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Fixed Point Regions, Unified Construction of Fixed Point Mappings for Integral, Quadratic, and Fractional Equations

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Abstract: This paper is a study of integral equations by means of mapping a closed bounded convex nonempty set G into its interior. This tells us that all possible fixed points reside in G which we then call a fixed point region. The study is restricted to convolution kernels $A(t-s)$ for which there is a transformation yielding an equivalent equation. We then devise a method whereby we can often find the above mentioned set G. This leads us to globally stable fixed points. The term which makes the equation of quadratic type is added in after the transformation, whereas existing theory along these lines adds it in directly to the Volterra equation. That method produces difficulties with compactness of the mapping. In our work compactness is never an issue.

Keywords: *fixed point regions; integral equations; quadratic equations; fractional equations; fixed points; transformations.*

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1 Introduction

This paper was motivated by the fact that many fixed point theorems begin with an integral equation and the preemptive assumption that there is a closed convex nonempty bounded set G in a Banach or normed space of bounded continuous functions $\phi : [0, \infty) \to \Re$ with the supremum norm, together with a continuous mapping of $G \to G$. Often the first mapping which comes to mind is the natural one defined by the integral equation.

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Then there are several additional conditions which would then yield a fixed point being a solution of the integral equation.

First, if we say that G is bounded and the mapping maps G into itself, then we are requiring that

The mapping maps bounded sets into bounded sets.

Accordingly, the first task of this paper is to give a coherent method for transforming an integral equation of the type under consideration here into one which maps bounded sets into bounded sets. In fact, we have long worked with exactly that transformation.

Second, there is such a wide class of integral equations that a paper of finite length needs to have a way of driving diverse equations into a single flexible type whether we start with a heat equation, a fractional differential equation of either Riemann-Liouville or Caputo type, or a quadratic integral equation.

In fact, we do exactly that and the final form into which all can fit after several known transformations is

$$x(t) = a(t) + g(t, x(t)) \int_0^t R(t-s)x(s)\left[1 - \frac{f(s, x(s))}{J}\right] ds.$$

Here, $R > 0$ is the resolvent of a kernel and $\int_0^\infty R(t)dt = 1$. It is this property which assures us that the natural mapping

maps bounded sets into bounded sets.

Third, many fixed point theorems require mappings with compactness or contraction conditions. This is known to fail for products $g(t, x(t)) \int_0^t R(t-s)f(s, x(s))ds$. We are not bothered by this because we never mention either compactness or contractions.

Fourth, we continue with the idea raised in several earlier papers that there is a conclusion which is competitive with the conclusion that there is a fixed point. Our conclusion is always that there is a closed bounded convex nonempty set G of such a nature that any fixed point which exists will reside entirely in G.

2 A Roadmap to Unification

We will begin with several very different kinds of equations. Through a series of transformations each will be brought down to the common Volterra integral equation of the form

$$x(t) = a(t) - \int_0^t A(t-s)f(s, x(s))ds \tag{1}$$

although the functions will change for each of them. But all of the functions are continuous and $A(t) > 0$ and often

$$\int_0^\infty A(s)ds = \infty.$$

Clearly, something must be done because it cannot map bounded sets into bounded sets.

Next, through a variation of parameters formula and substantial theory going back to Friedman in 1963 and organized by Miller in 1971, every one of those equations will now pass through (1) and become

$$x(t) = b(t) + \int_0^t R(t-s)\left[x(s) - \frac{f(s, x(s))}{J}\right] ds, \tag{2}$$

where

$$b(t) = a(t) - \int_0^t R(t-s)a(s)ds$$

and R will vary from one equation to the next being the resolvent of A and depending on the choice of J which can be any positive number. Regardless of the differences between various A and J it will always be true that R is continuous on $(0, \infty)$, $R(t) > 0$ and

$$\int_0^\infty R(t)dt \leq 1.$$

That condition will give our primary victory in having (2) map bounded sets into bounded sets.

At this point we reach back in history to 1950 when Chandrasekhar [9] offered the equation

$$x(t) = 1 + x(t) \int_0^1 \phi(s)x(s)ds$$

to describe radiation transfer which jolted fixed point theory investigators into a cottage industry called "quadratic integral equations". They knew that

$$\int_0^t A(t-s)f(s, x(s))ds$$

was often a compact map (see [8] for example), as required in many fixed point theorems, but a product

$$g(t, x(t)) \int_0^t A(t-s)f(s, x(s))ds$$

is not compact even with g and f continuous. By simply Googling "quadratic integral equations", we find a myriad of publications including many in pdf format freely available for download. Our bibliography contains several.

