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Alternative Legendre Functions for Solving Nonlinear Fractional Fredholm Integro-Differential Equations

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Abstract: This paper mainly focuses on the numerical technique based on a new set of functions called the fractional alternative Legendre functions for solving the nonlinear Fredholm integro-differential equations of fractional order. Also, the convergence analysis of the proposed technique is carried out. Finally, an example is included to demonstrate the validity and applicability of the proposed technique.

Keywords: *Fredholm integro-differential equations, alternative Legendre polynomials, Caputo fractional derivative, operational matrix.*

Mathematics Subject Classification (2010): 26A33, 45J05, 35C11.

1 Introduction

In recent years, fractional calculus and differential equations have found enormous applications in mathematics, physics, chemistry and engineering because of the fact that a realistic modeling of a physical phenomenon having dependence not only on the time instant but also on the previous time history can be successfully achieved by using fractional calculus. The developed analytical solutions are very few and are restricted to the solution of simple fractional Volterra integro-differential equations, therefore the development of effective and easy to use numerical schemes for solving such equations has acquired an increasing interest in recent years. Some fundamental works on various aspects of the fractional calculus are given by [2, 3, 9, 12, 15–20, 22].

Several numerical schemes have been presented for solving these problems, for example,
Mittal and Nigam [21] used the Adomian decomposition method for solving

$$D^{\alpha}u(x) = f(x)u(x) + g(x) + \int_0^x k(x,s)G(u(s))ds, \quad 0 < \alpha < 1.$$

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$$u(0) = \Upsilon.$$

In [24] a computational method was employed for the numerical solution of the following equation:

$$D^{\alpha}u(x) = f(x) + \lambda \int_0^x k(x,s)G(u(s))ds, \quad n-1 < \alpha \leq n,$$

$$u^{(i)}(0) = \Upsilon_i, \ i = 0, 1, \cdots, n-1.$$

Hamoud and Ghadle [3] used the Adomian decomposition method and modified Laplace Adomian decomposition method for the following equation:

$$D^{\alpha}u(x) = f(x)u(x) + g(x) + \int_0^x k_1(x,s)G_1(u(s))ds + \int_0^1 k_2(x,s)G_2(u(s))ds,$$

$$u^{(i)}(0) = \Upsilon_i, \quad n-1 < \alpha \leq n, \ i = 0, 1, \cdots, n-1.$$

Motivated by the above works, in this paper we discuss a new set of functions called the fractional alternative Legendre functions for solving the nonlinear Fredholm integro-differential equations of fractional order of the form

$$D^{\alpha}u(x) = F\Big(x, u(x) + \int_0^1 K(x,s)G(s,u(s))ds\Big), \quad n-1 < \alpha \leq n, \tag{1}$$

with the initial conditions

$$u^{(i)}(x) = \Upsilon_i, \quad i = 0, 1, \cdots, n-1. \tag{2}$$

During the last decades, several methods have been used for solving fractional differential equations, fractional integro-differential equations, fractional partial differential equations and dynamic systems containing fractional derivatives such as: the homotopy analysis method [2], Chebyshev wavelets [15], Sinc functions [17], Legendre wavelets [19], shifted second kind Chebyshev polynomials [20], Legendre collocation method [23].
For considering existence and uniqueness of the solutions of fractional integro-differential equations we refer the reader to [1, 4–8, 14].

The main objective of the present paper is to study the new fractional-order functions based on the alternative Legendre polynomials for solving the nonlinear fractional Fredholm integro-differential equations (FFIDEs). This method is accurate and easy to implement in solving the FVIDEs. First, the fractional derivative of the unknown function in the underlying FFIDE is approximated by finite linear combinations of the fractional-order alternative Legendre functions (FALFs). Then, we obtain the FALFs operational matrix of fractional integration. Finally, the problem is converted to a system of algebraic equations by using the FALFs operational matrix together with the collocation method.

