hold for all regular $x \in \widetilde{K}$. It follows from Corollary 4.1 that $\dim_H K < 2 + d_1$. It is well-known (see [39, 45]) that a sufficient condition for the dimension estimate $\dim_H K < 2 + d_1$ is the inequality

$$\alpha_1(x) + \alpha_2(x) + d_1\alpha_3(x) < 0 \quad \text{for all} \quad x \in \widetilde{K}. \tag{6.6}$$

It is easy to show ([16]) that our condition (6.5) is always satisfied supposed that (6.6) is satisfied.

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Solution of the Problem of Constructing Liapunov Matrix Function for a Class of Large Scale Systems

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Abstract: New sufficient conditions for the Liapunov stability of a class of large scale systems described by ordinary differential equations are established. In all cases we proposed a new construction for matrix-valued Liapunov function and the objective is the same: to analyze the stability of large scale systems (nonautonomous and autonomous) in terms of sign definiteness of specific matrices. In order to demonstrate the usefulness of the presented results several examples are considered.

Keywords: *Large scale systems; Liapunov function construction; stability; asymptotic stability; nonautonomous oscillator.*

Mathematics Subject Classification (2000): 34D20, 93A15, 93D05, 93D30.

1 Introduction

The methods of stability analysis of large-scale dynamical systems via one-level decomposition of the system and a vector Liapunov functions were summarized in a series of monographs. The necessity of further development of the known approaches for the mentioned class of dynamical systems and creation of new ones is caused by the fact that the methods of qualitative analysis based on vector Liapunov function yield, as a rule, "super-sufficient" stability conditions.

The aim of this paper is to present a new method of constructing the matrix-valued function and then to obtain efficient stability conditions for one class of large scale systems admitting one-level decomposition.

2 A Class of Large Scale System

We consider a system with finite number of degrees of freedom whose motion is described by the equations (2.1)

$$\frac{dx_i}{dt} = f_i(x_i) + g_i(t, x_1, \dots, x_m), \qquad i = 1, 2, \dots, m \tag{2.1}$$

where $x_i \in R^{n_i}$, $t \in \mathcal{T}_\tau$, $\mathcal{T}_\tau = [\tau, +\infty)$, $f_i \in C(R^{n_i}, R^{n_i})$, $g_i \in C(\mathcal{T}_\tau \times R^{n_1} \times \dots \times R^{n_m}, R^{n_i})$.

Introduce the designation

$$G_i(t, x) = g_i(t, x_1, \dots, x_m) - \sum_{j=1,\, j\neq i}^{m} g_{ij}(t, x_i, x_j), \tag{2.2}$$

where $g_{ij}(t, x_i, x_j) = g_i(t, 0, \dots, x_i, \dots, x_j, \dots, 0)$ for all $i \neq j$; $i, j = 1, 2, \dots, m$. Taking into consideration (2.2) system (2.1) is rewritten as

$$\frac{dx_i}{dt} = f_i(x_i) + \sum_{j=1,\, j\neq i}^{m} g_{ij}(t, x_i, x_j) + G_i(t, x). \tag{2.3}$$

Actually equations (2.3) describe the class of large-scale nonlinear nonautonomously connected systems. It is of interest to extend the method of matrix Liapunov functions to this class of equations in view of the new method of construction of nondiagonal elements of matrix-valued functions.

3 On Construction of Nondiagonal Elements of Matrix-Valued Function

In order to extend the method of matrix Liapunov functions to systems (2.3) it is necessary to estimate variation of matrix-valued function elements and their total derivatives along solutions of the corresponding systems. Such estimates are provided by the assumptions below.

Assumption 3.1 There exist open connected neighborhoods $\mathcal{N}_i \subseteq R^{n_i}$ of the equilibriums state $x_i = 0$, functions $v_{ii} \in C^1(R^{n_i}, R_+)$, the comparison functions φ_{i1}, φ_{i2} and ψ_i of class $K(KR)$ and real numbers $\underline{c}_{ii} > 0$, $\bar{c}_{ii} > 0$ and γ_{ii} such that

(1) $v_{ii}(x_i) = 0$ for all $(x_i = 0) \in \mathcal{N}_i$;
(2) $\underline{c}_{ii}\varphi_{i1}^2(\|x_i\|) \le v_{ii}(x_i) \le \bar{c}_{ii}\varphi_{i2}^2(\|x_i\|)$;
(3) $(D_{x_i} v_{ii}(x_i))^{\mathrm{T}} f_i(x_i) \le \gamma_{ii}\psi_i^2(\|x_i\|)$ for all $x_i \in \mathcal{N}_i$, $i = 1, 2, \dots, m$.

It is clear (see [3, 5]) that under conditions of Assumption 3.1 the equilibrium states $x_i = 0$ of nonlinear isolated subsystems

$$\frac{dx_i}{dt} = f_i(x_i), \qquad i = 1, 2, \dots, m \tag{3.1}$$

are

(a) uniformly asymptotically stable in the whole, if $\gamma_{ii} < 0$ and $(\varphi_{i1}, \varphi_{i2}, \psi_i) \in KR$-class;
(b) stable, if $\gamma_{ii} = 0$ and $(\varphi_{i1}, \varphi_{i2}) \in K$-class;
(c) unstable, if $\gamma_{ii} > 0$ and $(\varphi_{i1}, \varphi_{i2}, \psi_i) \in K$-class.

The approach proposed in this section takes large scale systems (2.3) into consideration, subsystems (3.1) having various dynamical properties specified by conditions of Assumption 3.1.

Assumption 3.2 There exist open connected neighborhoods $\mathcal{N}_i \subseteq R^{n_i}$ of the equilibrium states $x_i = 0$, functions $v_{ij} \in C^{1,1,1}(\mathcal{T}_\tau \times R^{n_i} \times R^{n_j}, R)$, comparison functions $\varphi_{i1}, \varphi_{i2} \in K(KR)$, positive constants $(\eta_1, \dots, \eta_m)^{\mathrm{T}} \in R^m$, $\eta_i > 0$ and arbitrary constants $\underline{c}_{ij}$, $\bar{c}_{ij}$, $i, j = 1, 2, \dots, m$, $i \neq j$ such that

(1) $v_{ij}(t, x_i, x_j) = 0$ for all $(x_i, x_j) = 0 \in \mathcal{N}_i \times \mathcal{N}_j$, $t \in \mathcal{T}_\tau$, $i, j = 1, 2, \dots, m$, $(i \neq j)$;
(2) $\underline{c}_{ij}\varphi_{i1}(\|x_i\|)\varphi_{j1}(\|x_j\|) \le v_{ij}(t, x_i, x_j) \le \bar{c}_{ij}\varphi_{i2}(\|x_i\|)\varphi_{j2}(\|x_j\|)$ for all $(t, x_i, x_j) \in \mathcal{T}_\tau \times \mathcal{N}_i \times \mathcal{N}_j$, $i \neq j$;
(3)
$$\begin{aligned} &D_t v_{ij}(t, x_i, x_j) + (D_{x_i} v_{ij}(t, x_i, x_j))^{\mathrm{T}} f_i(x_i) \\ &+(D_{x_j} v_{ij}(t, x_i, x_j))^{\mathrm{T}} f_j(x_j) + \tfrac{\eta_i}{2\eta_j}(D_{x_i} v_{ii}(x_i))^{\mathrm{T}} g_{ij}(t, x_i, x_j) \\ &+\tfrac{\eta_j}{2\eta_i}(D_{x_j} v_{jj}(x_j))^{\mathrm{T}} g_{ji}(t, x_i, x_j) = 0; \end{aligned} \tag{3.2}$$

It is easy to notice that first order partial equations (3.2) are a somewhat variation of the classical Liapunov equation proposed in [8] for determination of auxiliary function in the theory of his direct method of motion stability investigation. In a particular case these equations are transformed into the systems of algebraic equations whose solutions can be constructed analytically.

Assumption 3.3 There exist open connected neighbourhoods $\mathcal{N}_i \subseteq R^{n_i}$ of the equilibrium states $x_i = 0$, comparison functions $\psi \in K(KR)$, $i = 1, 2, \dots, m$, real numbers α^1_{ij}, α^2_{ij}, α^3_{ij}, ν^1_{ki}, ν^1_{kij}, μ^1_{kij} and μ^2_{kij}, $i, j, k = 1, 2, \dots, m$, such that

(1) $(D_{x_i} v_{ii}(x_i))^{\mathrm{T}} G_i(t, x) \le \psi_i(\|x_i\|) \sum\limits_{k=1}^{m} \nu^1_{ki}\psi(\|x_k\|) + R_1(\psi)$
for all $(t, x_i, x_j) \in \mathcal{T}_\tau \times \mathcal{N}_i \times \mathcal{N}_j$;
(2) $(D_{x_i} v_{ij}(t, \cdot))^{\mathrm{T}} g_{ij}(t, x_i, x_j) \le \alpha^1_{ij}\psi^2_i(\|x_i\|) + \alpha^2_{ij}\psi_i(\|x_i\|)\psi_j(\|x_j\|) + \alpha^3_{ij}\psi^2_j(\|x_j\|)$
$+R_2(\psi)$ for all $(t, x_i, x_j) \in \mathcal{T}_\tau \times \mathcal{N}_i \times \mathcal{N}_j$;
(3) $(D_{x_i} v_{ij}(t, \cdot))^{\mathrm{T}} G_i(t, x) \le \psi_j(\|x_j\|) \sum\limits_{k=1}^{m} \nu^2_{ijk}\psi_k(\|x_k\|) + R_3(\psi)$
for all $(t, x_i, x_j) \in \mathcal{T}_\tau \times \mathcal{N}_i \times \mathcal{N}_j$;
(4) $(D_{x_i} v_{ij}(t, \cdot))^{\mathrm{T}} g_{ik}(t, x_i, x_k) \le \psi_j(\|x_j\|)(\mu^1_{ijk}\psi_k(\|x_k\|) + \mu^2_{ijk}\psi_i(\|x_i\|)) + R_4(\psi)$
for all $(t, x_i, x_j) \in \mathcal{T}_\tau \times \mathcal{N}_i \times \mathcal{N}_j$.

