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Unified Continuous and Discrete Lur'e Systems Stability Analysis Based on Augmented Model Description

A. Ayadi [1*], K. Dchich [2], B. Sfaihi [1] and M. Benrejeb [1]

[1] *Reseach Laboratory of Automatic Control (LA.R.A), University of Tunis El Manar, National Engineering School of Tunis, PO Box 37, 1002 Tunis Le Belvdre, Tunisia,*
[2] *Laborotory LISIER, University of Tunis, ENSIT, PO Box 56, 1008, Tunis, Tunisia*

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Abstract: The proposed unified approach for stability analysis of nonlinear Lur'e continuous- and discrete-time systems is based on a unified Borne-Gentina practical stability criterion and augmented systems description. New Lur'e systems stability conditions are developed and compared with the original ones. An illustrative example is considered to show the efficiency of the proposed stability approaches.

Keywords: *Lur'e systems; augmented models; stability; vector norms; arrow form matrix.*

Mathematics Subject Classification (2010): 93C55, 93D09, 93D15.

1 Introduction

The presence of model nonlinearities in most control problems is still a big challenge for modern control theory [2, 6, 7] since there is no universal design procedure for nonlinear systems. Lur'e systems [3] represent an important and common class of nonlinear systems and refer to such systems that consist of a linear dynamical system and a nonlinear feedback loop satisfying certain sector conditions.

The stability of Lur'e systems is stated, first, as an absolute stability problem of the equilibrium point at the origin, then, as the asymptotic stability for any nonlinearity belonging to certain section conditions. Later, different stability criteria are derived via different forms of Lyapunov functions (LFs): the classical quadratic LF [16], non-quadratic Lur'e-type LF [3], the piecewise quadratic LF [1] and fuzzy LF [20].

* Corresponding author: mailto:anis.ayadi@enit.utm.tn

Despite these advances, most results of stability analysis of Lur'e systems are based on the Lyapunov approach and presented in continuous-time and discrete-time, separately. Moreover, to the best knowledge of the authors, there is no unified technique that develops stability conditions in the continuous- and discrete-time domain.

Among stability analysis techniques, the comparison principle is an efficient solution to the stability problem of dynamical systems [7, 9]. The method is powerful and efficient and has been applied successfully to continuous- and discrete-time systems. The comparison principle is based on defining an ordinary differential equation or functional differential equation, called the comparison system, whose stability properties imply the stability properties of the initial system. The idea is to approximate the available differential equation from above or below through relations that ensure guaranteed upper or lower solution estimates by operating with functions simpler than those in the original equation. Two main approaches have been used to construct the comparison system. The first approach is based on the construction of a Lyapunov function for the comparison system. Lyapunov theory is then used to synthesize system stability criteria [9, 16, 17]. However, this approach is generally conservative and is not systematic because of the chosen Lyapunov function. The second approach is to associate a second level comparison system to the original one. Stability criteria of the original system can be established based on the comparison system. The method is simpler and can be combined with other techniques (i.e., vector norms) leading to systematic stability approaches [9–12, 16].

Arrow form representations provide a straightforward method to describe linear and nonlinear systems, that show effective results when integrated in the systems analysis process. The arrow form state space matrices, i.e., the Benrejeb arrow form matrix, have been introduced by Benrejeb in the early seventies [13, 15]. Since then, combined with aggregation techniques [10,15], it has become a systematic procedure for stability analysis and synthesis of large classes of nonlinear systems [10, 11, 15]. The arrow form matrix representations were successfully applied to the stability/stabilization study of important classes of nonlinear systems: fuzzy models [11, 20], singularly perturbed systems [4, 12], time-delay systems [19], interconnected systems [5] and choatic systems [18].