2 Basic Definitions

The mathematical definitions of fractional derivative and fractional integration are the subject of several different approaches. The most frequently used definitions of the fractional calculus involve the Riemann-Liouville fractional derivative and Caputo derivative [3, 9–11, 13, 15].

Definition 2.1 [3] **(Riemann-Liouville fractional integral).** The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function f is defined as

$$\begin{aligned} J^\alpha f(x) &= \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt, \qquad x>0, \quad \alpha \in \mathbb{R}^+, \\ J^0 f(x) &= f(x), \end{aligned} \tag{3}$$

where $\mathbb{R}^+$ is the set of positive real numbers.

Definition 2.2 [3] **(Caputo fractional derivative).** The fractional derivative of $f(x)$ in the Caputo sense is defined by

$$\begin{aligned} {}^cD_x^\alpha f(x) &= J^{m-\alpha} D^m f(x) \\ &= \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} \frac{d^m f(t)}{dt^m} dt, & m-1<\alpha<m, \\ \frac{d^m f(x)}{dx^m}, & \alpha = m, \quad m \in N, \end{cases} \end{aligned} \tag{4}$$

where the parameter α is the order of the derivative and is allowed to be real or even complex. In this paper, only real and positive α will be considered. Hence, we have the following properties:

1. $J^\alpha J^v f = J^{\alpha+v} f, \quad \alpha, v > 0.$
2. $J^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} x^{\beta+\alpha}.$
3. $D^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha}, \quad \alpha>0, \quad \beta>-1, \quad x>0.$
4. $J^\alpha D^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+)\frac{x^k}{k!}, \quad x>0, \quad m-1<\alpha\le m.$

Definition 2.3 [3] **(Riemann-Liouville fractional derivative).** The Riemann-Liouville fractional derivative of order $\alpha > 0$ is normally defined as

$$D^\alpha f(x) = D^m J^{m-\alpha} f(x), \qquad m-1<\alpha\le m, \quad m \in \mathbb{N}. \tag{5}$$

3 Fractional Alternative Legendre Polynomials

Let m be a fixed non-negative integer. The set $P_m = \{p_{m,i}(t)\}_{i=0}^m$ of alternative Legendre polynomials is

$$\begin{aligned} p_{m,i}(t) &= \sum_{r=0}^{m-i} (-1)^r \binom{m-i}{r} \binom{m+i+r+1}{m-i} t^{i+r} \\ &= \sum_{r=i}^{m} (-1)^{r-i} \binom{m-i}{r-i} \binom{m+r+1}{m-i} t^r, \quad i=0,1,\cdots,m. \end{aligned} \tag{6}$$

These polynomials are orthogonal on the interval $[0,1]$ with respect to the weight function $w(t) = 1$, and satisfy the orthogonality relationships

$$\int_0^1 p_{m,k}(t) p_{m,l}(t) dt = \frac{1}{k+l+1} \delta_{k,l}, \quad k,l = 0,1,\cdots,m. \tag{7}$$

Here $\delta_{k,l}$ denotes the Kronecker delta [23]. It should be noted that, in contrast to common sets of orthogonal polynomials, every member in P_m has degree m. For example, when $m = 3$, we have

$$\begin{aligned} p_{3,0}(t) &= 4 - 30t + 60t^2 - 35t^3, \\ p_{3,1}(t) &= 10t - 30t^2 + 21t^3, \\ p_{3,2}(t) &= 6t^2 - 7t^3, \\ p_{3,3}(t) &= t^3. \end{aligned} \tag{8}$$

Eq. (6) obtains Rodrigues's type representation

$$p_{m,i}(t) = \frac{1}{(m-i)!}\frac{1}{t^{i+1}}\frac{d^{m-i}}{dt^{m-i}}(t^{m+i+1}(1-t)^{m-i}), \quad i = 0, 1, \cdots, m. \tag{9}$$

