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# Increased Order Generalized Combination Synchronization of Non-Identical Dimensional Fractional-Order Systems by Introducing Different Observable Variable Functions

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**Abstract:** An increased order generalized combination synchronization (IOGCS) of non-identical dimensional fractional-order systems with suitable different observable variable functions is proposed and analyzed in this paper. This synchronization scheme is applied for the combination of two fractional-order unified drive systems and the fractional-order Liu response system. In view of the stability property of linear fractional-order systems, an effective nonlinear control scheme is designed to achieve the desired synchronization. Theoretical analysis and numerical simulations are shown to demonstrate the effectiveness of the proposed method.

**Keywords:** *increased order generalized combination synchronization; chaotic system; fractional-order system; stability property of fractional-order system.*

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## 1 Introduction

Fractional calculus can be dated back to the 17th century as studied by Podlubny [1]. Over the last decades, fractional calculus has applied in various fields such as control processing [2], reaction difusion equation [3], biological phenomena [4] and so on.

Chaos synchronization schemes for fractional-order dynamical systems have also been investigated in several fields such as secure communication and data encryption [5, 6]. Up to now, a variety of approaches of chaos synchronization have been developed, such as complete synchronization [7], generalized synchronization [8], inverse matrix projective synchronization [9], modified projective synchronization [10], coexistence of different types of chaos synchronization [11], and $Q - S$ synchronization [12].

However, most of researchers mainly focused on the usual drive-response synchronization model, which has one drive system and one response system.

Recently, studying synchronization between the combination of two (or more) drive systems and one response system becomes an interesting problem due to its potential applications in secure communication [13].

Now, some results on the combination synchronization of several chaotic fractional order systems are obtained. For example, the combination synchronization of three classic chaotic systems using active backstepping design is investigated in [14]. The combination synchronization of three identical or different nonlinear complex hyperchaotic systems is achieved in [15]. The reduced order function projective combination synchronization of three Josephson junctions using the backstepping technique is investigated in [16]. An adaptive function projective combination synchronization of three different fractional order chaotic systems is investigated in [17]. The generalized combination complex synchronization for fractional-order chaotic complex systems is investigated in [18]. And the generalized combination synchronization of three different dimensional fractional chaotic and hyperchaotic systems by using three scaling matrices is achieved in [19]. However, these studies are mainly concerned with the combination synchronization between chaotic systems with respect to the scaling matrices. Therefore the combination synchronization of non-identical dimensional chaotic fractional order systems with respect to the variable functions becomes an interesting and challenging work.

By exploiting the idea of the stability property of linear fractional order systems, an effective nonlinear controller for the IOGCS of three fractional-order chaotic systems with suitable different observable variable functions is designed in this paper, and the stability criterion for the above-mentioned systems is found. To simplify our discussions, the synchronization scheme is applied for the combination of two fractional-order unified drive systems and the fractional-order Liu response system.

The rest of the paper is organized as follows. In Section 2, based on the stability property of linear fractional order systems, a powerful scheme is proposed to realize the IOGCS of non-identical fractional order dynamical chaotic systems. In Section 3, numerical simulations show that the method can ensure the occurrence of the IOGCS between the fractional-order unified chaotic system and fractional-order Liu system. Finally, conclusion is given in Section 4.

## 2 Problem Formulation of the IOGCS

In this section, we introduce the concept of the IOGCS of three non-identical dimension fractional-order systems with suitable different observable variable functions. The model can be given as follows

$$D^{\alpha}x = f(x), \tag{1}$$

$$D^{\alpha}y = g(y), \tag{2}$$

$$D^{\alpha}z = h(z) + u, \tag{3}$$

where $D^{\alpha}$ is the Caputo differential operator [1] which is defined as

$$D^{\alpha}\xi(t) = J^{n-\alpha}\xi^{(n)}(t), \ \ \alpha \in (n-1, n)\,, \tag{4}$$

where $J^{\alpha}$ is the $\alpha$ -order Riemann–Liouville integral operator which is defined as

$$J^{\alpha}\xi(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\xi(\tau)d\tau, \tag{5}$$