Here $R_s(\psi)$ are polynomials in $\psi = (\psi_1(\|x_1\|, \dots, \psi_m(\|x_m\|))$ in a power higher than three, $R_s(0) = 0$, $s = 1, \dots, 4$.

Under conditions (2) of Assumptions 3.1 and 3.2 it is easy to establish for function

$$v(t, x, \eta) = \eta^{\mathrm{T}} U(t, x)\eta = \sum_{i,j=1}^{m} v_{ij}(t, \cdot)\eta_i\eta_j \tag{3.3}$$

the bilateral estimate (cf. [4])

$$u_1^{\mathrm{T}} H^{\mathrm{T}} \underline{C} H u_1 \le v(t, x, \eta) \le u_2^{\mathrm{T}} H^{\mathrm{T}} \bar{C} H u_2, \tag{3.4}$$

where

$$u_1 = (\varphi_{11}(\|x_1\|, \dots, \varphi_{m1}(\|x_m\|))^{\mathrm{T}},$$

$$u_2 = (\varphi_{12}(\|x_1\|, \dots, \varphi_{m2}(\|x_m\|))^{\mathrm{T}},$$

which holds true for all $(t,x)\in\mathcal{T}_\tau\times\mathcal{N}$, $\mathcal{N}=\mathcal{N}_1\times\cdots\times\mathcal{N}_m$.

Based on conditions (3) of Assumptions 3.1, 3.2 and conditions (1) – (4) of Assumption 3.3 it is easy to establish the inequality estimating the auxiliary function variation along solutions of system (2.3). This estimate reads

$$Dv(t,x,\eta)\big|_{(2.1)}\le u_3^{\mathrm{T}}Mu_3, \tag{3.5}$$

where $u_3=(\psi_1(\|x_1\|),\dots,\psi_m(\|x_m\|)$ and holds for all $(t,x)\in\mathcal{T}_\tau\times\mathcal{N}$.

Elements σ_{ij} of matrix M in the inequality (3.8) have the following structure

$$\sigma_{ii}=\eta_i^2\gamma_{ii}+\eta_i^2\nu_{ii}+\sum_{k=1,\,k\ne i}^{m}(\eta_k\eta_i\nu_{kii}^2+\eta_i^2\nu_{kii}^2)+2\sum_{j=1,\,j\ne i}^{m}\eta_i\eta_j(\alpha_{ij}^1+\alpha_{ji}^3);$$

$$\sigma_{ij}=\frac{1}{2}\,(\eta_i^2\nu_{ji}^1+\eta_j^2\nu_{ij}^1)+\sum_{k=1,\,k\ne j}^{m}\eta_k\eta_j\nu_{kij}^2+\sum_{k=1,\,k\ne i}^{m}\eta_i\eta_j\nu_{kij}^2$$

$$+\eta_i\eta_j(\alpha_{ij}^2+\alpha_{ji}^2)+\sum_{\substack{k=1,\,k\ne i,\\ k\ne j}}^{m}(\eta_k\eta_j\mu_{kji}^1+\eta_i\eta_j\mu_{ijk}^2+\eta_i\eta_k\mu_{kij}^1+\eta_i\eta_j\mu_{jik}^2),$$

$$i=1,2,\dots,m,\qquad i\ne j.$$

4 Test for Stability Analysis

Sufficient criteria of various types of stability of the equilibrium state $x=0$ of system (2.3) are formulated in terms of the sign definiteness of matrices $\underset{\sim}{C}$, $\bar{C}$ and M from estimates (3.4), (3.5). We shall show that the following assertion is valid.

Theorem 4.1 *Assume that the perturbed motion equations are such that all conditions of Assumptions 3.1 – 3.3 are fulfilled and moreover*

(1) *matrices $\underset{\sim}{C}$ and $\bar{C}$ in estimate (3.4) are positive definite;*
(2) *matrix M in inequality (3.5) is negative semi-definite (negative definite).*

Then the equilibrium state $x=0$ of system (2.1) is uniformly stable (uniformly asymptotically stable).

If, additionally, in conditions of Assumptions 3.1 – 3.3 all estimates are satisfied for $\mathcal{N}_i=R^{n_i}, R_k(\psi)=0, k=1,\dots 4$ and comparison functions $(\varphi_{i1},\varphi_{i2})\in KR$-class, then the equilibrium state of system (2.1) is uniformly stable in the whole (uniformly asymptotically stable in the whole).

Proof If all conditions of Assumptions 3.1 – 3.2 are satisfied, then it is possible for system (2.1) to construct function $v(t,x,\eta)$ which together with total derivative $Dv(t,x,\eta)$ satisfies the inequalities (3.4), and (3.5). Condition (1) of Theorem 4.1 implies that function $v(t,x,\eta)$ is positive definite and decreasing for all $t\in\mathcal{T}_\tau$. Under condition (2) of Theorem 4.1 function $Dv(t,x,\eta)$ is negative semi-definite (definite). Therefore all conditions of Theorem 2.3.1, 2.3.3 from [9] are fulfilled. The proof of the second part of Theorem 4.1 is based on Theorem 2.3.4 from the same monograph [9].

5 Nonautonomous Oscillator

We shall study the motion of two non-autonomously connected oscillators whose behaviour is described by the equations

$$\begin{aligned}
\frac{dx_1}{dt} &= \gamma_1 x_2 + v\cos\omega t y_1 - v\sin\omega t y_2,\\
\frac{dx_2}{dt} &= -\gamma_1 x_1 + v\sin\omega t y_1 + v\cos\omega t y_2,\\
\frac{dy_1}{dt} &= \gamma_2 y_2 + v\cos\omega t x_1 + v\sin\omega t x_2,\\
\frac{dy_2}{dt} &= -\gamma_2 y_2 + v\cos\omega t x_2 - v\sin\omega t x_1,
\end{aligned} \tag{5.1}$$

where $\gamma_1,\ \gamma_2,\ v,\ \omega,\ \omega+\gamma_1-\gamma_2\neq 0$ are some constants.

For the independent subsystems

$$\begin{aligned}
\frac{dx_1}{dt} &= \gamma_1 x_2, \qquad \frac{dx_2}{dt} = -\gamma_1 x_1\\
\frac{dy_1}{dt} &= \gamma_2 y_2, \qquad \frac{dy_2}{dt} = -\gamma_2 y_1
\end{aligned} \tag{5.2}$$

the auxiliary functions $v_{ii},\ i=1,2$, are taken in the form

$$\begin{aligned}
v_{11}(x) &= x^{\mathrm T}x, \qquad x=(x_1,x_2)^{\mathrm T},\\
v_{22}(y) &= y^{\mathrm T}y, \qquad y=(y_1,y_2)^{\mathrm T}.
\end{aligned} \tag{5.3}$$

We use the equation (3.2) (see Assumption 3.2) to determine the non-diagonal element $v_{12}(x,y)$ of the matrix-valued function $U(t,x,y)=[v_{ij}(\cdot)],\ i,\,j=1,2$. To this end set $\eta=(1,1)^{\mathrm T}$ and $v_{12}(x,y)=x^{\mathrm T}P_{12}y$, where $P_{12}\in C^1(\mathcal{T}_\tau,R^{2\times 2})$. For the equation

$$\begin{aligned}
&\frac{dP_{12}}{dt} + \begin{pmatrix} 0 & -\gamma_1\\ \gamma_1 & 0\end{pmatrix} P_{12}\\
&+ P_{12}\begin{pmatrix} 0 & \gamma_2\\ -\gamma_2 & 0\end{pmatrix} + 2v\begin{pmatrix}\cos\omega t & -\sin\omega t\\ \sin\omega t & \cos\omega t\end{pmatrix} = 0,
\end{aligned} \tag{5.4}$$

the matrix

$$P_{12} = -\frac{2v}{\omega+\gamma_1-\gamma_2}\begin{pmatrix}\sin\omega t & \cos\omega t\\ -\cos\omega t & \sin\omega t\end{pmatrix}$$

is a partial solution bounded for all $t\in\mathcal{T}_\tau$.