It follows from (9) that

$$\int_0^1 p_{m,i}(t)dt = \int_0^1 t^m dt = \frac{1}{m+1}, \quad i = 0, 1, 2, \cdots, m. \tag{10}$$

Now, we define a new set of fractional functions based on the alternative Legendre polynomials to obtain the solution of NVIDEs. The FALFs are obtained by a change of variable t to $x^\alpha (\alpha > 0)$, on the alternative Legendre polynomials. We denote $p_{m,i}(x^\alpha)$ by $p^\alpha_{m,i}(x)$. Therefore we have

$$\begin{aligned} p^\alpha_{m,i}(x) &= \sum_{r=0}^{m-i}(-1)^r \binom{m-i}{r}\binom{m+i+r+1}{m-i} x^{(i+r)\alpha} \\ &= \sum_{r=i}^{m}(-1)^{r-i} \binom{m-i}{r-i}\binom{m+r+1}{m-i} x^{r\alpha}, \quad i = 0, 1, \cdots, m. \end{aligned} \tag{11}$$

The set of FALFs is orthogonal with respect to the weight function $w(x) = x^{\alpha-1}$ on the interval $[0, 1]$ with the orthogonality property

$$\int_0^1 p^\alpha_{m,k}(x)p^\alpha_{m,l}(x)x^{\alpha-1}dx = \frac{1}{(k+l+1)\alpha}\delta_{k,l}, \quad k, l = 0, 1, \cdots, m. \tag{12}$$

For example, when $m = 3$, we have

$$\begin{aligned} p^\alpha_{3,0}(x) &= 4 - 30x^\alpha + 60x^{2\alpha} - 35x^{3\alpha}, \\ p^\alpha_{3,1}(x) &= 10x^\alpha - 30x^{2\alpha} + 21x^{3\alpha}, \\ p^\alpha_{3,2}(x) &= 6x^{2\alpha} - 7x^{3\alpha}, \\ p^\alpha_{3,3}(x) &= x^{3\alpha}. \end{aligned} \tag{13}$$

Any $f \in L^2[0, 1]$ may be expanded in terms of the fractional-order alternative Legendre functions as

$$f(x) = \sum_{i=0}^{\infty} c_i p^\alpha_{m,i}(x), \tag{14}$$

where the coefficients c_i are given by

$$c_i = < f, p^{\alpha}_{m,i} > = (2i+1)^{\alpha} \int_0^1 f(x) p^{\alpha}_{m,i}(x) x^{\alpha-1} dx,$$

where $<,>$ denotes the inner product in $L^2[0,1]$. If the infinite series in Eq. (14) is truncated, then it can be written as

$$f(x) \simeq \sum_{i=0}^{m} c_i p^{\alpha}_{m,i}(x) = C^T \Phi^{\alpha}(x), \tag{15}$$

where T indicates transposition, C and $\Phi^{\alpha}(x)$ are $(m+1)\times 1$ vectors given by

$$C = [c_0, c_1, c_2, \cdots, c_m]^T, \tag{16}$$

and

$$\Phi^{\alpha}(x) = [p^{\alpha}_{m,0}(x), p^{\alpha}_{m,1}(x), p^{\alpha}_{m,2}(x), \cdots, p^{\alpha}_{m,m}(x)]^T. \tag{17}$$

Now we will derive the fractional-order alternative Legendre functions operational matrix of the fractional integration. The Riemann-Liouville fractional integration of the vector $\Phi^{\alpha}(x)$ given in equation (17) is obtained by

$$I^{\nu}\Phi^{\alpha}(x) = F^{(\nu,\alpha)}\Phi^{\alpha}(x), \tag{18}$$

where $F^{(\nu,\alpha)}$ is the $(m+1)\times(m+1)$ operational matrix of the fractional integration of order α in the Riemann-Liouville sense.

