Nonlinear Dynamics and Systems Theory, **20** (3) (2020) 253–266

Dynamical Analysis, Stabilization and Discretization of a Chaotic Financial Model with Fractional Order

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Received: October 6, 2019; Revised: May 26, 2020

Abstract: In this paper, we use the discretization process of a fractional-order financial system. The conditions of the local asymptotic stability of the equilibrium points of the discretized system are analyzed. Through numerical reproductions, we brighten some dynamical behaviors, such as the chaotic attractor, bifurcation for different values of step size and fractional order parameters. Moreover, the numerical simulations confirm the validity of our theoretical results relative to the time step parameter.

Keywords: *fractional-order differential equations; bifurcation; numerical methods; equilibrium; chaotic behavior; discrete-time systems; asymptotic stability.*

Mathematics Subject Classification (2010): 34A08, 34C23, 49Mxx, 74Gxx, 74H65, 93C55, 93D20.

1 Introduction

In the previous two decades, fractional calculus and its applications have attracted a lot of attention. Many models or systems can be described by the fractional order dynamics, among these are viscoelastic models [16], diffusion wave equations [5], equations of electronic circuits [7], energy supply-request equations [21], muscular blood vessel model [22], equations of seismic tremors [1], image encryption scheme models [15], models for nonlocal scourges [11], and nonlinear dynamical model of finance system [28]. In fact, classical differential equations of integer order are generalized by fractional-order differential equations. Meanwhile, chaos is an important dynamical phenomenon which has received an increased attention of scientists since the pioneering work of Lorenz in 1963 [6]. More recently, chaotic behaviors have been found in some nonlinear fractional-order systems [25]. Moreover, the applications of chaos theory such as synchronization [4, 9] and chaos control of fractional-order hyperchaotic and chaotic systems have recently become central topics for research [10, 29].

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The fractional derivatives have sundry definitions. One of the most commonly used definitions in real applications of fractional derivatives is the Caputo definition [14, 19]

$$D^{\alpha} f(t) = I^{n-\alpha} D^{n} f(t), \qquad \alpha > 0, \tag{1}$$

where $D^n f$ represents the n order derivative of $f(t)$, where $n = [\alpha]$ is to round to the nearest integer of α, and I^{β} is the Riemann-Liouville integral operator of order β

$$I^{\beta} f(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s)\, ds, \qquad \beta > 0, \tag{2}$$

where $\Gamma(.)$ is the gamma function. And D^{α} is the Caputo differential operator of order α. It has been shown that chaos in fractional-order autonomous systems can occur for the orders less than three and this cannot happen in their integer order counterparts according to the Poincar–Bendixon theorem, and there are two popular methods for solving differential equations for fractional order: the frequency domain method [13] and time domain method [17]. In the last years, the second method has proved to be more effective because the first method is not always reliable in showing chaos [23, 24]. An analytical solution of differential equations is often undesirable, and one uses numerical or computational methods.

In [29], a numerical method for nonlinear fractional-order differential equations with constant or time-varying delay is devised. We must indicate that the fractional differential equations tend to inferior the dimensionality of the differential equations, however, introducing time retard in differential equations makes them infinite dimensional. Thus, even a single ordinary differential equation with delay can display chaos.

Many discretization methods have been used to construct the discrete-time model utilizing continuous-time methods such as Euler's method, the Runge–Kutta method, predictor-corrector method and nonstandard finite difference methods [2, 26]. Some of them are approximation for the integral and some for the derivative.

In [8, 27], the discretization method is applied to the logistic fractional-order differential equations. This method is an approximation to the right part direction of the differential equation with the formula: $D^{\alpha} X(t) = f(X(t)),\, t > 0,\, \alpha \in (0, 1]$.

The organization of this paper is as follows. In Section 2, the equilibrium points are found and the discretized-time financial model with fractional order parameter is established. In Section 3 we studied the local stability of all the equilibrium points of the discretized fractional-order system. In Section 4, we present the numerical simulations, which not only illustrate the validity of the theoretical analysis relative to the time step parameter, but also exhibit the complex dynamical behaviors such as the Hopf bifurcation and chaos phenomenon.

2 The Fractional-Order Financial Model and Its Discretization

The fractional-order finance model [28] is given by the following dynamical system:

$$\begin{cases} D^{\alpha} x(t) = (y(t) - a)\, x(t) + z(t), \\ D^{\alpha} y(t) = 1 - x^2(t) - b y(t), \\ D^{\alpha} z(t) = -(x(t) + c z(t)), \end{cases} \tag{3}$$

where x is the interest rate, y is the investment demand, z is the price index, $a > 0$ denotes the saving amount, $b > 0$ denotes the cost per investment, and $c > 0$ denotes the elasticity of the demand of commercial markets. Besides, $t > 0$, and α is the fractional order satisfying $\alpha \in (0, 1]$. The equilibrium points of system (3) are given as follows: $E_0 = \left(0, \frac{1}{b}, 0\right)$, $E_1 = \left(\sqrt{1 - b\left(a + \frac{1}{c}\right)}, \left(a + \frac{1}{c}\right), -\frac{1}{c}\sqrt{1 - b\left(a + \frac{1}{c}\right)}\right)$, and $E_2 = \left(-\sqrt{1 - b\left(a + \frac{1}{c}\right)}, \left(a + \frac{1}{c}\right), \frac{1}{c}\sqrt{1 - b\left(a + \frac{1}{c}\right)}\right)$.

Now, the discretization process of the fractional-order financial system is given as follows.