and

$$\Gamma(\alpha) = \int_0^{+\infty} z^{\alpha-1}\exp(-z)dz \tag{6}$$

is the gamma function, $x = (x_1, x_2, ..., x_n)^T \in \mathbb{R}^n$ and $y = (y_1, y_2, ..., y_n)^T \in \mathbb{R}^n$ are the state variables of two drive systems, $z = (z_1, z_2, ..., z_m)^T \in \mathbb{R}^m (n < m)$ is the state variable of the response system, $f, g : \mathbb{R}^n \to \mathbb{R}^n$ and $h : \mathbb{R}^m \to \mathbb{R}^m$ are the continuous vector-valued functions and $u = (u_1, u_2, ..., u_m)^T \in \mathbb{R}^m$ is the controller vector which will be designed.

The definition of the proposed synchronization is given as follows.

**Definition 2.1** The two drive systems (1)-(2) and the response system (3) are said to achieve the IOGCS if there exists a suitable controller $u$ and three continuous smooth vector functions $Q$, $R : \mathbb{R}^n \to \mathbb{R}^m$ and $S : \mathbb{R}^m \to \mathbb{R}^m$, such that the error vector

$$e(t) = Q\left(x(t)\right) + R\left(y(t)\right) - S\left(z(t)\right) \tag{7}$$

will approach zero for large enough $t$, i.e.,

$$\lim_{t\to+\infty} \left\|Q\left(x(t)\right) + R\left(y(t)\right) - S\left(z(t)\right)\right\| = 0, \tag{8}$$

where $\|.\|$ represents the matrix norm.

**Remark 2.1** From Definition 2.1, one can show that the IOGCS of three different fractional-order chaotic systems can be extended to more chaotic systems.

In this paper, we consider the fractional-order unified system (Lorenz, Chen and Lü systems) [20] as the first drive system, which is described by

$$\begin{cases} D^{\alpha}x_1 = (25\delta + 10)(x_2 - x_1), \\ D^{\alpha}x_2 = (-35\delta + 28)x_1 - x_1x_3 + (29\delta - 1)x_2, \\ D^{\alpha}x_3 = x_1x_2 - (\dfrac{\delta + 8}{3})x_3. \end{cases} \tag{9}$$

The second drive system is described also by the fractional-order unified system

$$\begin{cases} D^{\alpha}y_1 = (25\delta + 10)(y_2 - y_1), \\ D^{\alpha}y_2 = (-35\delta + 28)y_1 - y_1y_3 + (29\delta - 1)y_2, \\ D^{\alpha}y_3 = y_1y_2 - (\dfrac{\delta + 8}{3})y_3, \end{cases} \tag{10}$$

and the controlled response system is chosen as the fractional-order Liu system [21]

$$\begin{cases} D^{\alpha} z_1 = a(z_2 - z_1) + u_1, \\ D^{\alpha} z_2 = bz_1 + z_1 z_3 - z_4 + u_2, \\ D^{\alpha} z_3 = -cz_3 - z_1 z_2 + z_4 + u_3, \\ D^{\alpha} z_4 = dz_1 + z_2 + u_4, \end{cases} \tag{11}$$

where $x_i$ , $y_i$ $(i = 1,\ 2,\ 3)$ and $z_j$ $(j = 1,\ 2,\ 3, 4)$ are the state variables of the master systems and the slave system, respectively, $\delta \in [0, 1]$, $D^{\alpha}$ is the Caputo differential operator $(0 < \alpha \leq 1)$,$u_1$, $u_2$ , $u_3$ and $u_4$ are the nonlinear controllers to be designed.
To simplify our discussions, we take the observable variable functions $Q, R : \mathbb{R}^3 \to \mathbb{R}^4$ and $S : \mathbb{R}^4 \to \mathbb{R}^4$ as

$$Q(x_1, x_2, x_3) = (x_1 - x_2, x_2, x_3 + 1, 2), \tag{12}$$

$$R(y_1, y_2, y_3) = (y_1 - y_2, y_2, y_3, 0) \tag{13}$$

and

$$S(z_1, z_2, z_3, z_4) = (z_1, z_2, z_3 + 1, z_4 - cz_3 + 2). \tag{14}$$

The error states are defined by

$$\begin{cases} e_1 = x_1 - x_2 + y_1 - y_2 - z_1, \\ e_2 = x_2 + y_2 - z_2, \\ e_3 = x_3 + y_3 - z_3, \\ e_4 = -z_4 + cz_3. \end{cases} \tag{15}$$