By using Eq. (11) and linearity of the Riemann-Liouville fractional integral, for $i = 0, 1, \cdots, m$, we get

$$\begin{aligned} I^{\nu} p^{\alpha}_{m,i}(x) &= \sum_{r=i}^{m} (-1)^{r-i} \binom{m-i}{r-i} \binom{m+r+1}{m-i} I^{\alpha} x^{r\alpha} \\ &= \sum_{r=i}^{m} (-1)^{r-i} \binom{m-i}{r-i} \binom{m+r+1}{m-i} \frac{\Gamma(r\alpha+1)}{\Gamma(r\alpha+\nu+1)} x^{r\alpha+\nu} \\ &= \sum_{r=i}^{m} \gamma^{(\nu,\alpha)}_{mi,r} x^{r\alpha+\nu}, \end{aligned} \tag{19}$$

where

$$\gamma^{(\nu,\alpha)}_{mi,r} = (-1)^{r-i} \binom{m-i}{r-i} \binom{m+r+1}{m-i} \frac{\Gamma(r\alpha+1)}{\Gamma(r\alpha+\nu+1)}.$$

Now, approximating $x^{r\alpha+\nu}$ by $m+1$ terms of the fractional-order alternative Legendre functions, we get

$$x^{r\alpha+\nu} \simeq \sum_{j=0}^{m} \delta^{(\nu,\alpha)}_{r,j} p^{\alpha}_{m,j}(x). \tag{20}$$

Substituting Eq. (20) into Eq. (19) for $i = 0, 1, \cdots, m$, we obtain

$$I^{\nu} p^{\alpha}_{m,i}(x) \simeq \sum_{r=i}^{m} \gamma^{(\nu,\alpha)}_{mi,r} \sum_{j=0}^{m} \delta^{(\nu,\alpha)}_{r,j} p^{\alpha}_{m,j}(x) = \sum_{j=0}^{m} \Big(\sum_{r=i}^{m} \omega^{(\nu,\alpha)}_{mi,j,r} \Big) p^{\alpha}_{m,j}(x), \tag{21}$$

where

$$\omega_{mi,j,r}^{(\nu,\alpha)} = \gamma_{mi,r}^{(\nu,\alpha)} \delta_{r,j}^{(\nu,\alpha)}.$$

Eq. (21) can be rewritten as

$$I^{\nu} p_{m,i}^{\alpha}(x) \simeq [\sum_{r=i}^{m} \omega_{mi,0,r}^{(\nu,\alpha)}, \sum_{r=i}^{m} \omega_{mi,1,r}^{(\nu,\alpha)}, \cdots, \sum_{r=i}^{m} \omega_{mi,m,r}^{(\nu,\alpha)}] \Phi^{\alpha}(x).$$

Finally, we get

$$F^{(\nu,\alpha)} = \begin{bmatrix} \sum_{r=0}^{m} \omega_{m0,0,r}^{(\nu,\alpha)} & \sum_{r=0}^{m} \omega_{m0,1,r}^{(\nu,\alpha)} & \cdots & \sum_{r=0}^{m} \omega_{mi,m,r}^{(\nu,\alpha)} \\ \sum_{r=1}^{m} \omega_{m1,0,r}^{(\nu,\alpha)} & \sum_{r=1}^{m} \omega_{m1,1,r}^{(\nu,\alpha)} & \cdots & \sum_{r=1}^{m} \omega_{m1,m,r}^{(\nu,\alpha)} \\ \vdots & \vdots & \cdots & \vdots \\ \omega_{mm,0,r}^{(\nu,\alpha)} & \omega_{mm,1,r}^{(\nu,\alpha)} & \cdots & \omega_{mm,m,r}^{(\nu,\alpha)} \end{bmatrix}.$$