The main contribution of this paper is to develop a new technique to solve the stability analysis problem of Lur'e type systems in a unified and systematic manner for continuous- and discrete-time descriptions. In this context, based on the Borne-Gentina practical stability criteria [8, 9], an advanced and unified formulation for stability criteria of nonlinear systems is synthesized. The case of Lur'e systems is considered for investigations and new stability conditions are synthesized in a systematic manner by the definition of a unified augmented model description and the use of the comparison principle. Convenient further results are developed based on the arrow form matrix representation.

The paper is organized as follows. In Section 2, the Lur'e augmented model is developed and the stability problem is formally stated. In Section 3, a unified formulation of the Borne-Gentina practical stability criteria is introduced. In Section 4, new unified continuous and discrete Lur'e systems asymptotic stability conditions are provided. The case of diagonal characteristic matrix of the linearized system is considered. In Section 5, a second-order Lur'e system and the associated discretized model are considered to illustrate the efficiency of the proposed approaches. Concluding remarks are found in Section 6.

2 Problem Statement

Let us consider the continuous-time and discrete-time nonlinear systems described in a unified state space description as follows:

$$D[x(\tau)] = A(.)x(\tau). \tag{1}$$

$D[.]$ is the derivative operator $\frac{d}{dt}$ and Δ is the shift operator for the continuous-time and the discrete-time systems, respectively, $x(\tau) \in D \subset R^n$ is the state vector and $A(.) = \{a_{i,j}(.)\} \in R^{n\times n}$ is the instantaneous characteristic matrix.

When the study of system (1) turns out to be complex, the comparison principle gives a way of comparing system stability property with a simpler system one, for which it can be easier to establish algebraic stability conditions.

The aggregation technique using a regular vector norm $p(x)$ enables one to construct in a systematic way the corresponding comparison system defined as [16]

$$D\left[z(\tau)\right] = M(A(.))z(\tau), z_0 = p(x_0), \tag{2}$$

$M(A(.))$, called the overvaluing of the matrix $A(.)$, is such that

$$D\left[p(x(\tau))\right] \leq M(A(.))z(\tau). \tag{3}$$

When its off diagonal elements are non-negative and nonconstant elements are regrouped in one column or one row, the stability condition can be easily deduced from the application of the Borne-Gentina practical stability criterion based on the M-matrices technique [16].

A change of basis remained an abundant solution to bypass this structured condition problem on the matrix $A(.)$, and the transformation of the caracteristic matrix to an arrow form matrix appears to be a well-adapted model description to the use of this method, in particular, when the model $A(.)$ is in the companion or Frobenius form [10–15].

Let us consider both continuous- and discrete-time Lur'e system [3] described by

$$\begin{cases} D\left[x(\tau)\right] = Ax(\tau) + Bu(\tau), \\ u(\tau) = f\left(\varepsilon(\tau)\right)\varepsilon(\tau), \\ \varepsilon(\tau) = r(\tau) - C^T x(\tau), \end{cases} \tag{4}$$

$A \in R^{n\times n}$, $B \in R^{n\times 1}$ and $C \in R^{n\times 1}$ are constant matrices, $x(\tau) \in R^n$ is the state vector, $u(\tau) \in R$ is the control input, $r \in R$ is the reference input, $\varepsilon(\tau) \in R$ is the error of the closed-loop system, and $f(\varepsilon(\tau)) : R \to R$ is a nonlinear function.

Use the analytical relationship linking together the nonlinear equation description of the system (4) and its linearized model (5), for which the nonlinearity $f(\varepsilon(\tau))$ is considered constant and equal to f_l.

$$D[x(\tau)] = A_l x(\tau) \tag{5}$$

with

$$A_l = A - f_l BC^T. \tag{6}$$

Let us introduce an augmented model description for the autonomous Lur'e system (4) or (7) by choosing $\bar{X} \in R^{n+1}$ as the new state space vector such that $\bar{X} = \begin{bmatrix} x^T & \varepsilon \end{bmatrix}^T$.