Then the error dynamical systems between the drive systems (9), (10) and the response system (11) can be written as

$$\begin{cases} D^{\alpha} e_1 = (10\delta - 38)e_1 + (7\delta - 27)e_2 + (10\delta + a - 38)z_1 + \\ \qquad + (7\delta - a - 27)z_2 + x_1 x_3 + y_1 y_3 - u_1, \\ D^{\alpha} e_2 = (29\delta - 1)e_2 - e_4 + (29\delta - 1)z_2 + (-35\delta + 28)(x_1 + y_1) + \\ \qquad - (x_1 x_3 + y_1 y_3) - bz_1 - z_1 z_3 + cz_3 - u_2, \\ D^{\alpha} e_3 = -(\dfrac{\delta + 8}{3})e_3 + e_4 + x_1 x_2 + y_1 y_2 + z_1 z_2 + -(\dfrac{\delta + 8}{3})z_3 - u_3, \\ D^{\alpha} e_4 = -ce_4 - dz_1 - z_2 - cz_1 z_2 - u_4 + cu_3. \end{cases} \tag{16}$$

To get the IOGCS to occur, the zero solutions of the error system must be stable, that is to say, the error evolution of the systems (9), (10) and (11) should tend to zero as $t \to +\infty$. So, a suitable controller $u_i, i = 1, 2, 3, 4$ should be designed which guarantees that system (16) stabilizes towards the origin. To this end, we need the following theorem and hypothesis.

**Theorem 2.1** *[22] Consider the fractional-order linear system*

$$D^{\alpha} x = Ax, \tag{17}$$

*where $x \in \mathbb{R}^n$ is the state vector. The previous system is asymptotically stable if and only if $|\arg(\lambda_i\ (A))| > \alpha \frac{\pi}{2}$ for $i = 1, 2, ..., n$, where $\arg(\lambda_i\ (A))$ denotes the argument of the eigenvalue $\lambda_i$ of $A$.*

***Hypothesis:*** *We assume that the controllers $u_i, i = 1, 2, 3, 4$ are chosen as*

$$\begin{cases} u_1 = (10\delta + a - 38)z_1 + (7\delta - a - 27)z_2 + x_1x_3 + y_1y_3 + k_1e_1, \\ u_2 = (29\delta - 1)z_2 + (-35\delta + 28)(x_1 + y_1) - (x_1x_3 + y_1y_3) - bz_1 - (z_1 - c)z_3 + k_2e_2, \\ u_3 = x_1x_2 + y_1y_2 + z_1z_2 - (\frac{\delta+8}{3})z_3, \\ u_4 = -dz_1 - z_2 + c\left(x_1x_2 + y_1y_2 + -(\frac{\delta+8}{3})z_3\right), \end{cases} \tag{18}$$

*where $k_1$ and $k_2$ are the feedback gains satisfying*

$$k_1 > 10\delta - 38 \text{ and } k_2 > 29\delta - 1. \tag{19}$$

Now, due to Theorem 2.1, we have the following results.

**Theorem 2.2** *If the controllers $u_i, i = 1, 2, 3, 4$ are given by (18), and the feedback gains $k_1$ and $k_2$ are given by (19), then*

$$\lim_{t\to+\infty} \|Q(x(t)) + R(y(t)) - S(z(t))\| = 0,$$

*that is to say, the IOGCS occurs between the systems (9), (10) and (11) with respect to the variable functions $Q, R$ and $S$.*