4 Description of the Method

In this section, we present a numerical method for solving the fractional Fredholm integro-differential equation (1)-(2). To solve this equation, we first expand $D^{\alpha}u(x)$ by the fractional-order alternative Legendre functions as

$$D^{\nu} u(x) \simeq D^{\nu} u_m(x) = C^T \Phi^{\alpha}(x), \tag{22}$$

with C and $\Phi^{\alpha}(x)$ defined in the previous section. By applying I^{α} on both sides of (22), we obtain

$$u(x) \simeq u_m(x) = C^T F^{(\nu,\alpha)} \Phi^{\alpha}(x) + \sum_{k=0}^{n-1} \frac{x^k}{k!} \Upsilon_k, \tag{23}$$

where $F^{(\nu,\alpha)}$ is the operational matrix of fractional integration of order α of the fractional-order alternative Legendre functions. Now, by substituting Eqs.(22)-(23) into (1), we have

$$\begin{aligned} C^T \Phi^{\alpha}(x) &= F\Big(x, C^T F^{(\nu,\alpha)} \Phi^{\alpha}(x) + \sum_{k=0}^{n-1} \frac{x^k}{k!} \Upsilon_k, \\ &\quad \int_0^1 K(x,s) G(s, C^T F^{(\nu,\alpha)} \Phi^{\alpha}(s) + \sum_{k=0}^{n-1} \frac{x^k}{k!} \Upsilon_k) ds\Big) \\ &\quad + Res_m(x), \end{aligned} \tag{24}$$

where $Res_m(x),\ x \in [0,1]$, is a residual error; that is, the error made when substituting the approximate solution into the governing equation. By using the Gauss-Legendre numerical integration for evaluating the integral in Eq. (24), we get

$$\begin{aligned} C^T\Phi^\alpha(x) &= F\Big(x, C^TF^{(\nu,\alpha)}\Phi^\alpha(x) + \sum_{k=0}^{n-1}\frac{x^k}{k!}\Upsilon_k, \\ &\quad \frac{1}{2}\sum_{j=1}^{\tilde{m}}\omega_j K(x,\frac{1+\zeta_j}{2})G(\frac{1+\zeta_j}{2}, C^TF^{(\nu,\alpha)}\Phi^\alpha(\frac{1+\zeta_j}{2}) \\ &\quad +\sum_{k=0}^{n-1}\frac{(\frac{1+\zeta_j}{2})^k}{k!}\Upsilon_k)\Big) \\ &\quad +E_{\tilde{m}} + Res_m(x), \end{aligned} \tag{25}$$

where $\zeta_j, j = 1, 2, \cdots, \tilde{m}$ are zeros of the Legendre polynomial $P_{\tilde{m}}(x)$ and $\omega_j = \frac{-2}{(\tilde{m}+1)P_{\tilde{m}}(\zeta_j)P_{\tilde{m}}+1(\zeta_j)}$ and $E_{\tilde{m}}$ is the error between the Gauss-Legendre rule and the exact integral given in [24]. By collocating Eq. (25) at the zeros of the shifted Legendre polynomials $L_{m+1}(x); (x_i,\ i = 0, 1, \cdots, m)$ we have

$$\begin{aligned} C^T\Phi^\alpha(x_i) &= F\Big(x_i, C^TF^{(\nu,\alpha)}\Phi^\alpha(x_i) + \sum_{k=0}^{n-1}\frac{x_i^k}{k!}\Upsilon_k, \\ &\quad \frac{1}{2}\sum_{j=1}^{\tilde{m}}\omega_j K(x_i,\frac{1+\zeta_j}{2})G(\frac{1+\zeta_j}{2}, C^TF^{(\nu,\alpha)}\Phi^\alpha(\frac{1+\zeta_j}{2}) \\ &\quad +\sum_{k=0}^{n-1}\frac{(\frac{1+\zeta_j}{2})^k}{k!}\Upsilon_k)\Big). \end{aligned} \tag{26}$$

Eqs. (26) are nonlinear equations which can be solved for the unknown C using Newton's iterative method. By determining C, the values of $u(x)$ can be obtained from Eq. (23).

5 Convergence Analysis

In this section we investigate the convergence of the proposed method for solving FFIDEs. Before starting and proving the main results, we introduce the following hypotheses:

(H1) There exists a constant K_1 such that $K_1 = \max|K(x, s)|;\ (x, s) \in [0, 1] \times [0, 1]$.