To take all of this into account we return to (2) and perturb it with $g(t, x)$ obtaining our final product as

$$x(t) = b(t) + g(t, x(t)) \int_0^t R(t-s) \left[x(s) - \frac{f(s, x(s))}{J} \right] ds \tag{3}$$

with the assumption that $g(t, x)$ is continuous and

$$|g(t, x)| \leq 1. \tag{4}$$

Thus, if $g(t, x)$ is missing in an equation of interest, then it is simply replaced by 1. In spite of all the equations which we are going to describe here, it is only (3) that we must treat.

This is the unity.

But we will take it one step further. We will offer a result concerning a closed bounded convex nonempty set G which will contain any possible fixed point of the natural mapping defined by (3). It will be a single theorem encompassing all the problems treated.

3 Some Related Problems

It is not at all necessary for the reader to be acquainted with fractional equations to get the point that in each case we start with one type of equation, but always transform it into one specific class. We will start with a fractional differential equation of Riemann-Liouville type and continue down to its relationship with very common Volterra equations. In preparation for that we offer some references. A modern introduction to fractional differential equations is found in Diethelm [12]. A very readable history of the theory up to 1974 is found in Chapter 1 of Oldham and Spanier [16] and a continuation up to the present time is found in Abbas, Benchohra, Graef, and Henderson [1]. However, we quickly see in the work below that the fractional calculus is immediately left behind and it is translated into an integral equation of common type going back almost 100 years. We start with the fractional differential equation, translate it into an integral equation with more than one singularity, introduce a shift of length T, and then transform it into a classical integral equation. The reader may wish to simply start with that final translation and follow the chain down from there. The next equation will be a fractional differential equation of Caputo type which was introduced because the Riemann-Liouville equation has a troublesome initial condition. In one step we translate it into a common Volterra integral equation. From there on we are all on familiar ground and the reader will need no guidance as we descend the chain.

We will see $R(t)$ throughout this list. It is the resolvent of the kernel. For fractional and some other kinds of equations it is the continuous resolvent of the kernel $(t-s)^{q-1}$ satisfying

$$0 < R(t), \int_0^\infty R(s)ds = 1$$

and will be discussed in detail later. It is found in Miller [15, pp. 212-213]. It has the most useful property found in fixed point theory, namely, it enables us to show that the natural mapping defined by the integral equation maps bounded sets into bounded sets. We frequently find that we need a closed bounded convex nonempty set G and a continuous mapping of G into G.

(i) We begin with the Riemann-Liouville fractional differential equation

$$D^q x(t) = f(t, x(t)), \lim_{t\to 0^+} t^{1-q}x(t) = x^0 \neq 0, \quad q \in (0,1).$$

It is inverted as

$$x(t) = x^o t^{q-1} + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s, x(s))ds$$

and transformed in two steps to

$$y(t) = F(t) + \int_0^t R(t-s)\left[y(s) + \frac{f(s+T, y(s))}{J}\right]ds,$$

where $J > 0$ is an arbitrary constant. Here, $T > 0$ is a constant arising in a local existence theorem and $y(t) = x(t+T)$. All of the details are found in [2, pp. 242-271] (see, especially, p. 252 and (4.4)). It is the recurring form of that last equation which enables us to find a closed bounded convex nonempty set G so that the natural mapping maps G into the interior of G and contains all possible fixed points.

(ii) Close to (i) is the fractional differential equation of Caputo type sought mainly for the more direct initial conditions. It is

$$^{c}D^{q}x(t) = f(t) - g(t, x(t)), \quad x(0) \in \Re, \quad 0 < q < 1,$$

with f and g continuous, which inverts as

$$x(t) = x(0) - \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}[g(s, x(s)) - f(s)]ds.$$

One more transformation brings it in line with (i) and the property that bounded sets are mapped into bounded sets owing again to the function R. The final form is

$$x(t) = x(0)\left[1 - \int_0^t R(s)ds\right] + \int_0^t R(t-s)\left[x(s) - \frac{g(s, x(s)) - f(s)}{J\Gamma(q)}\right]ds.$$

Details may be found in [3], pp. 442-456. See, especially, Example 5.6, p. 450.