Thus, for the function $v(t,x,y)=\eta^{\mathrm T}U(t,x,y)\eta$ it is easy to establish the estimate of (3.4) type with matrices $\underline{C}$ and $\bar{C}$ in the form

$$\underline{C} = \begin{pmatrix}\underline{c}_{11} & \underline{c}_{12}\\ \underline{c}_{12} & \underline{c}_{22}\end{pmatrix}, \qquad \bar{C} = \begin{pmatrix}\bar{c}_{11} & \bar{c}_{12}\\ \bar{c}_{12} & \bar{c}_{22}\end{pmatrix},$$

where $\bar{c}_{11}=\underline{c}_{11}=1;\ \bar{c}_{22}=\underline{c}_{22}=1,\ \bar{c}_{12}=-\underline{c}_{12}=\frac{|2v|}{|\omega+\gamma_1-\gamma_2|}$. Besides, the vector $u_1^{\mathrm T}=(\|x\|,\|y\|)=u_2^{\mathrm T}$ since the system (5.1) is linear.

For system (5.1) the estimate (3.5) becomes

$$Dv(t,x,y)\big|_{(5.1)} = 0$$

for all $(x,y) \in R^2 \times R^2$ because $M = 0$.

Due to (5.4) the motion stability conditions for system (5.1) are established basing on the analysis of matrices $\underline{C}$ and $\bar{C}$ property of having fixed sign.

It is easy to verify that the matrices $\underline{C}$ and $\bar{C}$ are positive definite, if

$$1 - \frac{4v^2}{(\omega + \gamma_1 - \gamma_2)^2} > 0.$$

Consequently, the motion of nonautonomously connected oscillators is uniformly stable in the whole, if

$$|v| < \frac{1}{2}\,|\omega + \gamma_1 - \gamma_2|.$$

6 Large Scale Linear System

Assume that in the system

$$\begin{aligned} \frac{dx_1}{dt} &= A_{11}x_1 + A_{12}x_2 + A_{13}x_3, \\ \frac{dx_2}{dt} &= A_{21}x_1 + A_{22}x_2 + A_{23}x_3, \\ \frac{dx_3}{dt} &= A_{31}x_1 + A_{32}x_2 + A_{33}x_3, \end{aligned} \tag{6.1}$$

the state vectors $x_i \in R^{n_i}$, $i = 1,2,3$, and $A_{ij} \in R^{n_i \times n_j}$ are constant matrices for all i, $j = 1,2,3$.

For the independent systems

$$\frac{dx_i}{dt} = A_{ii}x_i, \quad i = 1,2,3 \tag{6.2}$$

we construct auxiliary functions $v_{ii}(x_i)$ as the quadratic forms

$$v_{ii}(x_i) = x_i^{\mathrm{T}} P_{ii} x_i, \quad i = 1,2,3 \tag{6.3}$$

whose matrices P_{ii} are determined by

$$A_{ii}^{\mathrm{T}} P_{ii} + P_{ii} A_{ii} = -G_{ii}, \quad i = 1,2,3, \tag{6.4}$$

where G_{ii} are prescribed matrices of definite sign. In order that to construct non-diagonal elements $v_{ij}(x_i,x_j)$ of matrix-valued function $U(x)$ we employ equation (3.2) from Assumption 3.2. Note that for system (6.1)

$$\begin{gathered} f_i(x_i) = A_{ii}x_i, \quad f_j(x_j) = A_{jj}x_j, \\ g_{ij}(x_i,x_j) = A_{ij}x_j, \quad G_i(t,x) = 0, \quad i = 1,2,3. \end{gathered}$$

Since for the bilinear forms

$$v_{ij}(x_i,x_j) = v_{ji}(x_j,x_i) = x_i^{\mathrm T} P_{ij} x_j, \tag{6.5}$$

the correlations

$$D_{x_i} v_{ij}(x_i,x_j) = x_j^{\mathrm T} P_{ij}^{\mathrm T}, \quad D_{x_j} v_{ij}(x_i,x_j) = x_i^{\mathrm T} P_{ij},$$

are true, equation (3.2) becomes

$$x_i^{\mathrm T}\left(A_{ii}^{\mathrm T} P_{ij} + P_{ij} A_{jj} + \frac{\eta_i}{\eta_j}\, P_{ii} A_{ij} + \frac{\eta_j}{\eta_i}\, A_{ji}^{\mathrm T} P_{ii}\right) x_j = 0.$$

From this correlation for determining matrices P_{ij} we get the system of algebraic equations

$$\begin{gathered} A_{ii} P_{ij} + P_{ij} A_{jj} = -\frac{\eta_i}{\eta_j}\, P_{ii} A_{ij} - \frac{\eta_j}{\eta_i}\, A_{ji}^{\mathrm T} P_{ii}, \\ i \neq j, \quad i,\, j = 1,2,3. \end{gathered} \tag{6.6}$$

Since for (6.3), and (6.5) the estimates (see [4,6])

$$\begin{aligned} &v_{ii}(x_i) \ge \lambda_m(P_{ii}) \|x_i\|^2, && x_i \in R^{n_i}; \\ &v_{ij}(x_i,x_j) \ge -\lambda_M^{1/2}(P_{ij} P_{ij}^{\mathrm T}) \|x_i\| \|x_j\|, && (x_i,x_j) \in R^{n_i} \times R^{n_j}, \end{aligned}$$

hold true, for function $v(x,\eta) = \eta^{\mathrm T} U(x) \eta$ the inequality

$$w^{\mathrm T} H^{\mathrm T} C H w \le v(x,\eta) \tag{6.7}$$

is satisfied for all $x \in R^n$, $w = (\|x_1\|, \|x_2\|, \|x_3\|)^{\mathrm T}$ and the matrix

$$C = \begin{pmatrix} \lambda_m(P_{11}) & -\lambda_M^{1/2}(P_{12}P_{12}^{\mathrm T}) & -\lambda_M^{1/2}(P_{13}P_{13}^{\mathrm T}) \\ -\lambda_M^{1/2}(P_{12}P_{12}^{\mathrm T}) & \lambda_m(P_{22}) & -\lambda_M^{1/2}(P_{23}P_{23}^{\mathrm T}) \\ -\lambda_M^{1/2}(P_{13}P_{13}^{\mathrm T}) & -\lambda_M^{1/2}(P_{23}P_{23}^{\mathrm T}) & \lambda_m(P_{33}) \end{pmatrix}.$$

For system (6.1) the constants from Assumption 3.3 are:

$$\alpha_{ij}^1 = \alpha_{ij}^2 = 0; \quad \alpha_{ij}^3 = \lambda_M(A_{ij}^{\mathrm T} P_{ij} + P_{ij}^{\mathrm T} A_{ij}),$$

$$\nu_{ki}^1 = \nu_{ijk}^2 = 0; \quad \nu_{ijk}^1 = \lambda_M^{1/2}[(P_{ij}^{\mathrm T} A_{ik})(P_{ij}^{\mathrm T} A_{ik})], \quad \mu_{ijk}^2 = 0.$$

Therefore the elements σ_{ij} of matrix M in (3.5) for system (6.1) have the structure

$$\sigma_{ii} = -\eta_i^2 \lambda_m(G_{ii}) + 2 \sum_{j=1,\, j\neq i}^{3} \eta_i \eta_j \alpha_{ij}^3, \quad i = 1,2,3,$$

$$\sigma_{ij} = \sum_{\substack{k=1,\, k\neq i, \\ k\neq j}}^{3} (\eta_k \eta_j \nu_{ijk}^1 + \eta_i \eta_k \nu_{kij}^1), \quad i,\, j = 1,2,3, \quad i \neq j.$$

Consequently, the function $Dv(x,\eta)$ variation along solutions of system (6.1) is estimated by the inequality

$$Dv(x,\eta)\big|_{(6.1)} \le w^{\mathrm T} M w \tag{6.8}$$

for all $(x_1,x_2,x_3) \in R^{n_1} \times R^{n_2} \times R^{n_3}$.

We summarize our presentation as follows.

Corollary 6.1 *Assume for system (6.1) the following conditions are satisfied:*

(1) *algebraic equations (6.4) have the sign-definite matrices P_{ii}, $i = 1,2,3$ as their solutions;*
(2) *algebraic equations (6.6) have constant matrices P_{ij}, for all i, $j = 1,2,3$, $i \neq j$ as their solutions;*
(3) *matrix C in (6.7) is positive definite;*
(4) *matrix M in (6.8) is negative semi-definite (negative definite).*

Then the equilibrium state $x = 0$ of system (6.1) is uniformly stable (uniformly asymptotically stable).

This corollary follows from Theorem 4.1 and hence its proof is obvious.

7 Discussion and Numerical Example

To start to illustrate the possibilities of the proposed method of Liapunov function construction we consider a system of two connected equations that was studied earlier by the Bellman-Bailey approach (see [7, 8], etc.).