(H2) u is a bounded function for all x in $[0, 1]$.

(H3) F and G satisfy the Lipschitz conditions with the Lipschitz constants η and η_1, respectively.

Theorem 5.1 *Assume that (H1)–(H3) hold, and let u and u_m be the exact and approximate solution of* (1)-(2)*, respectively. If* $\Gamma(\alpha)-\eta-K_1\eta\eta_1 \neq 0$*, then* $\|u-u_m\| \longrightarrow 0$.

Proof. Let e_m denote the error function as

$$e_m(x) = u(x) - u_m(x),$$

so from (1) we can write

$$\begin{aligned} D^{\alpha} e_m(x) &= F\Big(x, u(x), \int_0^1 K(x,s)G(s,u(s))ds\Big) \qquad (27)\\ &\quad -F\Big(x, u_m(x), \int_0^1 K(x,s)G(s,u_m(s))ds\Big) - E_{\tilde{m}} - Res_m(x). \end{aligned}$$

Using the definitions of the fractional derivative and integral, it is suitable to rewrite (27) in the integral form

$$\begin{aligned} e_m(x) &= I^{\alpha}\Big(F\Big(x, u(x), \int_0^1 K(x,s)G(s,u(s))ds\Big) \qquad (28)\\ &\quad -F\Big(x, u_m(x), \int_0^1 K(x,s)G(s,u_m(s))ds\Big)\Big) - I^{\alpha}E_{\tilde{m}} - I^{\alpha}Res_m(x). \end{aligned}$$

It follows from (28) that

$$e_m(x) = \Lambda_1(x) - \Lambda_2(x) - \Lambda_3(x), \qquad (29)$$

where

$$\begin{aligned} \Lambda_1(x) &= I^{\alpha}\Big(F\Big(x, u(x), \int_0^1 K(x,s)G(s,u(s))ds\Big)\\ &\quad -F\Big(x, u_m(x), \int_0^1 K(x,s)G(s,u_m(s))ds\Big)\Big), \qquad (30)\\ \Lambda_2(x) &= I^{\alpha}E_{\tilde{m}}, \qquad (31)\\ \Lambda_3(x) &= I^{\alpha}Res_m(x). \qquad (32) \end{aligned}$$

We now estimate the three terms one by one. For Λ_1, we have

$$\begin{aligned} \Big|\Lambda_1(x)\Big| &= \Big|\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\Big(F\Big(t, u(t), \int_0^1 K(t,s)G(s,u(s))ds\Big)\\ &\quad -F\Big(t, u_m(t), \int_0^1 K(t,s)G(s,u_m(s))ds\Big)\Big)dt\Big|\\ &\le \frac{1}{\Gamma(\alpha)}\int_0^x |x-t|^{\alpha-1}\Big|\Big(F\Big(t, u(t), \int_0^1 K(t,s)G(s,u(s))ds\Big)\\ &\quad -F\Big(t, u_m(t), \int_0^1 K(t,s)G(s,u_m(s))ds\Big)\Big)\Big|dt. \qquad (33) \end{aligned}$$

Since $|x-t| \le 1$ and F and G satisfy the Lipschitz conditions, we obtain

$$\Big|\Lambda_1(x)\Big| = \frac{1}{\Gamma(\alpha)}\int_0^1 (\eta + K_1\eta\eta_1)|u(t) - u_m(t)|dt. \qquad (34)$$

Using $0 \le t \le x \le 1$ leads to

$$\Big|\Lambda_1(x)\Big| = \frac{1}{\Gamma(\alpha)}\int_0^1 (\eta + K_1\eta\eta_1)|e_m(t)|dt. \qquad (35)$$

So, we have

$$\|\Lambda_1\| \leq \frac{1}{\Gamma(\alpha)}(\eta + K_1\eta\eta_1)\|e_m\|. \tag{36}$$