$$\begin{cases} D[x(\tau)] = \left(A - f\left(\varepsilon(\tau)\right)BC^T\right)x(\tau), \\ D[\varepsilon(\tau)] = -C^T D[x(\tau)]. \end{cases} \tag{7}$$

Adding and removing $f_l BC^T$ in the first equation of (7), we obtain the following model:

$$\begin{cases} D[x(\tau)] = \left(A - f_l BC^T\right) x(\tau) - \left(f\left(\varepsilon(\tau)\right) - f_l\right) BC^T x(\tau), \\ D[\varepsilon(\tau)] = -C^T \left(A - f\left(\varepsilon(\tau)\right) BC^T\right) x(\tau), \end{cases} \tag{8}$$

which can be written, in a compact form, as

$$D[\bar{X}(\tau)] = A_a\left(.\right) \bar{X}(\tau) \tag{9}$$

with

$$A_a\left(.\right) = \left(\begin{array}{c|c} A_l & B(f\left(\varepsilon(\tau)\right) - f_l) \\ \hline -C^T A & -C^T B f\left(\varepsilon(\tau)\right) \end{array}\right). \tag{10}$$

The instantaneous characteristic matrix of the augmented system (9), $A_a(.) = \{a_{i,j}(.)\} \in R^{(n+1)\times(n+1)}$, highlights the characteristic matrix of the linearized system (5).

Now, the stability analysis of this system is synthesized, in Section 4, by using the Borne-Gentina stability criterion [8, 9], introduced in Section 3, and the arrow form matrices for system description.

3 Proposed Unified Formulation of the Borne-Gentina Practical Stability Criterion

Consider the dynamic systems (1) and introduce two parameters δ_1 and δ_2 and the matrix $M(A(.)) = \{m_{i,j}(.)\}$ defined such that

for the continuous-time case

$$\delta_1 = 0, \ \ \delta_2 = -1, \tag{11}$$

and

$$M(A(.)) = M_1(A(.)) \tag{12}$$

with

$$M_1(A(.)) : \begin{cases} m_{i,i}(.) = a_{i,i}(.) \quad \forall i = 1, 2, \ldots, n, \\ m_{i,j}(.) = |a_{i,j}(.)| \quad \forall i \neq j, \, \forall i, j = 1, 2, \ldots, n, \end{cases} \tag{13}$$

and for the discrete-time case

$$\delta_1 = \ \delta_2 = 1 \tag{14}$$

and

$$M(A(.)) = M_2(A(.)) \tag{15}$$

with

$$M_2(A(.)) : m_{i,j}(.) = |a_{i,j}(.)| \quad \forall i, j = 1, 2, \ldots, n. \tag{16}$$

The following theorem is proposed.

Theorem 3.1 *The nonlinear system (1) is asymptotically stable if the matrix*

$$M^*(.) = \delta_2 \left(\delta_1 I_n - \delta_2 M(.)\right) \tag{17}$$

satisfies the following conditions:

(i) the non-constant elements of $M^*(.)$ are isolated in only one row,

(ii) the successive minors $\Delta_j(M^*(.))$ of $M^*(.)$ are positive, i.e.,

$$(\delta_2)^j \Delta_j(M^*(.)) > 0 \ \ \forall j = 1, ..., n \forall x \in \ D \subset \ R^n. \tag{18}$$

If $D = R^n$, the stability property is global.

Proof. Theorem 3.1 constitutes a direct application of the Borne-Gentina practical stability criterion [8, 9], for which a unified formulation is proposed here. The choice of the vector norm

$$p\left(x(\tau)\right)=\left[\left|x_1(\tau)\right|,\left|x_2(\tau)\right|,\ldots,\left|x_n(\tau)\right|\right]^T,\ \forall\, x(\tau)=[x_1(\tau),x_2(\tau),\ldots,x_n(\tau)]^T, \tag{19}$$

as a vector Lyapunov function, leads to the corresponding comparison system

$$D\left[z(\tau)\right]=M(A(.))z(\tau) \tag{20}$$

characterized by the overvaluing matrix $M(A(.))$ defined previously.