Assume that $x(0) = x_0$, $y(0) = y_0$ and $z(0) = z_0$ are the initial conditions of system (3). So, the discretization of system (3) with piecewise constant arguments is given as

$$\begin{cases} D^{\alpha} x(t) = \left(y\left(\left[\frac{t}{h}\right]h\right) - a\right) x\left(\left[\frac{t}{h}\right]h\right) + z\left(\left[\frac{t}{h}\right]h\right), \\ D^{\alpha} y(t) = 1 - x^2\left(\left[\frac{t}{h}\right]h\right) - by\left(\left[\frac{t}{h}\right]h\right), \\ D^{\alpha} z(t) = -\left(x\left(\left[\frac{t}{h}\right]h\right) + cz\left(\left[\frac{t}{h}\right]h\right)\right). \end{cases} \tag{4}$$

First, let $t \in [0, h)$, so $t/h \in [0, 1)$. Thus, we obtain

$$\begin{cases} D^{\alpha} x(t) = (y_0 - a) x_0 + z_0, \\ D^{\alpha} y(t) = 1 - x_0^2 - by_0, \\ D^{\alpha} z(t) = -(x_0 + cz_0), \end{cases} \tag{5}$$

and the solution of (5) is reduced to

$$\begin{cases} x_1(t) = x_0 + J^{\alpha}((y_0 - a) x_0 + z_0) = x_0 + \frac{t^{\alpha}}{\Gamma(1+\alpha)}[(y_0 - a) x_0 + z_0], \\ y_1(t) = y_0 + J^{\alpha}\left(1 - x_0^2 - by_0\right) = y_0 + \frac{t^{\alpha}}{\Gamma(1+\alpha)}\left[1 - x_0^2 - by_0\right], \\ z_1(t) = z_0 + J^{\alpha}(-(x_0 + cz_0)) = z_0 + \frac{t^{\alpha}}{\Gamma(1+\alpha)}[-(x_0 + cz_0)]. \end{cases} \tag{6}$$

Second, let $t \in [h, 2h)$, which makes $t/h \in [1, 2)$. Hence, we get

$$\begin{cases} D^{\alpha} x(t) = (y_1 - a) x_1 + z_1, \\ D^{\alpha} y(t) = 1 - x_1^2 - by_1, \\ D^{\alpha} z(t) = -(x_1 + cz_1), \end{cases} \tag{7}$$

which have the following solution

$$\begin{cases} x_2(t) = x_1(h) + J_h^{\alpha}((y_1 - a) x_1 + z_1) = x_1(h) + \frac{(t-h)^{\alpha}}{\Gamma(1+\alpha)}[(y_1 - a) x_1 + z_1], \\ y_2(t) = y_1(h) + J_h^{\alpha}\left(1 - x_1^2 - by_1\right) = y_1(h) + \frac{(t-h)^{\alpha}}{\Gamma(1+\alpha)}\left[1 - x_1^2 - by_1\right], \\ z_2(t) = z_1(h) + J_h^{\alpha}(-(x_1 + cz_1)) = z_1(h) + \frac{(t-h)^{\alpha}}{\Gamma(1+\alpha)}[-(x_1 + cz_1)], \end{cases} \tag{8}$$

where $J_h^\alpha = \int_h^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds$, $\alpha > 0$. Thus, after repeating the discretization process n times, we obtain

$$\begin{cases} x_{n+1}(t) = x_n(nh) + \dfrac{(t-nh)^\alpha}{\Gamma(1+\alpha)} \left[(y_n(nh) - a) x_n(nh) + z_n(nh) \right], \\ y_{n+1}(t) = y_n(nh) + \dfrac{(t-nh)^\alpha}{\Gamma(1+\alpha)} \left[1 - x_n^2(nh) - b y_n(nh) \right], \\ z_{n+1}(t) = z_n(nh) + \dfrac{(t-nh)^\alpha}{\Gamma(1+\alpha)} \left[-(x_n(nh) + c z_n(nh)) \right], \end{cases} \tag{9}$$

where $t \in [nh, (n+1)h)$. For $t \to (n+1)h$, system (9) is reduced to

$$\begin{cases} x_{n+1} = x_n + \dfrac{h^\alpha}{\Gamma(1+\alpha)} \left[(y_n - a) x_n + z_n \right], \\ y_{n+1} = y_n + \dfrac{h^\alpha}{\Gamma(1+\alpha)} \left[1 - x_n^2 - b y_n \right], \\ z_{n+1} = z_n + \dfrac{h^\alpha}{\Gamma(1+\alpha)} \left[-(x_n + c z_n) \right], \end{cases} \tag{10}$$

which can be expressed as

$$\begin{cases} x_{n+1} = x_n + s \left[(y_n - a) x_n + z_n \right], \\ y_{n+1} = y_n + s \left[1 - x_n^2 - b y_n \right], \\ z_{n+1} = z_n + s \left[-(x_n + c z_n) \right], \end{cases} \tag{11}$$

in which $s = \dfrac{h^\alpha}{\Gamma(1+\alpha)}$.

3 Stability of the Fixed Points of Discretized Systems

In the following, we discuss the local stability of the equilibrium points of system (11). In fact, the local stability of the discrete-time system (11) is determined by calculating the eigenvalues of the Jacobian matrices corresponding to its equilibrium points. Hence, the Jacobian matrix of system (11) is given as follows:

$$J_{E_{eq}} = \begin{pmatrix} 1 + s(y_n - a) & s x_n & s \\ -2 s x_n & 1 - bs & 0 \\ -s & 0 & 1 - cs \end{pmatrix}. \tag{12}$$

In order to study stability of the equilibrium points of system (11) , we recall the following two lemmas and theorem.

Lemma 3.1 *[18] Let $F(\lambda) = \lambda^2 - Tr\lambda + Det$. Suppose that $F(1) > 0$, λ_1, λ_2 are the two roots of $F(\lambda) = 0$. Then*

1. *$|\lambda_1| < 1$ and $|\lambda_2| < 1$ if and only if $F(-1) > 0$ and $Det < 1$.*
2. *$|\lambda_1| < 1$ and $|\lambda_2| > 1$ or $|\lambda_1| > 1$ and $|\lambda_2| < 1$ if and only if $F(-1) < 0$.*
3. *$|\lambda_1| > 1$ and $|\lambda_2| > 1$ if and only if $F(-1) > 0$ and $Det > 1$.*
4. *$\lambda_1 = -1$ and $\lambda_2 \neq 1$ if and only if $F(-1) = 0$ and $Tr \neq 0, 2$.*

5. *λ_1 and λ_2 are complex and $|\lambda_1| = |\lambda_2|$ if and only if $Tr^2 - 4Det < 0$ and $Det = 1$.*