***Proof.*** By hypothesis (18), the error system (16) becomes

$$\begin{cases} D^\alpha e_1 = (10\delta - 38 - k_1)e_1 + (7\delta - 27)e_2, \\ D^\alpha e_2 = (29\delta - 1 - k_2)e_2 - e_4, \\ D^\alpha e_3 = -(\frac{\delta+8}{3})e_3 + e_4, \\ D^\alpha e_4 = -ce_4, \end{cases} \tag{20}$$

and the characteristic equation is

$$\frac{1}{3}(\lambda + c)(3\lambda + \delta + 8)(\lambda - 29\delta + k_2 + 1)(\lambda - 10\delta + k_1 + 38) = 0. \tag{21}$$

It is easy to obtain its characteristic roots as

$$\lambda_1 = -c,\ \lambda_2 = -(\frac{\delta+8}{3}),\ \lambda_3 = 29\delta - 1 - k_2 \text{ and } \lambda_4 = 10\delta - 38 - k_1. \tag{22}$$

Since $\delta \in [0, 1]$ and by hypothesis (19), all roots of (21) are negative. Therefore,

$$|\arg \lambda_i\ | > \alpha\frac{\pi}{2} \text{ for } i = 1, 2, 3, 4 \text{ and } 0 < \alpha < 1.$$

In view of Theorem 2.1, the error system (20) is asymptotically stable, which implies that the desired synchronization is achieved.

## 3 Numerical Simulations

In order to verify the theoretical results obtained in the above section, the corresponding numerical simulations will be performed. In the simulations, we take: $\alpha = 0.97$, $\delta = 1$,

$k_1 = -27$, $k_2 = 29$. The initial values of the two drive and the response systems are chosen as $(x_1(0), x_2(0), x_3(0))^T = (0.1, 0.1, 0.1)^T$, $(y_1(0), y_2(0), y_3(0))^T = (0.1, 0.1, 0.1)^T$ and $(z_1(0), z_2(0), z_3(0), z_4(0))^T = (0.3, 0.3, 0, -0.3)^T$ , respectively. The initial conditions for the error system are thus $(e_1(0), e_2(0), e_3(0), e_4(0))^T = (-0.3, -0.1, 0.2, 0.3)^T$.

Figures 1, 2 , 3 and 4 display the chaotic behaviors of the Lorenz system (9) (when $\delta = 0$), the Lü system (9) (when $\delta = 0.8$), the Chen system (9) (when $\delta = 1$), and the Liu system (11) (when $a = 10$, $b = 35$, $c = 1.4$ and $d = 5$), respectively. Figure 5 shows the curves of the synchronization errors (20) under the controllers (18).