If the nonlinear elements of that matrix are isolated in only one row, the stability conditions given by the application of the Borne-Gentina criterion are based on the verification of the positivity of n principal minors of the matrix $M^*(.)$, i.e.,

- $$(-M_1(A(.))\begin{pmatrix} 1 & 2 & \cdots & j \\ 1 & 2 & \cdots & j \end{pmatrix} > 0\ \ \forall j=1,2,\ldots,n, \tag{21}$$

for the continuous-time system case with

$$M^*(.)=-M_1(A(.)); \tag{22}$$

- $$(I-M_2(A(.))\begin{pmatrix} 1 & 2 & \cdots & j \\ 1 & 2 & \cdots & j \end{pmatrix} \succ 0\ \ \forall j=1,2,\ldots,n, \tag{23}$$

for the discrete-time system case with

$$M^*(.)=I-M_2(A(.)). \tag{24}$$

This completes the Theorem 3.1 proof.

4 New Unified Continuous- and Discrete-time Lur'e System Asymptotic Stability Conditions

Let us consider the augmented system described by ((9)-(10)) and the corresponding pseudo-overvaluing matrix $\bar{M}(.)$ obtained by the choice of the vector norm $p\left(\bar{X}\right)=\left[\left|\bar{X}_1\right|,\left|\bar{X}_2\right|,\ldots,\left|\bar{X}_n\right|,\left|\bar{X}_{n+1}\right|\right]^T$ such that

$$D\left[p(\bar{X})\right]\leq\bar{M}\left(A_a(.)\right)p(\bar{Z}). \tag{25}$$

We have the following comparison system:

$$D\left[\bar{Z}\right]=\bar{M}(A_a(.))\bar{Z},\bar{Z}_0=p(\bar{X}_0), \tag{26}$$

characterized by the matrix $\bar{M}(A_a(.))=\{\bar{m}_{i,j}(.)\}\in R^{(n+1)\times(n+1)}$, such that

for the continuous-time system case

$$\begin{cases} \bar{m}_{i,i}(.)=a_{i,i}(.)\ \ \forall i=1,2,\ldots,n+1, \\ \bar{m}_{i,j}(.)=\left|a_{i,j}(.)\right|\ \ \forall i\neq j, \end{cases} \tag{27}$$

and for the discrete-time system case

$$\bar{m}_{i,j}(.)=\left|a_{i,j}(.)\right|\ \ \forall i,j=1,2,\ldots,n+1. \tag{28}$$

Theorem 4.1 *The continuous-time system, i.e.,* $\delta_1 = 0$, $\delta_2 = -1$ *, (resp. the discrete-time system,* $\delta_1 = \delta_2 = 1$ *) defined by (1) is asymptotically stable if the following conditions are satisfied:*

1. The linearized system is stable, i.e., the first n *successive principal minors of the matrix* $(\delta_1 I_{n+1} - \delta_2 \bar{M}(.))$ *are such that*

$$\delta_2{}^j (\delta_1 I_{n+1} - \delta_2 \bar{M}(.)) \begin{pmatrix} 1 & 2 & \cdots & j \\ 1 & 2 & \cdots & j \end{pmatrix} \succ 0 \;\; \forall j = 1, ..., n, \tag{29}$$

2. The nonlinearity $f(\varepsilon(\tau))$ *satisfies the inequality*

$$\delta_2{}^{n+1} \det(\delta_1 I_{n+1} - \delta_2 \bar{M}(.)) \succ 0 \; . \tag{30}$$

Proof. Since the nonlinearities of the comparison Lur'e system (26-28) are isolated in the last column of the matrix, stability conditions are obtained by the application of the BorneGentina stability criterion [8,9] based on the verification of the positivity definition of $n+1$ principal minors of the matrix $-\bar{M}(.)$ (for the continuous-time system case) and of $(I_{n+1} - \bar{M}(.))$ (for the discrete-time system case), the first n ones corresponding to the sufficient stability conditions of the linearized system characterized by the matrix A_l. This completes the Theorem 4.1 proof.