Lemma 3.2 *[20] Let equation $\lambda^3 + a\lambda^2 + b\lambda + c = 0$, where a, b, $c \in \mathbb{R}$. Let further $A = a^2 - 3b$, $B = ab - 9c$, $C = b^2 - 3ac$, and $\Delta = B^2 - 4AC$. Then*

1. *The equation has three real roots if and only if $\Delta \leq 0$.*
2. *The equation has one real root λ_1 and a pair of conjugate complex roots if and only if $\Delta > 0$. Furthermore, the conjugate complex roots $\lambda_{2,3}$ are*

$$\lambda_{2,3} = \frac{1}{6}\left[\sqrt[3]{y_1} + \sqrt[3]{y_2} - 2a \pm \sqrt{3}i\left(\sqrt[3]{y_1} - \sqrt[3]{y_2}\right)\right],$$

where

$$y_{1,2} = aA + \frac{3}{2}\left(-B \pm \sqrt{\Delta}\right).$$

Theorem 3.1 *[20] The equilibrium point E_{eq} has the following topological types of its all values of parameters:*

1. *E_{eq} is asymptotically stable if one of the following conditions holds:*
   - *1.1. $\Delta \leq 0$, $P(1) > 0$, $P(-1) < 0$ and $-1 < \lambda^*_{1,2} < 1$,*
   - *1.2. $\Delta > 0$, $P(1) > 0$, $P(-1) < 0$ and $|\lambda_{2,3}| < 1$.*
2. *E_{eq} is unstable if one of the following conditions holds:*
   - *2.1. $\Delta \leq 0$, and one of the following conditions holds:*
     - *2.1.1. $P(1) > 0$, $P(-1) > 0$ and $\lambda^*_2 < -1$ or $\lambda^*_2 > 1$,*
     - *2.1.2. $P(1) < 0$, $P(-1) < 0$ and $\lambda^*_2 < -1$ or $\lambda^*_2 > 1$,*
   - *2.2. $\Delta > 0$, and one of the following conditions holds:*
     - *2.2.1. $P(1) < 0$ and $|\lambda_{2,3}| > 1$,*
     - *2.2.2. $P(-1) > 0$ and $|\lambda_{2,3}| > 1$.*
3. *E_{eq} is one-dimensional if one of the following conditions holds:*
   - *3.1. $\Delta \leq 0$, and one of the following conditions holds:*
     - *3.1.1. $P(1) > 0$, $P(-1) < 0$, and $\lambda^*_1 < -1$ or $\lambda^*_2 > 1$,*
     - *3.1.2. $P(1) < 0$, $P(-1) > 0$,*
   - *3.2. $\Delta > 0$, and one of the following conditions holds:*
     - *3.2.1. $P(1) > 0$, $P(-1) < 0$ and $|\lambda_{2,3}| > 1$,*
     - *3.2.2. $P(1) < 0$ and $|\lambda_{2,3}| < 1$,*
     - *3.2.3. $P(-1) > 0$ and $|\lambda_{2,3}| < 1$.*
4. *E_{eq} is two-dimensional if one of the following conditions holds:*
   - *4.1. $\Delta \leq 0$, and one of the following conditions holds:*
     - *4.1.1. $P(1) > 0$, $P(-1) > 0$ and $-1 < \lambda^*_2 < 1$,*
     - *4.1.2. $P(1) < 0$, $P(-1) < 0$ and $-1 < \lambda^*_1 < 1$,*
   - *4.2. $\Delta > 0$, and one of the following conditions holds:*

4.2.1. $P(-1) < 0$ *and* $|\lambda_{2,3}| < 1$,
4.2.2. $P(1) > 0$ *and* $|\lambda_{2,3}| < 1$.

5. E_{eq} *is non-hyperbolic if one of the following conditions holds*:

5.1. $\Delta \leq 0$, *and* $P(1) = 0$ *or* $P(-1) = 0$,
5.2. $\Delta > 0$, *and* $P(1) = 0$ *or* $P(-1) = 0$ *or* $|\lambda_{2,3}| = 1$.

3.1 Stability of fixed point E_0

The Jacobian matrix of E_0 of system (11) is

$$J_{E_0} = \begin{pmatrix} 1 + s\left(\frac{1}{b} - a\right) & 0 & s \\ 0 & 1 - bs & 0 \\ -s & 0 & 1 - cs \end{pmatrix}. \tag{13}$$

The characteristic equation of the Jacobian matrix (13) is

$$(\lambda - (-bs + 1))\left(\lambda^2 - Tr_0\lambda + Det_0\right) = 0, \tag{14}$$

where $Tr_0 = \left(2 - \frac{1}{b}s\,(ab + bc - 1)\right)$,

$Det_0 = \left(\frac{1}{b}\left((b - c + abc)\,s^2 + (1 - bc - ab)\,s + b\right)\right)$, and $s_1 = \dfrac{ab + bc - 1}{b - c + abc}$, , $s_4 = \dfrac{2}{b}$,

$s_2 = -\dfrac{\left(\sqrt{-(2b + ab - bc - 1)(2b - ab + bc + 1)} - ab - bc + 1\right)}{b - c + abc}$,

and $s_3 = \dfrac{\left(\sqrt{-(2b + ab - bc - 1)(2b - ab + bc + 1)} + ab + bc - 1\right)}{b - c + abc}$.

Theorem 3.2 *If the equilibrium point* E_0 *exists and* $c \geq a + 2$, *and* $b > \dfrac{c}{ac + 1}$ *with* $a \geq 0$, *then it has the following topological types of its all values of parameters:*

(i) E_0 *is asymptotically stable if* $0 < h < \min\left(\sqrt[\alpha]{s_2\Gamma(1+\alpha)},\ \sqrt[\alpha]{s_4\Gamma(1+\alpha)}\right)$.

(ii) E_0 *is unstable if* $h > \max\left(\sqrt[\alpha]{s_3\Gamma(1+\alpha)},\ \sqrt[\alpha]{s_4\Gamma(1+\alpha)}\right)$.