Partial case of system (6.1) is the system

$$\begin{aligned} \frac{dx_1}{dt} &= Ax_1 + C_{12}x_2, \\ \frac{dx_2}{dt} &= Bx_2 + C_{21}x_1, \end{aligned} \tag{7.1}$$

where $x_1 \in R^{n_1}$, $x_2 \in R^{n_2}$, and A, B, C_{12} and C_{21} are constant matrices of corresponding dimensions. For independent subsystems

$$\begin{aligned} \frac{dx_1}{dt} &= Ax_1, \\ \frac{dx_2}{dt} &= Bx_2 \end{aligned} \tag{7.2}$$

the functions $v_{11}(x_1)$ and $v_{22}(x_2)$ are constructed as the quadratic forms

$$v_{11} = x_1^{\mathrm{T}} P_{11} x_1, \qquad v_{22} = x_2^{\mathrm{T}} P_{22} x_2, \tag{7.3}$$

where P_{11} and P_{22} are sign-definite matrices.

Function $v_{12} = v_{21}$ is searched for as a bilinear form $v_{12} = x_1^{\mathrm{T}} P_{12} x_2$ whose matrix is determined by the equation

$$A^{\mathrm{T}} P_{12} + P_{12} B = -\frac{\eta_1}{\eta_2} P_{11} C_{12} - \frac{\eta_2}{\eta_1} C_{21}^{\mathrm{T}} P_{22}, \quad \eta_1 > 0, \quad \eta_2 > 0. \tag{7.4}$$

According to Lankaster [7, p.240] equation (7.4) has a unique solution, provided that matrices A and $-B$ have no common eigenvalues.

Matrix C in (6.7) for system (7.1) reads

$$C = \begin{pmatrix} \lambda_m(P_{11}) & -\lambda_M^{1/2}(P_{12}P_{12}^{\mathrm{T}}) \\ -\lambda_M^{1/2}(P_{12}P_{12}^{\mathrm{T}}) & \lambda_m(P_{22}) \end{pmatrix}. \tag{7.5}$$

Here $\lambda_m(\cdot)$ are minimal eigenvalues of matrices P_{11}, P_{22}, and $\lambda_M^{1/2}(\cdot)$ is the norm of matrix $P_{12}P_{12}^{\mathrm{T}}$.

Estimate (6.7) for function $Dv(x,\eta)$ by virtue of system (7.1) is

$$Dv(x,\eta)\mid_{(7.1)} \leq w^{\mathrm{T}}\Xi w, \tag{7.6}$$

where $w = (\|x_1\|, \|x_2\|)^{\mathrm{T}}$, $\Xi = [\sigma_{ij}]$, $i,\ j = 1,2$;

$$\sigma_{11} = \lambda_1\eta_1^2 + \eta_1\eta_2\alpha_{22},$$
$$\sigma_{22} = \lambda_2\eta_2^2 + \eta_1\eta_2\beta_{22},$$
$$\sigma_{12} = \sigma_{21} = 0.$$

The notations are

$$\lambda_1 = \lambda_M(A^{\mathrm{T}}P_{11} + P_{11}A),$$
$$\lambda_2 = \lambda_M(B^{\mathrm{T}}P_{22} + P_{22}B),$$
$$\alpha_{22} = \lambda_M(C_{12}^{\mathrm{T}}P_{12} + P_{12}^{\mathrm{T}}C_{12}),$$
$$\beta_{22} = \lambda_M(C_{21}^{\mathrm{T}}P_{12}^{\mathrm{T}} + P_{12}C_{21}),$$

$\lambda(\cdot)$ is maximal eigenvalue of matrix $(\cdot)$. Partial case of Assumption 3.1 is as follows.

Corollary 7.1 *For system (7.1) let functions $v_{ij}(\cdot)$, $i,\ j = 1,2$ be constructed so that matrix C for system (7.1) is positive definite and matrix Ξ in inequality (7.6) is negative definite. Then the equilibrium $x = 0$ of system (7.1) is uniformly asymptotically stable.*

We consider the numerical example. Let the matrices from system (7.1) be of the form

$$A = \begin{pmatrix} -2 & 1 \\ 3 & -2 \end{pmatrix}, \qquad B = \begin{pmatrix} -4 & 1 \\ 2 & -1 \end{pmatrix}, \tag{7.7}$$

$$C_{12} = \begin{pmatrix} -0.5 & -0.5 \\ 0.8 & -0.7 \end{pmatrix}, \qquad C_{21} = \begin{pmatrix} 1.1 & 0.5 \\ -0.6 & -0.3 \end{pmatrix}. \tag{7.8}$$

Functions v_{ii} for subsystems

$$\dot{x} = Ax, \quad x = (x_1, x_2)^{\mathrm{T}},$$
$$\dot{y} = By, \quad y = (y_1, y_2)^{\mathrm{T}}$$

are taken as the quadratic forms

$$\begin{aligned} v_{11} &= 1.75x_1^2 + x_1x_2 + 1.5x_2^2, \\ v_{22} &= 0.35y_1^2 + 0.9y_1y_2 + 0.95y_2^2. \end{aligned} \tag{7.9}$$

Let $\eta = (1,1)^{\mathrm{T}}$. Then $\lambda_1 = \lambda_2 = -1$,

$$P_{12} = \begin{pmatrix} -0.011 & 0.021 \\ -0.05 & -0.022 \end{pmatrix},$$

$$\alpha_{22} = 0.03, \qquad \beta_{22} = -0.002.$$

It is easy to verify that $\sigma_{11} < 0$, and $\sigma_{22} < 0$, and hence all conditions of Corollary 7.1, are fulfilled in view that

$$\lambda_M^{1/2}(P_{12}P_{12}^{\mathrm{T}}) \le (\lambda_m(P_{11})\lambda_m(P_{22}))^{1/2},$$

for the values of $\lambda_M^{1/2}(P_{12}P_{12}^{\mathrm{T}}) = 0.06$, $\lambda_m(P_{11}) = 1.08$, $\lambda_m(P_{22}) = 0.115$. This implies uniform asymptotic stability in the whole of the equilibrium state of system (7.1) with matrices (7.7), and (7.8).

Let us show now that stability of system (7.1) with matrices (7.7), and (7.8) can not be studied in terms of the Bailey [2] theorem.

We recall that in this theorem the conditions of exponential stability of the equilibrium state are

(1) for subsystems (7.2) there must exist functions (7.3) satisfying estimates
 (a) $c_{i1}\|x_i\|^2 \le v_i(t,x_i) \le c_{i2}\|x_i\|^2$,
 (b) $Dv_i(t,x_i) \le -c_{i3}\|x_i\|^2$,
 (c) $\|\partial v_i/\partial x_i\| \le c_{i4}\|x_i\|$ for $x_i \in R^{n_i}$,
 where c_{ij} are some positive constants, $i = 1, 2$, $j = 1, 2, 3, 4$;
(2) the norms of matrices C_{ij} in system (2.4.17) must satisfy the inequality (see Abdullin, *et al.* [1, p. 106])

$$\|C_{12}\|\|C_{21}\| < \left(\frac{c_{11}c_{21}}{c_{12}c_{22}}\right)^{1/2}\left(\frac{c_{13}c_{23}}{c_{14}c_{24}}\right). \tag{7.10}$$

We note that this inequality is refined as compared with the one obtained firstly by Bailey [2].

The constants $c_{11}, \dots, c_{24}$ for functions (7.9) and system (7.1) with matrices (7.7), and (7.8) take the values

$$c_{11} = 1.08, \quad c_{21} = 0.115, \quad c_{12} = 2.14,$$
$$c_{22} = 2.14, \quad c_{22} = 1.135, \quad c_{13} = c_{23} = 1, \quad c_{14} = 4.83, \quad c_{24} = 2.4.$$

Condition (7.10) requires that $\|C_{12}\|\|C_{21}\| < 0.0184$ whereas for system (7.1), (7.7), and (7.8) we have

$$\|C_{12}\|\|C_{21}\| = 1.75.$$

Thus, the Bailey theorem turns out to be nonapplicable to this system and the condition (7.10) is "super-sufficient" for the property of stability.

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Chaotic Control Systems

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Abstract: Generally speaking, it is relatively easy to design a feedback controller to eliminate the possibility of chaos in a nonlinear dynamical system. Here we examine chaotic control not from the prospective of eliminating chaos, but from the prospective of producing chaotic motion in order to take advantage of the random like "free ride" a chaotic attractor provides. The idea is stay with this free ride until the system moves into a target containing a desired fixed point. Once inside this target, feedback control is applied that provides asymptotic stability for the fixed point. A basic requirement with this approach is to determine an appropriate target. It must be a subset of the domain of attraction to the fixed point under state variable feedback control. In addition, the target must be large enough so that the time it takes for the system to reach it, under chaotic control, is not unreasonably large. After addressing the question as to why this might be a desirable approach for nonlinear control system design, the focus of this paper is on the presentation of a general method for applying chaotic control and then demonstrating its use in controlling an inverted pendulum and a bouncing ball.