Let us consider, now, the stability conditions reformulation for A_l diagonalizable. When the eigenvalues $\rho_i \, \forall i = 1, 2, \ldots, n$ of the characteristic matrix of the linearized system A_l are

$$\begin{cases} \rho_i \in R \;\; and \;\; \rho_i \neq 0 \;\; \forall i = 1, 2, \ldots, n, \\ \rho_i \neq \rho_j \, , \quad i \neq j \qquad \forall i = 1, 2, \ldots, n, \end{cases} \tag{31}$$

the matrix $P_d \in R^{n \times n}$ diagonalizing A_l, is such that

$$D = P_d^{-1} A_l P_d, \tag{32}$$

and the change of base (33) with $\tilde{X} = \left[\tilde{X}_1, \tilde{X}_2, \; \ldots, \tilde{X}_n \, , \tilde{X}_{n+1} \right]^T$, and the matrix $P \in R^{(n+1)\times(n+1)}$ is an invertible matrix such that

$$\tilde{X} = P\bar{X}, \, P = \begin{pmatrix} P_d & 0 \\ 0 & 1 \end{pmatrix}, \tag{33}$$

the system (9) can be characterized in the new state space by

$$D\left[\tilde{X}\right] = \tilde{A}_a(.)\tilde{X}, \tag{34}$$

such that the matrix $\tilde{A}_a(.)$ is in arrow form

$$\tilde{A}_a(.) = \quad P^{-1} A_a(.) P, \tag{35}$$

$$\tilde{A}_a(.) = \left(\begin{array}{c|c} D & P_d{}^{-1} B\left(f\left(\varepsilon\right) - f_l\right) \\ \hline -C^T A P_d & -C^T B f\left(\varepsilon\right) \end{array} \right) = \left\{ \tilde{a}_{ij}\left(.\right) \right\}, \tag{36}$$

$$\tilde{A}_a(.) = \left(\begin{array}{cccc|c} \tilde{a}_{1,1} & 0 & \cdots & 0 & \tilde{a}_{1,n+1}(.) \\ 0 & \tilde{a}_{2,2} & \ddots & \vdots & \tilde{a}_{2,n+1}(.) \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & \tilde{a}_{n,n} & \tilde{a}_{n,n+1}(.) \\ \hline \tilde{a}_{n+1,j} & \tilde{a}_{n+1,2} & \cdots & \tilde{a}_{n+1,n} & \tilde{a}_{n+1,n+1}(.) \end{array} \right), \tag{37}$$

$$\tilde{a}_{i,i} = \rho_i \ \forall i = 1, 2, \ldots, n. \tag{38}$$

For the stability study of this system, when $p\left(\tilde{X}\right) = \left[\left|\tilde{X}_1\right|, \left|\tilde{X}_2\right|, \ldots, \left|\tilde{X}_n\right|, \left|\tilde{X}_{n+1}\right|\right]^T$ as a vector norm, we have the comparison system

$$D\left[\tilde{Z}\right] = \tilde{M}(\tilde{A}_a(.))\tilde{Z},\ \tilde{Z}_0 = p(\tilde{X}_0), \tag{39}$$

where the elements $\tilde{m}_{i,j}(.)$ of the pseudo-overvaluing matrix $\tilde{M}(\tilde{A}_a(.))$

for the continuous-time system case, are such that

$$\begin{cases} \tilde{m}_{i,i}(.) = \tilde{a}_{i,i}(.) \ \forall i = 1, 2, \ldots, n+1, \\ \tilde{m}_{i,j}(.) = |\tilde{a}_{i,j}(.)| \quad \forall i \neq j, \end{cases} \tag{40}$$

and for the discrete-time system case, are such that

$$\tilde{m}_{i,j}(.) = |\tilde{a}_{i,j}(.)| \quad \forall i, j = 1, 2, \ldots, n+1. \tag{41}$$