Proof. By applying stability conditions and using Lemma 3.1 the results (i) and (ii) can be achieved with $s_2 < s_1 < s_3$. □

3.2 Stability of fixed points E_1 and E_2

The Jacobian matrix of model (11) at the equilibrium points E_1 and E_2 is

$$J_{E_1} = \begin{pmatrix} \frac{1}{c}(c + s) & s\sqrt{-\frac{1}{c}(b - c + abc)} & s \\ -2s\sqrt{-\frac{1}{c}(b - c + abc)} & 1 - bs & 0 \\ -s & 0 & 1 - cs \end{pmatrix} \tag{15}$$

and

$$J_{E_2} = \begin{pmatrix} \frac{1}{c}(c+s) & -s\sqrt{-\frac{1}{c}(b-c+abc)} & s \\ 2s\sqrt{-\frac{1}{c}(b-c+abc)} & 1-bs & 0 \\ -s & 0 & 1-cs \end{pmatrix}, \tag{16}$$

for the convenience of calculations, we denote
$\eta_1 = 2(c-b-abc)$, $\eta_2 = \frac{1}{c}\left(2c-3b+bc^2-2abc\right)$, and $\eta_3 = -\frac{1}{c}\left(c^2+bc-1\right)$.

The corresponding characteristic equation of J_{E_1} and J_{E_2} can be written as

$$P_1(\lambda) = \lambda^3 + b_1\lambda^2 + b_2\lambda + b_3, \tag{17}$$

where $b_1 = -\eta_3 s - 3$, $b_2 = \eta_2 s^2 + 2\eta_3 s + 3$, and $b_3 = \eta_1 s^3 - s^2\eta_2 - \eta_3 s - 1$.

By calculating, we further have
$A = b_1^2 - 3b_2 = -s^2\left(3\eta_2 - \eta_3^2\right)$,
$B = b_1 b_2 - 9b_3 = -s^3(9\eta_1 + \eta_2\eta_3) + s^2\left(6\eta_2 - 2\eta_3^2\right)$,
$C = b_2^2 - 3b_1 b_3 = \left(\eta_2^2 + 3\eta_1\eta_3\right)s^4 + (9\eta_1 + \eta_2\eta_3)s^3 + \left(\eta_3^2 - 3\eta_2\right)s^2$,
and
$\Delta = B^2 - 4AC = s^6\Delta^*$,
where $\Delta^* = 3\left(27\eta_1^2 + 18\eta_1\eta_2\eta_3 - 4\eta_1\eta_3^3 + 4\eta_2^3 - \eta_2^2\eta_3^2\right)$.

The derivative of $P_1(\lambda)$ is $P_1'(\lambda) = 3\lambda^2 + 2b_1\lambda + b_2$. Obviously, the equation $P_1'(\lambda) = 0$ has two roots as follows:

$$\lambda_{1,2}^* = \frac{1}{3}\left(-b_1 \pm \sqrt{b_1^2 - 3b_2}\right) = \frac{1}{3}\left(s\eta_3 + 3 \pm s\sqrt{\eta_3^2 - 3\eta_2}\right). \tag{18}$$

When $\Delta^* \leq 0$, namely, $\Delta \leq 0$, by Lemma 3.2, we have that equation (17) has three real roots λ_1, λ_2 and λ_3. From this, we can easily prove that two roots $\lambda_{1,2}^*$ (let $\lambda_1^* \leq \lambda_2^*$) of equation $P_1'(\lambda) = 0$ also are real.

When $\Delta^* > 0$, namely, $\Delta > 0$, by Lemma 3.2, we have that equation (17) has one real root λ_1 as follows: $\lambda_1 = -\frac{b_1 + y_1^{\frac{1}{3}} + y_2^{\frac{1}{3}}}{3}$, and a pair of conjugate complex roots $\lambda_{2,3}$, and the conjugate complex roots are $\lambda_{2,3} = \frac{-2b_1 + y_1^{\frac{1}{3}} + y_2^{\frac{1}{3}}}{6} \pm i\frac{\sqrt{3}\left(y_1^{\frac{1}{3}} - y_2^{\frac{1}{3}}\right)}{6}$, where

$y_{1,2} = b_1 A + \frac{3(-B \pm \sqrt{\Delta})}{2} = \frac{s^3}{2}\left(\left(-2\eta_3^3 + 9\eta_2\eta_3 + 27\eta_1\right) \pm \sqrt{\Delta^*}\right)$.

Further, we have $P_1(1) = \eta_1 s^3$, and $P_1(-1) = \eta_1 s^3 - 2\eta_2 s^2 - 4\eta_3 s - 8$, and we pose $s_1 = -\frac{\left(\eta_3 + \sqrt{-4\eta_2 + \eta_3^2}\right)}{\eta_2}$, $s_2 = -\frac{\left(\eta_3 - \sqrt{-4\eta_2 + \eta_3^2}\right)}{\eta_2}$ and $s_3 = \frac{\left(-3\eta_3 - \sqrt{3}\sqrt{-8\eta_2 + 3\eta_3^2}\right)}{2\eta_2}$, $s_4 = \frac{\left(-3\eta_3 + \sqrt{3}\sqrt{-8\eta_2 + 3\eta_3^2}\right)}{2\eta_2}$ and $s_5 = \frac{2\eta_2}{\eta_1}$, $s_6 = -\frac{\left(2\eta_3 + 2\sqrt{\eta_3^2 - 3\eta_2}\right)}{\eta_2}$ and $(\eta_1)_1 = \frac{2\left(\eta_3\left(\eta_3^2 - \frac{9}{2}\eta_2\right) - \sqrt{-\left(3\eta_2 - \eta_3^2\right)^3}\right)}{27}$, $(\eta_1)_2 = \frac{2\left(\eta_3\left(\eta_3^2 - \frac{9}{2}\eta_2\right) + \sqrt{-\left(3\eta_2 - \eta_3^2\right)^3}\right)}{27}$.

Now, relative to the dynamical properties of the equilibrium points E_1 and E_2, we have the following result.

Theorem 3.3 *If the equilibrium point E_1 or E_2 exists and* $\frac{\sqrt{23}}{3} \leq a \leq \frac{\sqrt{27}}{3}$, $\frac{\sqrt{11}}{8} \leq b \leq \frac{\sqrt{18}}{10}$ *and* $\frac{\sqrt{35}}{3} \leq c \leq 2$, *then it has the following topological types of its all values of parameters:*

(i) *E_1 or E_2 is asymptotically stable if* $0 < h < \sqrt[\alpha]{s_3\Gamma(1+\alpha)}$ *or* $h > \sqrt[\alpha]{s_4\Gamma(1+\alpha)}$.