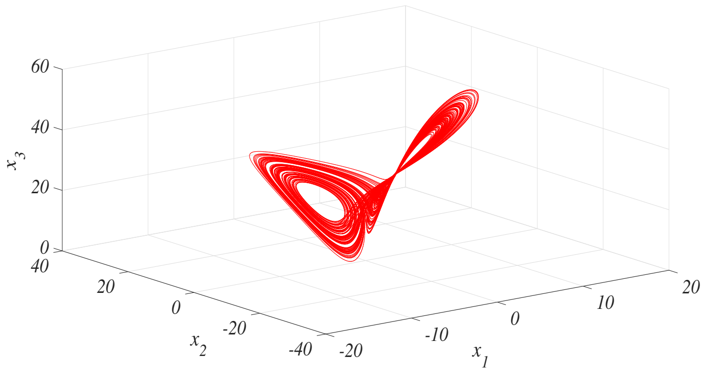


**Figure 1**: The chaotic attractor of the Lorenz system (9), when $\delta = 0$.

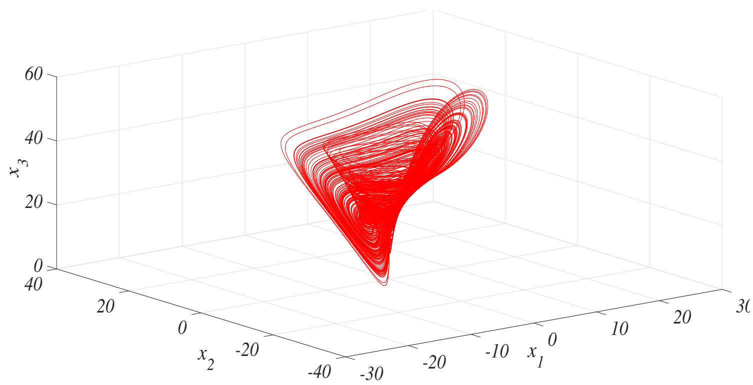


**Figure 2**: The chaotic attractor of the Lü system (9), when $\delta = 0.8$.

**Remark 3.1** From Figure 5, it can be seen that the components of the error system (20) decay towards zero as $t \to +\infty$, which implies that the desired synchronization is achieved with our designed scheme.

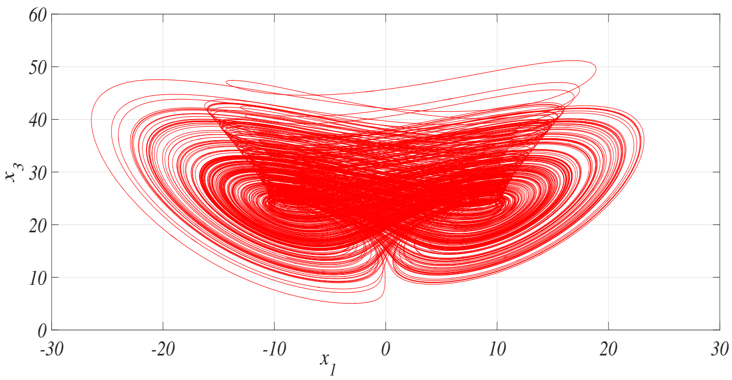


**Figure 3**: The chaotic attractor of the Chen system (9), when $\delta = 1$.

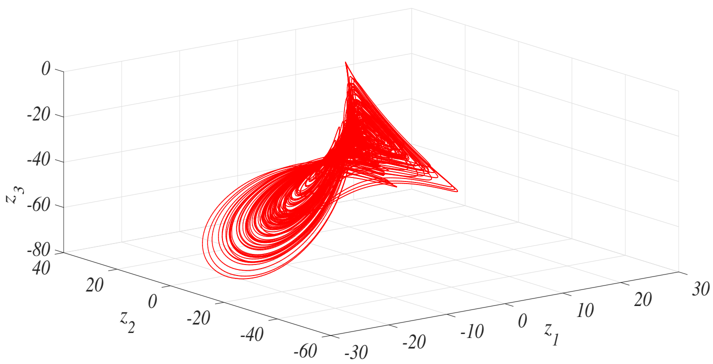


**Figure 4**: The hyperchaotic attractor of the Liu system (11).

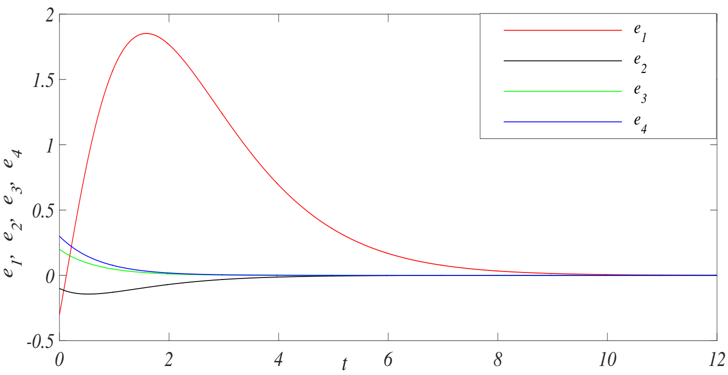


**Figure 5**: The curves of the synchronization errors (20).

## 4 Conclusion

In this paper, we have investigated a new type of combination synchronization, called IOGCS, between two drive systems of dimension 3 and a slave system of dimension 4 by introducing suitable observable variable functions. In view of the stability theory of linear fractional-order systems, a suitable controller is designed to achieve the desired synchronization. The method of this scheme has been applied for the combination of two fractional-order unified drive systems and the fractional-order Liu response system. Finally, numerical simulations are provided to verify the effectiveness of the proposed scheme.

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