Corollary 4.1 *The continuous nonlinear Lur'e system (4) ($\delta_1 = 0,\ \delta_2 = -1$), (the discrete system ($\delta_1 = \delta_2 = 1$)), is asymptotically stable if the following conditions are verified:*

1. The linearized system is stable, i.e., the first n successive principal minors of the matrix $(\delta_1 I_n - \delta_2 \tilde{M}(.))$ are such that

$$(\delta_2)^j \prod_{j=1}^{n} (\delta_1 - \delta_2 \tilde{m}_{j,j}) > 0 \ \forall j = 1, 2, \ldots, n, \tag{42}$$

2. The nonlinearity $f\left(\varepsilon(\tau)\right)$ satisfies the inequality

$$(\delta_2)\left[(\delta_1 - \delta_2 \tilde{m}_{n+1,n+1}(.)) - \sum_{j=1}^{n}\left(\tfrac{(-\delta_2 \tilde{m}_{n+1,j})(-\delta_2 \tilde{m}_{j,n+1}(.))}{(\delta_1 - \delta_2 \tilde{m}_{j,j})}\right)\right] > 0. \tag{43}$$

Proof. By applying the Borne-Gentina stability criterion to the comparison system (39), we obtain the following stability conditions:

$$\delta_2{}^j(\delta_1 I_{n+1} - \delta_2 \tilde{M}(.))\begin{pmatrix} 1 & 2 & \cdots & j \\ 1 & 2 & \cdots & j \end{pmatrix} \succ 0 \ \ \forall j = 1, ..., n+1, \tag{44}$$

with

$$(\delta_1 I_{n+1} - \delta_2 \tilde{M}(.)) = \left(\begin{array}{cccc|c} \delta_1 - \delta_2 \tilde{m}_{1.1} & 0 & \cdots & 0 & -\delta_2 \tilde{m}_{1,n+1}(.) \\ 0 & \delta_1 - \delta_2 \tilde{m}_{2.2} & \ddots & \vdots & -\delta_2 \tilde{m}_{2,n+1}(.) \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & \delta_1 - \delta_2 \tilde{m}_{n.n} & -\delta_2 \tilde{m}_{n,n+1}(.) \\ \hline -\delta_2 \tilde{m}_{n+1,1} & -\delta_2 \tilde{m}_{n+1,2} & \cdots & -\delta_2 \tilde{m}_{n+1,n} & \delta_1 - \delta_2 \tilde{m}_{n+1,n+1}(.) \end{array}\right). \tag{45}$$

It is clear that, for $j = 1, ..., n$, the first n minors of (44) correspond to condition (42). For $j = n+1$, the last condition is

$$(\delta_2)^{n+1}\left[\begin{array}{l}\left(\prod_{q=1}^{n}(\delta_1-\delta_2\tilde{m}_{q,q})\right)(\delta_1-\delta_2\tilde{m}_{n+1,n+1}(.)) - \\ \sum_{i=1}^{n}\left((-\delta_2\tilde{m}_{n+1,i})(-\delta_2\tilde{m}_{i,n+1}(.))\left(\prod_{\substack{j=1\\ j\neq q}}^{n}(\delta_1-\delta_2\tilde{m}_{j,j})\right)\right)\end{array}\right] > 0 \tag{46}$$

which is equivalent to (43). This ends Corollary 4.1 proof.