(ii) *E_1 or E_2 is unstable if* $\sqrt[\alpha]{s_3\Gamma(1+\alpha)} < h < \sqrt[\alpha]{s_1\Gamma(1+\alpha)}$ *or* $\sqrt[\alpha]{s_2\Gamma(1+\alpha)} < h < \sqrt[\alpha]{s_4\Gamma(1+\alpha)}$.

(iii) *E_1 or E_2 is non-hyperbolic if* $h = \sqrt[\alpha]{s_3\Gamma(1+\alpha)}$ *or* $h = \sqrt[\alpha]{s_4\Gamma(1+\alpha)}$.

Proof. Here $\eta_1, \eta_2 > 0$, $\eta_3 < 0$.

We have $\Delta > 0$ if $\eta_1 > (\eta_1)_2$. We also achieve these conditions $\left(-4\eta_2 + \eta_3^2\right) > 0$, $27\eta_1 + 3\eta_2\eta_3 > 0$ and $-\eta_2\eta_3 - \eta_1 > 0$.

We assume $E_1 = \frac{\sqrt[3]{y_1} + \sqrt[3]{y_2}}{s} - \eta_3$. And by calculation, we find

$$\|\lambda_{2,3}\|^2 - 1 = \left(E_1^2 + 3\eta_3 E_1 + 9\eta_2\right) s^2 + \left(9\eta_3 + 3E_1\right) s, \tag{19}$$

and

$$|\lambda_1| - 1 = \frac{1}{3} s E_1 - 2. \tag{20}$$

The conditions $\|\lambda_{2,3}\| = 1$, $|\lambda_1| = 1$, are realized if and only if $s = s_3$ or $s = s_4$.

And the conditions $\|\lambda_{2,3}\| < 1$, $|\lambda_1| < 1$ are verified if and only if $0 < s < \left(-3\frac{E_1 + 3\eta_3}{E_1^2 + 3\eta_3 E_1 + 9\eta_2}\right)$ and $0 < s < \left(\frac{6}{E_1}\right)$.

So

$$0 < s < 2s + s\frac{s\eta_3 + 2}{\eta_2 s^2 + 2\eta_3 s + 4} < -3\frac{E_1 + 3\eta_3}{E_1^2 + 3\eta_3 E_1 + 9\eta_2}, \tag{21}$$

$\frac{s\left(\eta_2 s^2 + 3\eta_3 s + 6\right)}{\eta_2 s^2 + 2\eta_3 s + 4}$ is positive if $\left(\eta_2 s^2 + 3\eta_3 s + 6\right) > 0$ and $\left(\eta_2 s^2 + 2\eta_3 s + 4\right) > 0$.

We finally obtain $\|\lambda_{2,3}\| < 1$, $|\lambda_1| < 1$, if $s \in (]0, s_1[\cup]s_2, +\infty[) \cap (]0, s_3[\cup]s_4, +\infty[)$.
And the conditions $\|\lambda_{2,3}\| < 1$, $|\lambda_1| > 1$ are verified if and only if $0 < s < \left(-3\frac{E_1 + 3\eta_3}{E_1^2 + 3\eta_3 E_1 + 9\eta_2}\right)$ and $s > \left(\frac{6}{E_1}\right)$. So

$$0 < s < -3\frac{E_1 + 3\eta_3}{E_1^2 + 3\eta_3 E_1 + 9\eta_2} < -s\frac{s\eta_3 + 2}{\eta_2 s^2 + 2\eta_3 s + 4}, \tag{22}$$

$\frac{s\left(\eta_2 s^2 + 3\eta_3 s + 6\right)}{\eta_2 s^2 + 2\eta_3 s + 4}$ is negative if $\left(\eta_2 s^2 + 3\eta_3 s + 6\right) < 0$ and $\left(\eta_2 s^2 + 2\eta_3 s + 4\right) > 0$.

We finally obtain $\|\lambda_{2,3}\| < 1$, $|\lambda_1| > 1$, if $s \in (]0, s_1[\cup]s_2, +\infty[) \cap (]s_3, s_4[)$.

The intersection of the previous fields gives us the following results.

E_1 or E_2 is asymptotically stable if $s \in]0, s_3[$ or $]s_4, +\infty[$, E_1 or E_2 is unstable if $s \in]s_3, s_1[$ or $]s_2, s_4[$, and E_1 or E_2 is non-hyperbolic if $s = s_3$ or $s = s_4$ with $s_3 < s_1 < s_2 < s_4$.

Theorem 3.4 *If the equilibrium point E_1 or E_2 exists and* $\frac{9}{20} \leq b \leq \frac{1}{2}$, $2 \leq c \leq \frac{5}{2}$ *and* $\frac{9}{5} \leq a < \frac{c-b}{bc}$, *then it has the following topological types of its all values of parameters:*

(i) E_1 *or* E_2 *is asymptotically stable if* $0 < h < \sqrt[\alpha]{s_6 \Gamma(1+\alpha)}$.

(ii) E_1 *or* E_2 *is unstable if* $h > \sqrt[\alpha]{s_5 \Gamma(1+\alpha)}$.

Proof. Here η_1, $\eta_2 > 0$, $\eta_3 < 0$.

We have $\Delta \leq 0$ if $\eta_1 \leq (\eta_1)_2$. We also achieve these conditions $\left(-4\eta_2 + \eta_3^2\right) > 0$, $27\eta_1 + \eta_2\eta_3 < 0$. We will rely on Theorem 3.1 to find the final conditions:

1. $P_1(1) > 0$ and $P_1(-1) < 0$ are verifed if $s \in \left]0, s_5\right[$.

2. $P_1(1) > 0$ and $P_1(-1) > 0$ are verifed if $s \in \left]s_5, +\infty\right[$.

3. $-1 < \lambda_{1,2}^* < 1$ is realized if $s \in \left]0, s_6\right[$.

4. $\lambda_1^* < -1$ is realized if $s \in \left]s_6, +\infty\right[$.