Keywords: *Chaos; control; inverted pendulum; bouncing ball.*

Mathematics Subject Classification (2000): 49J15, 70Q05, 93C15.

1 Introduction

The full potential of control system design is often overlooked since there exists a strong prejudice, in classical control text books, to only deal with linear systems. A major focus of control system design is to achieve asymptotic stability to an equilibrium point. Since nonlinear systems are generally "linearized" before applying linear control design methods, chaotic or other motion which can only occur in nonlinear systems is not generally considered to be a part of the design process. However, for years, controls engineers have been using traditional linear control methods to control systems which have the potential for chaos. This is not thought of as "controlling chaos" as chaos or any motion other than asymptotic stability or tracking is suppressed.

Poincaré [1] noted in 1892 that certain mechanical systems could display chaotic motion. However the notion that deterministic models of discrete or continuous nonlinear dynamical systems could behave chaotically did not attract wide attention until Lorenz [2] in (1963), May [3] in (1976), and others reported chaotic behavior in very simple dynamical models.

In 1990, Ott, Grebogi and York [4] published the first paper to point out that chaos could be advantageous in achieving control objectives. Their method, now called the OGY method, involves stabilizing one of the unstable periodic orbits embedded in the chaotic attractor using small time dependent perturbations of a system parameter. Chaotic motion allows this method to work since all of the unstable periodic orbits will eventually be visited. One simply waits until the chaotic motion brings the system near a neighborhood of the proper unstable periodic orbit, at which time the small control is applied. Many variants of this method have appeared in the literature [5]. A more traditional control approach to regulating the Lorenz equations appeared in 1991 [6].

Since that time there have been a number of other papers dealing with the control of chaotic systems [7 – 12] including feedback control of the Lorenz equations [13, 14]. These references tend to focus on the problem of designing a stabilizing controller for systems which, without control, would be chaotic. However, since chaos can be useful in moving a system to various points in state space, the systems of interest here are not necessarily chaotic but as in [15 – 17] chaotic motion can be created as a part of the total control design. We make use of the fact that for many nonlinear systems chaos is easy to create using open-loop control. In particular, see [18] for a discussion of producing chaos in the driven pendulum system.

In the **chaotic control algorithm** given below, two essential ingredients are needed: a chaotic attractor and a controllable target. It is assumed that chaos can be created using open loop control. A **controllable target** is any subset of the domain of attraction to a equilibrium point, under a corresponding feedback control law, that has a non-empty intersection with the chaotic attractor. Thus the equilibrium point itself need not be on the chaotic attractor. If we start the system at any point within the basin of attraction of the chaotic attractor, the resulting chaotic motion will ultimately arrive in the controllable target. The chaotic control algorithm simply has to keep track of when the system enters the controllable target. When it does, the open loop control used to create chaos is turned off and at the same time the closed-loop feedback control is turned on.

The chaotic control algorithm has some distinct advantages in designing controllers for nonlinear systems. Its main advantage is simplicity. Consider for a moment one of the alternatives. Optimal control [19 – 21] is well suited for nonlinear problems and numerical methods are available [22] for solving complex problems. However optimal control solutions, for nonlinear problems, obtained by applying Pontryagin's maximum principle are generally open-loop. One could use this open-loop control to drive the system to a controllable target in a direct fashion. However such a control program would not be robust in the sense that if a perturbation were to drive the system outside of the controllable target, one could not simply turn the open loop control back on. The only alternative would be to start over again.

2 Stability under Chaotic Control

Consider the class of dynamical systems, subject to control, which can be described by either nonlinear difference equations of the form (discrete system)

$$\mathbf{X}_{(t+1)} = \mathbf{F}(\mathbf{X}, \mathbf{U}) \tag{2.1}$$

or nonlinear differential equations of the form (continuous system)

$$\dot{\mathbf{X}} = \mathbf{F}[\mathbf{X}, \mathbf{U}], \tag{2.2}$$

where $\mathbf{F} = [F_1 \cdots F_{N_X}]$ is an N_X dimensional vector function of the state vector $\mathbf{X} = [X_1 \cdots X_{N_X}]$, and control vector $\mathbf{U} = [U_1 \cdots U_{N_U}]$. Current time is indicated by t and the subscript $(t+1)$ is used to denote one time unit latter. The dot $(\dot{\ })$ denotes differentiation with respect to time. For the discrete system, t is just a counter and need not denote actual time. The functions F_i are assumed to be continuous and continuously differentiable in their arguments. The control will, in general, be bounded and it is assumed that at every time t, the control $\mathbf{U}$ must lie in a subset of the control space $\mathcal{U}$ defined by the inequalities

$$U_{i_{\min}} \le U_i \le U_{i_{\max}}$$

for $i = 1 \cdots N_U$.

The control input $\mathbf{U}$ is either a specified function of time, $\mathbf{U}(t)$ (open-loop) or a specified function of the state, $\mathbf{U}(\mathbf{x})$ (closed-loop). Assume that for all t there exists a specified open-loop control input $\widehat{\mathbf{U}}(t)$ such that the system has a chaotic attractor. Furthermore assume that for a specified constant control, $\widehat{\mathbf{U}}(t) \equiv \bar{\mathbf{U}}$, there is a corresponding fixed point of interest which is near the chaotic attractor. The fixed point satisfies

$$\bar{\mathbf{X}} = (\bar{\mathbf{X}}, \bar{\mathbf{U}})$$

for the discrete difference equation system and it satisfies

$$\mathbf{F}(\bar{\mathbf{X}}, \bar{\mathbf{U}}) = 0$$

for the continuous differential equation system.

Given the above assumptions, the controllable target is obtained using a method of linearization and Lyapunov function estimates [16, 21]. The nonlinear system is first linearized about the fixed point. The resulting linear system is assumed to be controllable and the LQR method [23, 24] is used to design a full state variable feedback controller that will guarantee the origin for this system is asymptotically stable. This, in turn, implies that for the nonlinear system, the fixed point will be asymptotically stable in some neighborhood containing the fixed point. An under estimate for this neighborhood is then determined using a Lyapunov function obtained from the linear system. This under estimate is used as the controllable target. Full details of this approach is given in [17]. We will now illustrate the application of this chaotic control method to two systems. The first is continuous and the second is discrete.

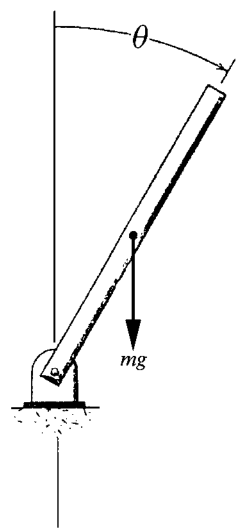

Figure 3.1. Inverted pendulum attached to a DC motor.

3 Inverted Pendulum

Consider the inverted pendulum, attached to a DC motor as shown in Figure 3.1. The pendulum is free to rotate through all angles so that it has stable downward equilibrium positions at $\theta = \pi(1 \pm 2n)$, $(n = 0, 1, 2, \dots)$ and unstable upright equilibrium position at $\theta = 2n\pi$, $(n = 0, 1, 2, \dots)$. Let $x_1 = \pi - \theta$ be the angle of the pendulum as measured from the downward position, $\dot{x}_1$ be the rate of change of this angle, and bu be the torque applied by the motor. Positive values are in the counter clockwise direction. In terms of these variables, the equations of motion are given by

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= a_1 \sin x_1 + a_2 x_2 + bu, \end{aligned} \tag{3.1}$$

where a_1, a_2, and b are constants associated with the system. The term $a_1 \sin x_1$ is the torque provided by gravity, $a_2 x_2$ is damping provided by friction and back EMF of the motor, bu is the control torque provided by a DC motor, and u is the voltage applied to the motor, with $|u| \leq u_{\max}$. The particular system that we will examine here has [21]

$$\begin{aligned} a_1 &= -17.627 \text{ rad/sec}^2 \\ a_2 &= -0.187 \text{ sec}^{-2} \\ b &= 0.6455 \text{ rad/volt-sec}^2 \\ u_{\max} &= 18 \text{ volts.} \end{aligned}$$

Our objective is to stabilize the inverted pendulum in the vertical upright position.