For Λ_2 and Λ_3, we have

$$\|\Lambda_2\| \leq \frac{1}{\Gamma(\alpha)}\|E_{\tilde{m}}\|, \quad \|\Lambda_3\| \leq \frac{1}{\Gamma(\alpha)}\|Res_m\|. \tag{37}$$

Then,

$$\|e_m\| \leq \frac{1}{\Gamma(\alpha)}(\eta + K_1\eta\eta_1)\|e_m\| + \frac{1}{\Gamma(\alpha)}\|E_{\tilde{m}}\| + \frac{1}{\Gamma(\alpha)}\|Res_m\|. \tag{38}$$

Consequently,

$$\|e_m\| \leq \frac{\|E_{\tilde{m}}\| + \|Res_m\|}{\Gamma(\alpha) - \eta - K_1\eta\eta_1}. \tag{39}$$

If we choose $\tilde{m}$ sufficiently large, then by [24], $E_{\tilde{m}}$ tends to 0. So, if Res_m tends to 0, then $\|e_m\| = \|u - u_m\| \longrightarrow 0$. The numerical results reveal that Res_m tends to 0.

6 Numerical Example

In this section, we give a numerical example and apply the technique for solving it.

Example 1. Consider the following nonlinear FFIDE:

$$D^\alpha u(x) = f(x) + \int_0^1 (x+s)^2 u^3(s)ds \tag{40}$$

with the initial conditions $u(0) = u'(0) = 0$, where $f(x) = \frac{6x^{\frac{1}{3}}}{\Gamma(\frac{1}{3})} - \frac{x^2}{7} - \frac{x}{4} - \frac{1}{9}$, and the exact solution is $u(x) = x^2$ when $\alpha = \frac{5}{3}$.

Table 1: The absolute errors with m = 6 for Example 1.

x	$\alpha = 1$	$\alpha = \frac{1}{2}$	$\alpha = \frac{1}{3}$	$\alpha = \frac{5}{3}$
0.1	2.96×10^{-4}	5.13×10^{-5}	1.12×10^{-14}	5.54×10^{-5}
0.2	4.73×10^{-4}	8.40×10^{-5}	2.11×10^{-14}	1.25×10^{-3}
0.3	6.61×10^{-4}	1.16×10^{-4}	3.23×10^{-14}	1.25×10^{-3}
0.4	8.60×10^{-4}	1.51×10^{-4}	4.53×10^{-14}	7.17×10^{-4}
0.5	1.07×10^{-3}	1.88×10^{-4}	6.03×10^{-14}	1.68×10^{-4}
0.6	1.28×10^{-3}	2.29×10^{-4}	7.75×10^{-14}	4.72×10^{-4}
0.7	1.53×10^{-3}	2.74×10^{-4}	9.74×10^{-14}	2.16×10^{-3}
0.8	1.82×10^{-3}	3.24×10^{-4}	1.20×10^{-13}	3.37×10^{-3}
0.9	2.15×10^{-3}	3.80×10^{-4}	1.45×10^{-13}	8.49×10^{-4}
1.0	2.46×10^{-3}	4.44×10^{-4}	1.74×10^{-13}	5.70×10^{-3}

Table 1 shows the absolute errors between the exact and approximate solutions $|u(x) - u_m(x)|$ for $m = 6$ and various choices of α.

7 Conclusion

In this paper, we derive a general formulation for the fractional alternative Legendre functions and obtain their operational matrix of fractional integration $F(\nu, \alpha)$. Then, a numerical method based on the FALFs expansions together with this matrix and the collocation method is proposed to obtain the numerical solution of the nonlinear fractional Fredholm integro-differential equations. Several examples are given to demonstrate the validity and applicability of the proposed method for solving the fractional Fredholm integro-differential equations. Some of the advantages of the present approach are summarized as follows. It is shown that only a small value of the fractional alternative Legendre functions is needed to achieve high accuracy and satisfactory results.

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