The intersection of the previous fields gives us the following results.

Condition (1.1.) of Theorem 3.1 is verified if $s \in \left]0, s_6\right[$, and condition (3.1.1.) of Theorem 3.1 is verified if $s \in \left]s_5, +\infty\right[$ with $s_6 < s_5$.

4 Numerical Simulations

In this section, we present the bifurcation diagrams, the phase portraits of the model (11), which confirm the analytical results above and illustrate the dynamic behaviors of our model using the digital relay. A bifurcation occurs when the stability of a point of equilibrium changes [12].

As discussed earlier in Section 3, this paper focuses on varying the time step size parameter h and the fractional-order parameter α in the model (11).

Based on the previous analysis, the parameters of the model (11) can be examined by:

varying h in the range $1.25 \leq h \leq 1.4$ and fixing $a = 1.63$, $b = 0.418$, $c = 1.98$, $\alpha = 0.99$ with the initial conditions $(x_0, y_0, z_0) = (0.3, 2.11, -0.1)$. The resulting points are plotted versus the parameter h (see Figure 1). According to Theorem 3.3, we have $\frac{\sqrt{23}}{3} < a < \frac{\sqrt{27}}{3}$, $\frac{\sqrt{11}}{8} < b < \frac{\sqrt{18}}{10}$ and $\frac{\sqrt{35}}{3} < c < 2$; and E_1 is asymptotically stable if $1.25 \leq h < \sqrt[\alpha]{s_3 \Gamma(1+\alpha)}$ (see Figure 2(a)), the neighbor trajectories converge to the point E_1. If $h = \sqrt[\alpha]{s_3 \Gamma(1+\alpha)} \simeq 1.3045$, system (11) undergoes a Hopf bifurcation as mentioned above (see Figure 2(b)); the fixed point E_1 becomes unstable if $\sqrt[\alpha]{s_3 \Gamma(1+\alpha)} < h < \sqrt[\alpha]{s_1 \Gamma(1+\alpha)}$ with $\sqrt[\alpha]{s_1 \Gamma(1+\alpha)} \simeq 1.6684$ (see Figure 2(c) and 2(d)).

Second, varying α in the range $0.1 \leq \alpha \leq 0.8$ and fixing $h = 1.393$, the resulting points are plotted versus the parameter α (see Figure 3).

We note that the increase of α causes instability of the fixed point. If $\alpha < 0.44$, the fixed point E_1 is locally asymptotically stable (see Figure 4(a)), the neighbor trajectories converge to the point E_1. If $\alpha = 0.44$, system (11) undergoes a Hopf bifurcation (see

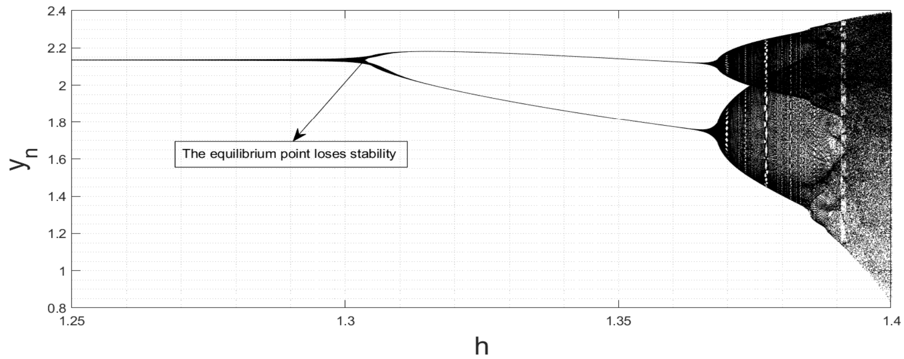
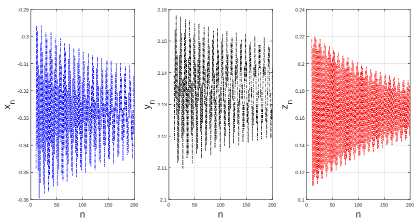
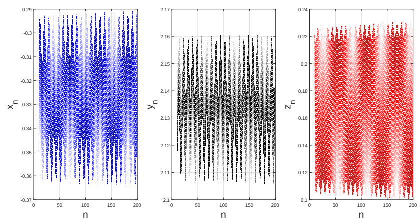


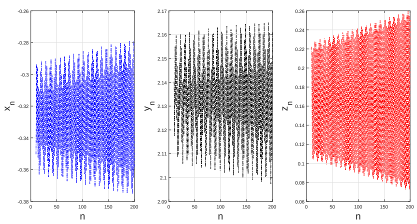
Figure 1: Bifurcation Diagram of Model (11) for $h \in [1.25, 1.4]$.



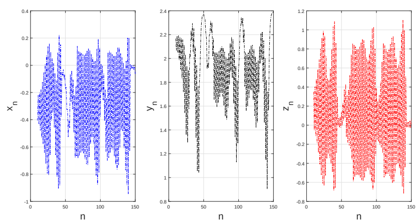
(a) Stable equilibria at $h = 1.3$.



(b) The equilibrium point loses stability at $h = 1.3045$.



(c) Unstable equilibria at $h = 1.306$.



(d) Unstable equilibria at $h = 1.4$.

Figure 2: The Trajectory Diagrams of Model (11) for Various h Corresponding to Figure (1).

Figure 4(b)). The fixed point E_1 becomes unstable if $0.44 < \alpha \leq 0.8$, as shown in Figure 4(c) and 4(d).

Third, varying b and fixing $h = 1.399$, $\alpha = 0.99$, the resulting points are plotted versus the parameter b (see Figure 5).