5 Illustrative Example

Let consider the second order autonomous Lur'e system described in state space by

$$\begin{cases} D[x(\tau)] = Ax(\tau) + Bu(\tau), \\ u(\tau) = f(\varepsilon(\tau))\,\varepsilon(\tau), \\ \varepsilon(\tau) = -C^T x(\tau). \end{cases} \tag{47}$$

The corresponding continuous-time system, shown in Figure 1, is such that

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} y(t) \\ \dot{y}(t) \end{pmatrix}, \quad A = A_c = \begin{pmatrix} 0 & 1 \\ -0.5 & -1.5 \end{pmatrix},$$
$$B = B_c = \begin{pmatrix} 0 \\ 0.5 \end{pmatrix}, \quad C = \begin{pmatrix} -0.5 \\ 5 \end{pmatrix}^T.$$

When using the sampler such that $T_e = 0.2s$ and the zero-order-holder characterised by

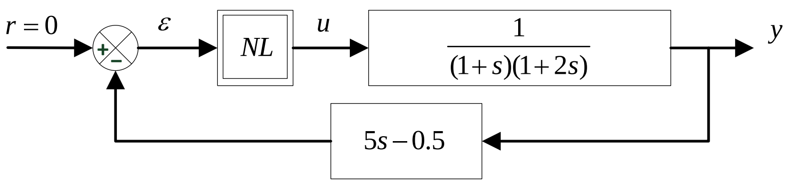

Figure 1: The continuous-time Lur'e-Postnikov system.

$H(s) = s^{-1}(1 - e^{-T_e s})$, the state space description of the associate discrete-time system is such that

$$x(k) = \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} = \begin{pmatrix} y(k) \\ \dot{y}(k) \end{pmatrix}, \quad A = A_d = \begin{pmatrix} 0.99 & 0.17 \\ -0.09 & 0.73 \end{pmatrix},$$
$$B = B_c = \begin{pmatrix} 0.01 \\ 0.04 \end{pmatrix}, \quad C = \begin{pmatrix} -0.5 \\ 5 \end{pmatrix}^T.$$

The closed-loop system descriptions of both continuous- and discrete-time are, respectively,

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0.25f(\varepsilon(t)) - 0.5 & -2.5f(\varepsilon(t)) - 1.5 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

and

$$\begin{pmatrix} x_1(k+1) \\ x_2(k+1) \end{pmatrix} = \begin{pmatrix} 0.99+0.004f(\varepsilon(k)) & 0.17-0.045f(\varepsilon(k)) \\ -0.861+0.0431f(\varepsilon(k)) & 0.73-0.431f(\varepsilon(k)) \end{pmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix}.$$

For $f(\varepsilon(\tau)) = f_l = 1$, the linearized continuous and discrete models are, respectively,

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -0.25 & -4 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

and

$$\begin{pmatrix} x_1(k+1) \\ x_2(k+1) \end{pmatrix} = \begin{pmatrix} 0.99 & 0.13 \\ -0.04 & 0.3 \end{pmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix},$$

According to (9-10), the instantaneous characteristic matrix of the augmented system is

$$A_{ca}(.) = \left(\begin{array}{cc|c} 0 & 1 & 0 \\ -0.25 & -4 & 0.5(f(\varepsilon(t))-1) \\ \hline 2.5 & 8 & -2.5f(\varepsilon(t)) \end{array}\right),$$

$$A_{da}(.) = \left(\begin{array}{cc|c} 0.99 & 0.13 & 0.009(f(\varepsilon(k))-1) \\ -0.04 & 0.3 & 0.09(f(\varepsilon(k))-1) \\ \hline 0.92 & -3.57 & -0.43f(\varepsilon(k)) \end{array}\right).$$