If we linearize this system about $\bar{\mathbf{X}} = [\,\pi \quad 0\,]^{\mathrm{T}}$, then we can use LQR design [24] with $\mathbf{Q}$ and $\mathbf{R}$ identity matrices to obtain the feedback gains

$$\mathbf{K} = \begin{bmatrix} 54.6333 & 12.7624 \end{bmatrix},$$

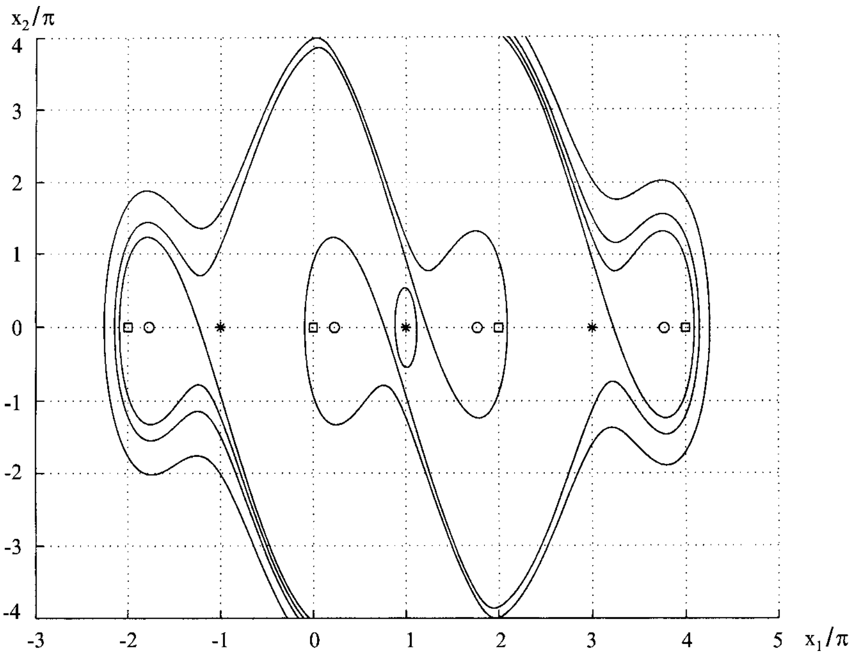

Figure 3.2. Domain of attraction for the inverted pendulum.

with the eigenvalues for the linearized controlled system given by $\lambda_1 = -4.5397$ and $\lambda_2 = -3.8855$. Under LQR design the feedback control law is given by

$$u(\mathbf{x}) = -\mathbf{K}(\mathbf{x} - \bar{\mathbf{X}}). \tag{3.2}$$

However since there are bounds on the control, the actual feedback control law used is that of saturating feedback control [21].

We can find the domain of attraction under saturating LQR control as applied to the nonlinear system (3.1) by integrating backward from points very close to the equilibrium points obtained when the control is set equal to $\pm u_{\max}$. The results are illustrated in Figure 3.2. In order to "see" the domain of attraction in this figure, first pinpoint the star located inside the ellipse. This represents $\bar{\mathbf{X}}$. The domain of attraction, to this point is the set of all points obtained by "flooding" with a color, from the star, or by drawing all continuous curves from the star which do not cross any of the curves contained in the figure other than the ellipse. Observe that, in places, the domain of attraction narrows down and becomes tubular.

Note that the x_1 and x_2 axis are in multiples of π. The equilibrium point $\bar{\mathbf{X}}$ and the first 2π multiple to the right and left are marked with a star, the squares locates stable equilibrium points, and the circles locate the equilibrium solutions obtained if saturating LQR control is used outside the domain of attraction.

Consider any one of the lob-like objects which contain a square and a small circle. If the system, under saturating LQR control is started anywhere inside the lob, including its tubular extension, it will remain in it, ultimately arriving at the equilibrium point contained in the lobe. Clearly these regions are not in the domain of attraction to $\bar{\mathbf{X}}$.

In fact, the domain of attraction to $\bar{\mathbf{X}}$ is simply all other points. Note that these other points include the stars located at 2π multiples of $\theta = 0$. Saturating LQR controller will not stabilize the system to these points since it does not recognize any vertical equilibrium position other than the one corresponding to $\theta = \dot{\theta} = 0$. In other words if the system were started at the upright position $x_1 = 3\pi,\ x_2 = 0$, the LQR controller would "unwind" the pendulum to bring it to $x_1 = x_2 = 0$.

3.1 Controllable targets

Under LQR control the linearized system is stable and we can solve the matrix Lyapunov equation [21] to find a Lyapunov function for this system

$$V = \mathbf{x}^{\mathrm{T}}\mathbf{P}\mathbf{x}, \quad \text{where} \quad \mathbf{P} = \begin{bmatrix} 1.3450 & 0.0283 \\ 0.0283 & 0.0627 \end{bmatrix}. \tag{3.3}$$

Since $\mathbf{P}$ is a positive definite matrix, lines of constant V are ellipses. We now seek the largest level curve $V = V_{\max}$ for which $\dot{V} < 0$ everywhere inside the ellipse using (3.1) under saturating LQR control. Since the problem is two dimensional, a value for $V_{\max}$ may be obtained by choosing a sufficiently small value for $V_{\max}$ so that integrating around the ellipse results in $\dot{V} < 0$ everywhere on this curve. A larger value for $V_{\max}$ may then be chosen and the process repeated until the inequality is satisfied with $\dot{V}$ very close to zero. Using this procedure results in $V_{\max} = 0.18$ for this problem. One advantage of this method for finding $V_{\max}$ is that it provides a numerical verification that the region inside the ellipse defined by (3.3) is a region of guaranteed asymptotic stability for the equilibrium point $x_1 = \pi,\ x_2 = 0$. The ellipse of Figure 3.2 is the one given by (3.3) with $V = V_{\max}$.

3.2 Chaotic attractor

In seeking an open loop control which will provide chaotic motion, it must be able to swing the pendulum to the upright vertical position from any given starting condition. An easy way to do this is to apply a sinusoidal voltage to the motor. For example, applying the control

$$u = 11\cos 3t \tag{3.4}$$

will drive the system "over the top". This is a necessary but not a sufficient condition for chaos. While there are many amplitude-frequency combinations which will produce chaos, a small change in one of these values, may result in motion which is not chaotic [18]. It is not proven here that the control given by (3.4) does actually produce chaos. What is obtained may be a long chaotic transient [6] with the possibility that after a sufficiently long time period the trajectory could settle down to a limit cycle. However as long as (3.4) produces a long term chaotic transient (if not true chaos) over a range of starting conditions of interest, it remains a viable chaotic controller.

Given a sufficiently long running time, a chaotic controller will wind up the pendulum for many revolutions in both directions, moving past the origin many times in a random way as depicted in Figure 3.3. The small circles represent the location of the pendulum in state space at every 0.1 second. Clearly the chaotic attractor is very large in comparison with the controllable target just obtained. This implies that the waiting time between chaotic control and feedback control may be large. One way to improve the odds is to

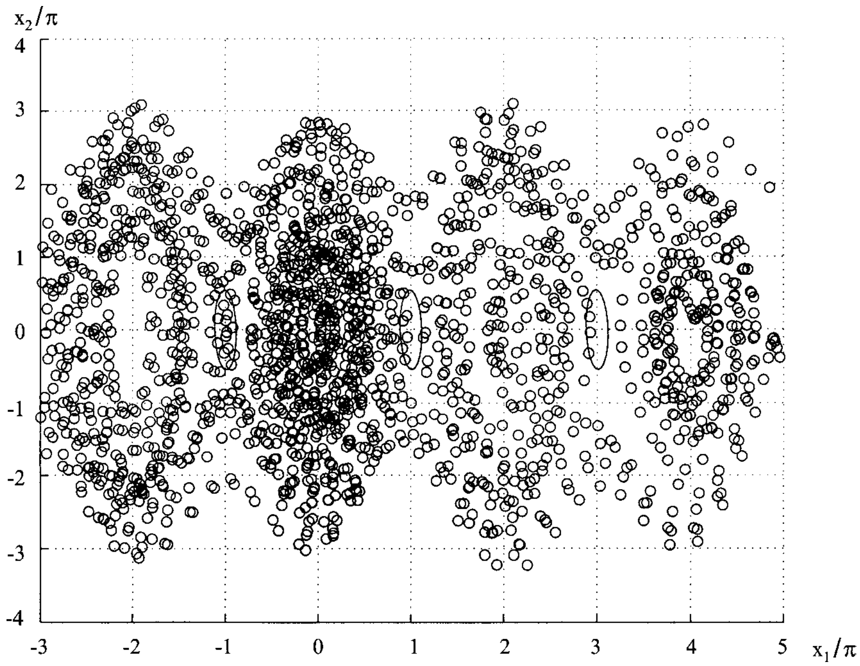

Figure 3.3. Chaotic attraction for the inverted pendulum.

simply increasing the number of targets. One obvious way of doing this is to introduce the ellipse just obtained at each of the points indicated by the stars in Figure 3.2. At each of these points the pendulum is in the upright position at zero velocity. The feedback control at these new targets must be adjusted accordingly. Three targets are illustrated in Figure 3.3. While the targets appear to have received many "hits" (especially the left one), there are only 21 points which lie inside the ellipses out of the 2,500 points (which lie inside the left/right limits of the figure and outside these limits). It should also be noted that the results shown in Figure 3.3 are only representative. It is possible that changing the initial conditions only slightly, using the same integration time, could result in more hits, no hits, or with most of the points lying outside the figure.