Attracting invariant circles and chaos appear when b decreases, so that the variable belongs to the domain $[0.417, 0.5]$. The phase portraits for various b-values corresponding

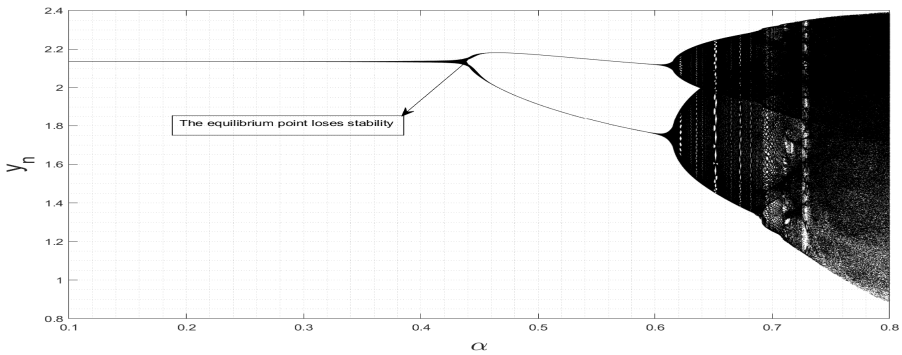
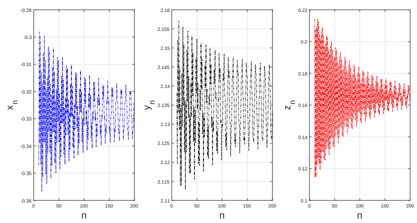
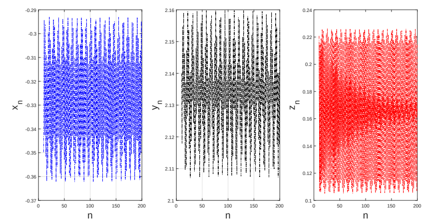


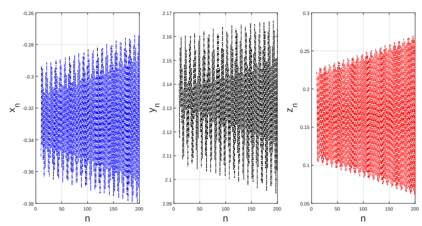
Figure 3: Bifurcation Diagram of Model (11) for $\alpha \in [0.1, 0.8]$.



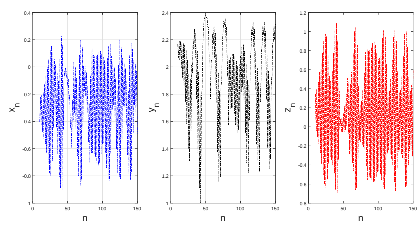
(a) Stable equilibria at $\alpha = 0.42$.



(b) The equilibrium point loses stability at $\alpha = 0.44$.



(c) Unstable equilibria at $\alpha = 0.445$.



(d) Unstable equilibria at $\alpha = 0.8$.

Figure 4: The Trajectory Diagrams of Model (11) for Various α Corresponding to Figure (3).

to Figure 5 are plotted.

Furthermore, the period-2 orbits ($b = 0.44$) are shown in Figure 6(d), and for the attracting invariant circles ($b = 0.4395$) see Figure 6(c). The quasi-periodic orbits ($b = 0.428$) are observed in Figure 6(b). Attracting chaotic sets are also seen if $b = 0.418$ and are plotted in Figure 6(a).

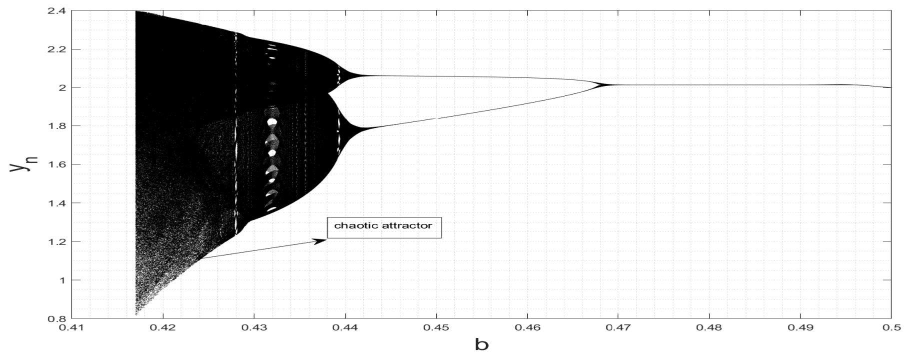
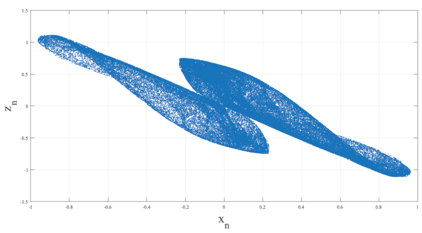
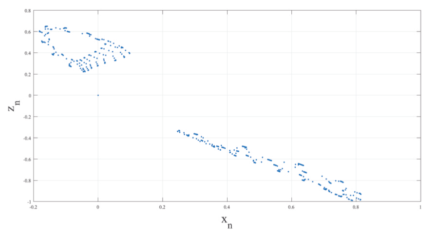


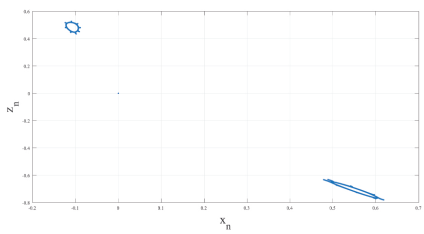
Figure 5: Bifurcation Diagram of Model (11) for $b \in [0.417, 0.5]$.



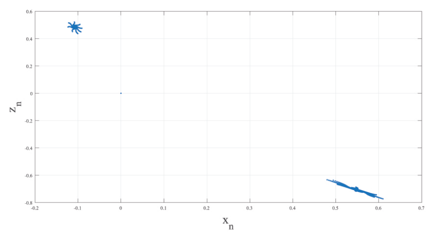
(a) Chaotic attractor at $b = 0.418$.



(b) Quasi-periodic orbits at $b = 0.428$.



(c) Attracting invariant circles at $b = 0.4395$.



(d) Period−2 orbits at $b = 0.44$.

Figure 6: Phase Portrait Diagrams of Model (11) for Various b Corresponding to Figure (5).