Due to (35), with the change of base P_c for the continuous system (P_d for the discrete system)

$$P_c = \left(\begin{array}{cc|c} 1 & 1 & 0 \\ -0.06 & -3.93 & 0 \\ \hline 0 & 0 & 1 \end{array}\right), \quad P_d = \left(\begin{array}{cc|c} 1 & -0.18 & 0 \\ -0.06 & 0.98 & 0 \\ \hline 0 & 0 & 1 \end{array}\right),$$

the matrices $A_{ca}(.)$ and $A_{da}(.)$ become in a thin arrow form, i.e., only the diagonal elements and last row and last column elements can be non zero, such that

$$\tilde{A}_{ca}(.) = \left(\begin{array}{cc|c} -0.06 & 0 & 0.12(f(\varepsilon(t))-1) \\ 0 & -3.93 & -0.12(f(\varepsilon(t))-1) \\ \hline 1.99 & -29 & -2.5f(\varepsilon(t)) \end{array}\right)$$

and

$$\tilde{A}_{da}(.) = \left(\begin{array}{cc|c} 0.99 & 0 & 0.02(f(\varepsilon(k))-1) \\ 0 & 0.31 & 0.09(f(\varepsilon(k))-1) \\ \hline 1.15 & -3.69 & -0.43f(\varepsilon(k)) \end{array}\right).$$

For the regular vector norm $p(X) = [\,|x_1(\tau)|\ ,\ |x_2(\tau)|\ ,\ |\varepsilon(\tau)|\,]^T$, the corresponding characteristics matrices of the comparison system are

$$\tilde{M}(\tilde{A}_{ca}(.)) = \left(\begin{array}{cc|c} -0.06 & 0 & 0.12\,|(f(\varepsilon)-1)| \\ 0 & -3.93 & 0.12\,|(f(\varepsilon)-1)| \\ \hline 1.99 & 29 & -2.5f(\varepsilon) \end{array}\right)$$

and

$$\tilde{M}(\tilde{A}_{da}(.)) = \left(\begin{array}{cc|c} 0.99 & 0 & 0.02\,|(f(\varepsilon(k))-1)| \\ 0 & 0.31 & 0.09\,|(f(\varepsilon(k))-1)| \\ \hline 1.15 & 3.69 & 0.43\,|f(\varepsilon(k))| \end{array}\right).$$

By Corollary 4.1, the system under study is asymptotically stable if the nonlinearity function $f(\varepsilon(\tau))$ is within the domain given in Figure 2,

– Stability domain for the continuous-time case: $0.67 < f(\varepsilon(t)) < 2$,
– Stability domain for the discrete-time case: $0.76 < f(\varepsilon(k)) < 1.17$.

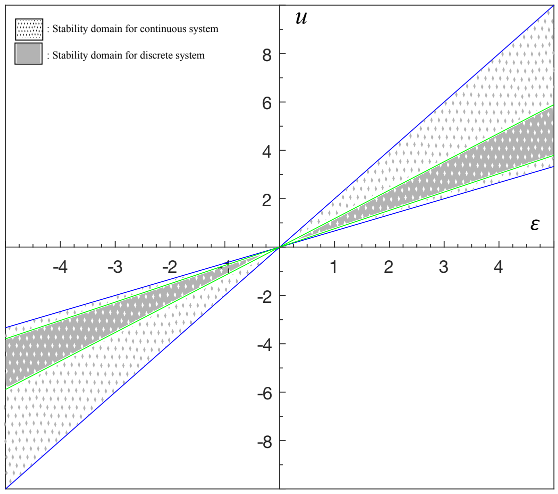

Figure 2: Continuous- and discrete-time stability domains.

6 Conclusion

In this work, a unified stability study of both continuous and discrete systems is presented. Based on the augmented model description and the Borne-Gentina stability criterion, new systematic systems stability conditions depending on the stability property of the linearized model and the nonlinearity are established. In the case of the diagonalizable characteristic matrix of linearised model, more convenient stabilty conditions are easily obtained with the use of arrow form characteristic matrices.

The studied example shows the simplicity of applying the proposed method to a unified stability study of the second order continuous Lur'e system and its associate discrete system.

It is expected that the approach will be extended to more general classes of nonlinear systems, in particular, interconnected nonlinear systems.

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