3.3 Intermediate targets

It is evident that it would be desirable to add additional targets. It has been previously shown that the waiting period required with the OGY method can be substantially reduced by using a "targeting method" [25]. This procedure uses intermediate targets in moving the system to a final target. We will use a similar procedure here. One way to arrive at additional intermediate targets is to simply integrate backward, under saturating LQR control from the neighborhood of the original equilibrium point $\bar{\mathbf{X}} = \begin{bmatrix} \pi & 0 \end{bmatrix}^{\mathrm{T}}$ using the initial conditions $\mathbf{x}(0) = \begin{bmatrix} \pi & \varepsilon \end{bmatrix}^{\mathrm{T}}$ and $\mathbf{x}(0) = \begin{bmatrix} \pi & -\varepsilon \end{bmatrix}^{\mathrm{T}}$ for a period of time, t_s. This results in two stopping points $\mathbf{x}_s^+$ and $\mathbf{x}_s^-$. At each of these stopping point we know that there exists some neighborhood about $\mathbf{x}_s^\pm$ such that if we integrate the system forward in time under saturating LQR control the system will be returned to $\bar{\mathbf{X}}$. Here we will choose t_s to be relatively small (e.g. so that the approximation $\sin x_1 = x_1$

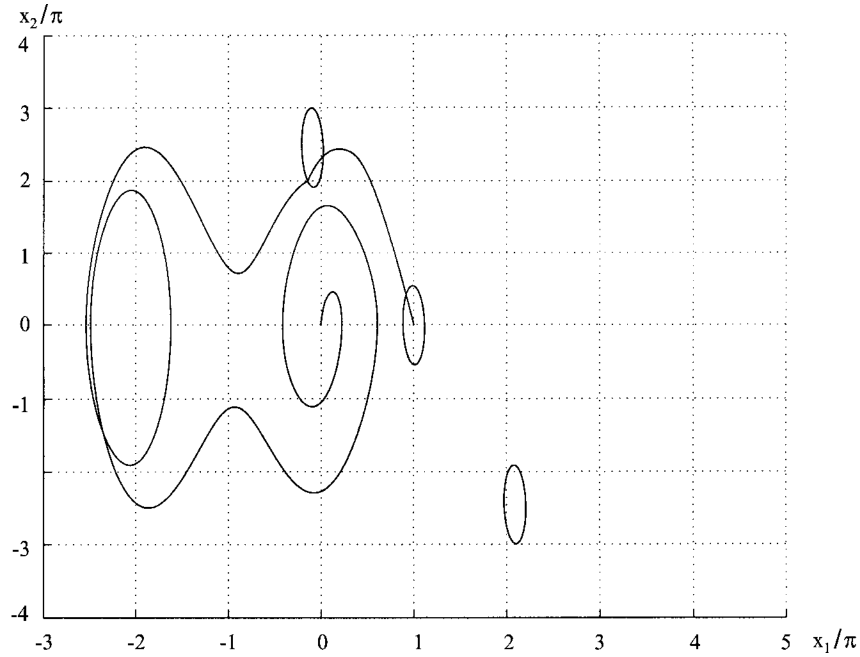

Figure 3.4. Utilizing chaos in the control of the inverted pendulum.

is valid) and use the controllable target ellipse centered at $\mathbf{x}_s^{\pm}$ to be an estimate of this additional neighborhood. If the system were indeed linear, the flow of all trajectories though the new intermediate target would converging to the line described by the retro trajectory. Unfortunately, for nonlinear systems there is no guarantee that all points in these intermediate targets will be controllable to the equilibrium point. This possibility does not defeat the method provided that we allow for it in the control algorithm. One way to do this is to make sure that chaotic control will be used in a small neighborhood of all equilibrium points other than the stars. In this way if the system enters a intermediate target, saturating LQR control is turned on, and the resultant trajectory does not arrive at a star, the chaotic control sequence will begin again. Clearly the majority of points in the intermediate target, under saturating LQR control must drive the system to a star in order for an intermediate target to be of any use.

A sample run using one control target at $\bar{\mathbf{X}}$ and two intermediate targets is illustrated in Figure 3.4. The system starts at the origin under (3.4) and remains under this control until it intersects the upper intermediate target. At this point saturating LQR control is applied which brings the system to $\bar{\mathbf{X}}$. In this case, since the chaotic trajectory is relatively short, it is shown as a solid line. Other example runs are similar and illustrate the usefulness of intermediate targets.

4 Bouncing Ball

A theoretical study of the motion of a ball bouncing on an oscillating plate is given in [26]. They construct an approximate map relating the velocity and phase between successive impacts of a ball bouncing on an oscillating plate of infinite mass, and show that the motion of the ball can be chaotic, by demonstrating the existence of a horseshoe. For certain frequencies of the plate and certain initial conditions of the ball it is possible to get periodic motion for any integer period. Some of these periodic orbits are stable and some are not.

Two variables, the position of the plate and the time interval between bounces, can be used as state variables used to model the ball map. Each time the ball bounces on the plate there is an opportunity to measure the state of the system and apply a control. We will be controlling the frequency of the plate in order to achieve a specified objective for the motion of the ball. In particular, our objective is to position the ball so as to maintain the system at a specified fixed point of the ball map when the plate is moving at a specified nominal frequency $\bar{\omega}$.

The command frequency applied just after a bounce (at time t_j) is given by

$$\omega_j = \bar{\omega} + u_j$$

where u_j is the change in frequency from the nominal value. Between the last bounce and the next, therefore, the plate is controlled according to

$$y(t) = A\sin(\omega_j \tau + \phi_j) \tag{4.1}$$

where $y(t)$ is the displacement of the plate in the vertical direction, A is the amplitude of the plate, $\tau = t - t_j$, is time since the last bounce and

$$\phi_j = \omega_j(t_j - t_{j-1}) + \phi_{j-1}$$

is the phase angle of the plate at the time of the last bounce. For this situation, it is not difficult to show that under the assumption that the bounce height is large compared to A, that the ball map is given by [27]

$$\begin{aligned} \phi_{j+1} &= \phi_j + \frac{\omega_j}{\bar{\omega}}\,\psi_j, \\ \psi_{j+1} &= -a_2\psi_j + \hat{a}_1\bar{\omega}\omega_j\,\cos\phi_{j+1}, \end{aligned} \tag{4.2}$$

where ψ_j is the change in phase between bounces (related to the time interval between bounces) a_2 and $\hat{a}_1$ are constants whose values depend on the mass ratio of the ball to the plate, the coefficient of restitution, the amplitude of the plate, and the acceleration of gravity.

When equations (4.2) are iterated to produce a dynamical solution, the first equation is evaluated modulo 2π since ϕ refers to a physical position of the plate. The second equation is not evaluated modulo 2π since the plate may go through more than one cycle before the next bounce.

There are many possible periodic solutions to (4.2). For example the ball can bounce to a fixed height at every n cycles of the plate. It can also bounce to m different heights at $n \times m$ cycles of the plate before repeating the pattern.

Figure 4.1 shows some of the kinds of dynamics possible with an actual bouncing ball system. In the top figure, the initial conditions were set very close to those corresponding to an unstable period-1 solution. We see the ball diverge from this solution and approaches a stable period-2 solution corresponding to $m = 2$, $n = 1$. In the middle

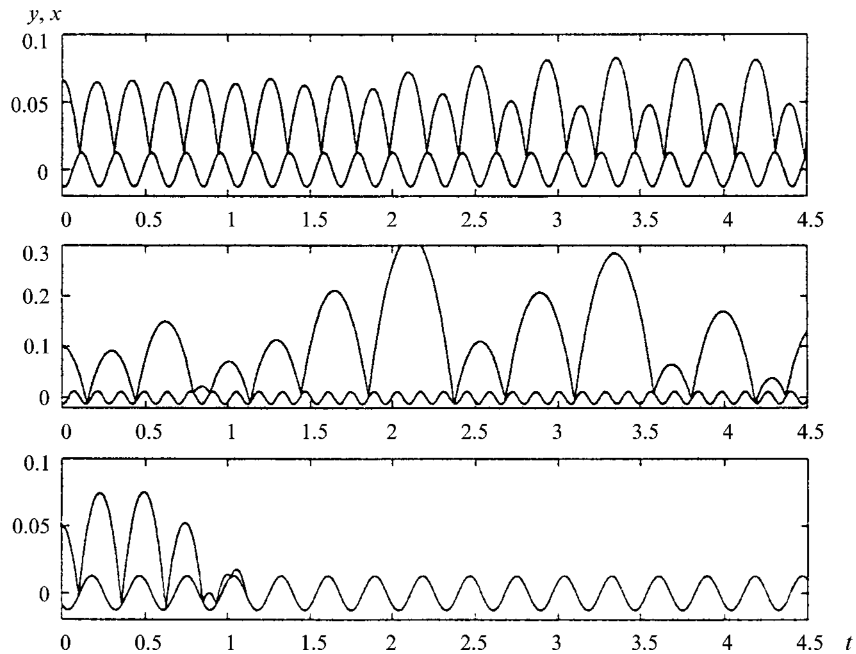

Figure 4.1. Ball and plate motion. (top) A stable period-2 solution at $\omega =$ 30 rad/sec. (middle) Chaotic motion at $\omega = 45$ rad/sec. (bottom) The ball can ride on the plate when $\omega = 22$ rad/sec.

figure, the ball bounces chaotically, with different n values between different steps. In the bottom figure, the plate frequency is such that a stable period one solution exists, however in this case, the initial conditions are not near this solution and the ball ends up riding on the plate after a few bounces.