5 Conclusion

In this paper, a discrete-time financial system has been discussed; such a discrete-time model is obtained from the discretization of the fractional-time financial system. The discretization method provides crucial terms such as the time step parameter (h) and fractional-order parameter (α), which are then varied in order to describe the dynamical behaviors of the model. Note that, whenever we diversify h and α, the system displays many dynamic behaviors including the appearance of bifurcation. Also, the variable b, whenever we change it, shows the bifurcation, attracting invariant circle and chaotic sets. Analytically, sufficient conditions are given to the parameter h for the local stability of equilibrium points. Moreover, numerical continuation is carried out to illustrate the validity of the analytical results.

References

[1] A. E. M. El-Misiery and E. Ahmed. On a fractional model for earthquakes. *Applied Mathematics and Computation* **178** (2) (2006) 207–211.

[2] A. J. Arenas, G. Gonzlez-Parra and B. M. Chen-Charpentier. A nonstandard numerical scheme of predictor–corrector type for epidemic models. *Computers & Mathematics with Applications* **59** (12) (2010) 3740–3749.

[3] A. Khan, N. Aneja and P. Tripathi, J. Biswas. A new hyper chaotic system and study of hybrid projective synchronization behavior. *Nonlinear Dynamics and Systems Theory* **17** (3) (2017) 266–278.

[4] A. Khan and P. Tripathi. Synchronization between a fractional order chaotic system and an integer order chaotic system. *Journal of Applied Mathematics* **13** (4) (2013) 425–436.

[5] A. M. A. El-Sayed. Fractional-order diffusion-wave equation. *International Journal of Theoretical Physics* **35** (2) (1996) 311–322.

[6] A. M. A. El-Sayed, H. M. Nour and A. Elsaid, A. E. Matouk and A. Elsonbaty. Circuit realization, bifurcations, chaos and hyperchaos in a new 4D system. *Applied Mathematics and Computation* **239** (2014) 333–345.

[7] A. M. A. El-Sayed, H. M. Nour and A. Elsaid A. E. Matouk and A. Elsonbaty. Dynamical behaviors, circuit realization, chaos control, and synchronization of a new fractional order hyperchaotic system. *Applied Mathematical Modelling* **40** (5-6) (2006) 3516–3534.

[8] A. M. A. El-Sayed and S. M. Salman. On a discretization process of fractional-order Riccati differential equation. *J. Fract. Calc. Appl.* **4** (2) (2013) 251–259.

[9] A. Ouannas. A new synchronization scheme for general 3D quadratic chaotic systems in discrete-time. *Nonlinear Dynamics and Systems Theory* **15** (2) (2015) 163–170.

[10] C. Li, X. Liao and J. Yu. Synchronization of fractional order chaotic systems. *Physical Review E.* **68** (6) (2003) 067203.

[11] E. Ahmed and A. S. Elgazzar. On fractional order differential equations model for nonlocal epidemics. *Physica A: Statistical Mechanics and its Applications* **379** (2) (2007) 607–614.

[12] H. Cao, Z. Yue and Y. Zhou. The stability and bifurcation analysis of a discrete Holling-Tanner model. *Advances in Difference Equations* **2013** (1) (2013) 330.

[13] H. Sun, A. Abdelwahab and B. Onaral. Linear approximation of transfer function with a pole of fractional power. *IEEE Transactions on Automatic Control* **29** (5) (1984) 441–444.

[14] I. Podlubny. *Fractional Differential Equations.* Academic Press. **198** (1998).

[15] J. Zhao, S. Wang and Y. Chang, X. Li. A novel image encryption scheme based on an improper fractional-order chaotic system. *Nonlinear Dynamics* **80** (4) (2015) 1721–1729.

[16] K. Adolfsson. Nonlinear fractional order viscoelasticity at large strains. *Nonlinear Dynamics* **38** (1-4) (2004) 233–246.

[17] K. Diethelm, N. J. Ford and A. D. Freed. A predictor-corrector approach for the numerical solution of fractional differential equations. *Nonlinear Dynamics* **29** (1-4) (2002) 3–22.

[18] M. A. Abdelaziz, A. I. Ismail, and F. A. Abdullah, M. H. Mohd. Bifurcations and chaos in a discrete SI epidemic model with fractional order. *Advances in Difference Equations* **2018** (1) (2018) 1–19.

[19] M. Caputo. Linear models of dissipation whose Q is almost frequency independent—II. *Geophysical Journal International* **13** (5) (1967) 529–539.

[20] M. El-Shahed, J. J. Nieto and A. M. Ahmed, I. M. E. Abdelstar. Fractional-order model for biocontrol of the lesser date moth in palm trees and its discretization. *Advances in Difference Equations* **2017** (1) (2017) 195.

[21] M. P. Aghababa. Fractional modeling and control of a complex nonlinear energy supply-demand system. *Complexity* **20** (6) (2015) 74–86.

[22] M. P. Aghababa and M. Borjkhani. Chaotic fractional-order model for muscular blood vessel and its control via fractional control scheme. *Complexity* **20** (2) (2014) 37–46.

[23] M. S. Tavazoei and M. Haeri. Limitations of frequency domain approximation for detecting chaos in fractional order systems. *Nonlinear Analysis: Theory, Methods & Applications* **69** (4) (2008) 1299–1320.

[24] M. S. Tavazoei and M. Haeri. Unreliability of frequency-domain approximation in recognising chaos in fractional-order systems. *IET Signal Processing* **1** (4) (2007) 171–181.

[25] P. Muthukumar, P. Balasubramaniam and K. Ratnavelu. Fast projective synchronization of fractional order chaotic and reverse chaotic systems with its application to an affine cipher using date of birth (DOB). *Nonlinear Dynamics* **80** (4) (2015) 1883–1897.

[26] R. E. Mickens. Numerical integration of population models satisfying conservation laws: NSFD methods. *Journal of Biological Dynamics* **1** (4) (2007) 427–436.

[27] R. P. Agarwal, A. M. A. El-Sayed and S. M. Salman. Fractional-order Chua's system: discretization, bifurcation and chaos. *Advances in Difference Equations* **2013** (1) (2013) 320.

[28] W. C. Chen. Nonlinear dynamics and chaos in a fractional-order financial system. *Chaos, Solitons & Fractals* **36** (5) (2008) 1305–1314.

[29] Z. Wang. A numerical method for delayed fractional-order differential equations. *Journal of Applied Mathematics* **2013** (2013), Article ID 256071.