4.1 Chaotic attractor

For the experimental bouncing ball system located at the University of Arizona non-linear control system laboratory, the parameters are given by $a_2 = -0.733$ and $\hat{a}_1 = 0.00459$. With these values, it turns out that the high-bounce map (4.2) produces chaotic motion with $\bar{\omega} = 45$ rad/sec as shown in the middle illustration of Figure 4.1. Figure 4.2 illustrates the corresponding chaotic attractor. The map was obtained by starting the system at $\phi(0) = 0$ and $\psi(0) = 9.1743$ radians and run for 1000 iterations. The values of ϕ and ψ are divided by 2π before plotting. The resulting chaotic attractor lies between an upper and lower bound as indicated in the figure. The upper bound is related to the height the ball would bounce, if it could bounce periodically at the phase angle ϕ and the lower bound is related to the height the ball would achieve if were to impact the plate at the phase angle ϕ with zero velocity. Under the high-bounce map (4.2) negative values for ψ are possible (corresponding to a negative bounce height). Since this is not physically possible, a slight modification of the map was used to produce Figure 4.2. Details of this modification are discussed in [17].

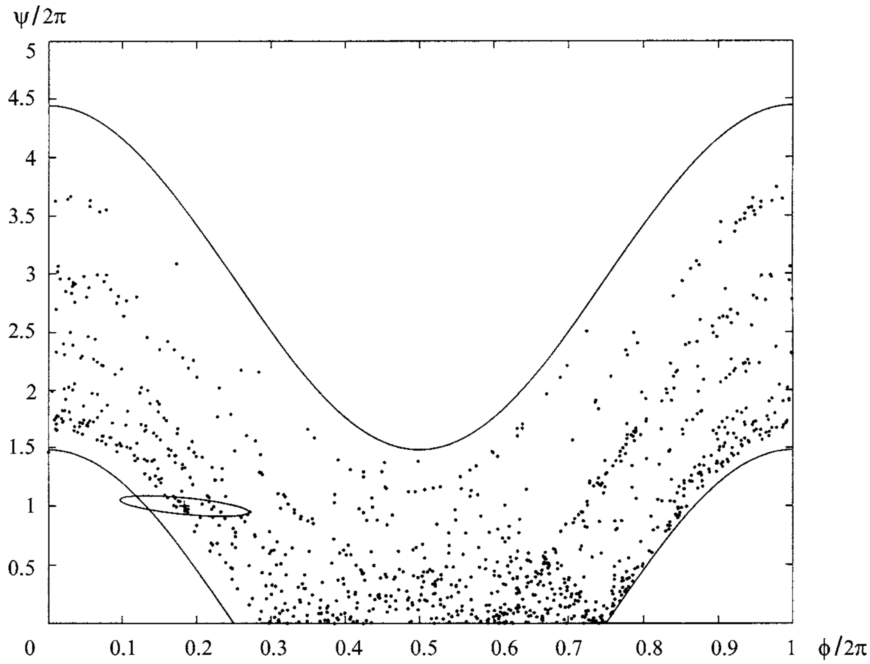

Figure 4.2. Chaotic attractor for the high-bounce map when $\omega = 45$ rad/sec.

4.2 Feedback controllers for the ball

We will consider only one case with $m = n = 1$ for a bounce pattern that is unstable in the absence of control.

We seek fixed point solutions (with $\omega_j = \bar{\omega}$) for

$$\begin{aligned} \bar{\phi} &= \bar{\phi} + \bar{\psi}, \\ \bar{\psi} &= -a_2\bar{\psi} + \hat{a}_1\bar{\omega}^2 \cos(\bar{\phi} + \bar{\psi}). \end{aligned} \tag{4.3}$$

In order to obtain equilibrium solutions from (4.3) the right hand side of the first equation must be evaluated modulo 2π, in which case we obtain

$$\begin{aligned} \bar{\psi} &= 2\pi, \\ \cos\bar{\phi} &= \frac{2\pi(1 + a_2)}{\hat{a}_1\bar{\omega}^2}. \end{aligned} \tag{4.4}$$

In order to design an LQR feedback controller, it is assumed that system is near enough to the fixed point to so that the motion may be approximated by a linear map. Let

$$\begin{aligned} x_1 &= \phi - \bar{\phi}, \\ x_2 &= \psi - \bar{\psi} \end{aligned}$$

be perturbations in ϕ and ψ from the nominal values given by (4.4) and let

$$u = \omega - \bar{\omega}$$

be a perturbation in frequency from the nominal value $\bar{\omega}$. First order changes in (4.2) are given by

$$x_{1_{j+1}} = x_{1_j} + \frac{\omega_j}{\bar{\omega}}\, x_2 + \frac{\psi_j}{\bar{\omega}}\, u_j,$$
$$x_{2_{j+1}} = -\hat{a}_1\bar{\omega}\omega_j\, \sin\phi_j x_{1_{j+1}} - a_2 x_{2_j} + \hat{a}_1\bar{\omega}\, \cos\phi_{j+1} u_j.$$

Evaluating these equations at $\omega_j = \bar{\omega},\ \psi_j = \bar{\psi},\ \phi_j = \bar{\phi}$, and $\phi_{j+1} = \bar{\phi} + \bar{\psi}$, results in

$$\mathbf{x}_{j+1} = \mathbf{A}x_j + \mathbf{B}u_j$$

with

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ a_{21} & -a_2 + a_{21} \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} \frac{\bar{\psi}}{\bar{\omega}} \\ a_{21}\,\frac{\bar{\psi}}{\bar{\omega}} + \hat{a}_1\bar{\omega}\, \cos\bar{\phi} \end{bmatrix},$$

where $a_{21} = -\hat{a}_1\bar{\omega}^2\, \sin\bar{\phi}$.

The LQR method for discrete systems may now be applied [23]. Once the gains have been determined by this process the system is controlled by adjusting the frequency every time the ball bounces according to

$$\omega = \bar{\omega} - k_1(\phi - \bar{\phi}) - k_2(\psi - \bar{\psi}). \tag{4.6}$$

4.3 Ball under control

With a plate frequency of $\bar{\omega} = 30$ there is an unstable period-1 $(m = 1,\ n = 1)$ solution to the high-bounce map, with $\bar{\phi} = 1.154$ and $\bar{\psi} = 2\pi$. The eigenvalues have magnitudes -0.4625 and -1.5855, so the solution is unstable. This is shown in the top illustration of Figure 4.1 where the initial condition was chosen very close to the fixed point solution. We see in this case that while the period-1 solution is unstable, the period-2 solution corresponding to $\bar{\omega} = 30$ is obviously stable. Our objective in this case is to stabilize the unstable solution.

The linearization of the high-bounce map results in

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ -3.780 & -3.047 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 0.2094 \\ -0.7358 \end{bmatrix}.$$

Using LQR design with $\mathbf{Q} = \mathbf{I}$, $R = 1$, the MATLAB command **dlqr** results in the feedback gains

$$\mathbf{k} = \begin{bmatrix} 3.931 & 2.843 \end{bmatrix}$$

with eigenvalues for the controlled system $\bar{\mathbf{A}} = \mathbf{A} - \mathbf{Bk}$ given by

$$\lambda = -0.3890 \pm 0.1975i \quad (|\lambda| = 0.4363)$$

all of which demonstrates that the system is locally asymptotically stable. Thus there must exist a 2×2 symmetric positive definite solution for $\mathbf{P}$ satisfying the discrete Lyapunov equation

$$\mathbf{P} = \hat{\mathbf{Q}} + \mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A},$$

where $\hat{\mathbf{Q}}$ is any 2×2 symmetric positive definite matrix such that

$$V = \mathbf{x}^{\mathrm{T}}\mathbf{P}\mathbf{x}$$

is a Lyapunov function for the system. Choosing $\hat{\mathbf{Q}} = \mathbf{I}$, we obtain

$$\mathbf{P} = \begin{bmatrix} 2.6290 & 1.5560 \\ 1.5560 & 2.5834 \end{bmatrix}.$$

Finding the largest V for which $\dot{V} < 0$ around a constant V contour results in $V_{\max} = 0.5$. This is the elliptical level curves illustrated in Figure 3.3. A small "+" locates the equilibrium solution.

5 Discussion

The procedures used here have actually been used to control both the inverted pendulum and the bouncing ball and have resulted in very satisfactory performance [16]. The bouncing ball is the more difficult system to control and it was found that due to inaccuracies in measuring ϕ and ψ, a somewhat larger value of $V_{\max}$ needs to be used. As a consequence, from time to time, the ball is not captured when the closed-loop feedback control is turned on. However the overall performance is quite satisfactory. The system is always started with the ball at rest on the plate. Generally fewer bounces under the $\omega = 45$ rad/sec control are needed to enter a controllable target than might be expected from the ball map results illustrated in Figure 4.2.

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