(iii) The next equation is the one which made this entire project possible by generating the resolvent R which is used in each of the transformations. We begin with a heat equation found in Miller [15, p. 209]

$$y(t) = -(\pi K)^{-1/2} \int_0^t (t-s)^{-1/2} g(s, y(s))ds,$$

where K is thermal conductivity of the medium. The construction of this equation can be traced back to pp. 356-361 of the book by Weinberger [21]. Miller [15, p. 209] picks it up and this is the beginning of his presentation which will lead us to the construction of that all important transformation and the introduction of the resolvent R. We will return to this and the construction of R later. In order to make the text consistent, we will write this equation as

$$x(t) = -\int_0^t (t-s)^{-1/2} f(s, x(s))ds.$$

Our transformation will yield

$$x(t) = \int_0^t R(t-s)\left[x(s) - \frac{f(s, x(s))}{J}\right]ds$$

and we will refer to it as our heat equation.

There can now be gathered into this a large and basic theory of heat equations generated near 1950 and published by Padmavally [17], Mann and Wolf [18], and Roberts and Mann [19]. We will not detail it here.

This will conclude the discussion of problems which can be put in the form of

$$x(t) = a(t) + \int_0^t R(t-s)x(s)\left[1 - \frac{f(s, x(s))}{Jx(s)}\right]ds.$$

When we consider that either of the fractional problems represent a theory about which entire books have been written, it is clear that we are dealing with a substantial fraction of problems from applied mathematics.

But there is one more type which does not fit snuggly in this set of problems which has generated an enormous literature. That is the so-called quadratic theory of integral equations. Interest in the problem was sparked by a paper written in 1950 by Chandrasekhar [9] in the study of radiative transfer governed by

$$x(t) = 1 + x(t) \int_0^1 \phi(s)x(s)ds$$

in which ϕ is a second order polynomial presumably giving rise to the name "quadratic" integral equations. We discussed aspects of it with (3). Many applications of off shoots may be found in Darwish [10] and Darwish and Henderson [11], for example, but the form of the equation with that product with the integral has been the driving force for a great many investigations in fixed point theory. At this time the area of study is so active that one can get an idea of the amount of work done in it by asking "google" for "quadratic integral or integro-differential equations".

4 Transformations

In 1963, Friedman [13] isolated a wide and important set of kernels and the entire process is well formulated by Miller [15, pp. 207-213] with refinements continuing to p. 224 and with more recent refinements given by Gripenberg [14]. It is required that $A(t)$ satisfy the following three conditions (A1)–(A3) which are present in many important problems in heat conduction and throughout fractional differential equations.

It is to be noted that if A satisfies (A1)–(A3) and $\int_0^\infty A(s)ds = \infty$ and if g and$f$ are bounded by the real numbers $\|g\|$ and $\|f\|$, then in the equation

$$x(t) = a(t) - \frac{g(t, x(t))}{\|g\|} \int_0^t \|g\| A(t-s) f(s, x(s))ds$$

the kernel will still satisfy (A1)–(A3) and it will still be the same equation if $\|g\|$ is a fixed positive number, being the supremum on $[0, \infty)$. We are going to assume that f and g are bounded because the entire focus here will be on a natural mapping which maps bounded sets into bounded sets and we are interested in solutions on $[0, \infty)$.

Conditions (A1)–(A3) are defined as follows:
(A1) $A(t) \in C(0, \infty) \cap L^1(0, 1)$.
(A2) $A(t)$ is positive and non-increasing for $t > 0$.
(A3) For each $T > 0$ the function $A(t)/A(t+T)$ is non-increasing in t for $0 < t < \infty$.
In those references above it is shown that the resolvent equation is

$$R(t) = A(t) - \int_0^t A(t-s)R(s)ds \tag{5}$$

and that its solution R is continuous on $(0, \infty)$ and

$$0 < R(t) \le A(t), \qquad \int_0^\infty R(t)dt = 1 \tag{6}$$

when the integral of A is infinite. When the integral of A is finite, then the integral of R is less than one.

Notice that if J is a positive constant, then $JA(t)$ still satisfies (A1)–(A3). We have noted just now that $\int_0^t A(t-s)f(s, x(s))ds$ may map bounded sets into unbounded sets. If we could possibly exchange $R(t)$ for $A(t)$, then we could map bounded sets into bounded sets. That is exactly what we do and the transformation can be reversed so that the transformed equation has the same solutions as the original equation.

In a sequence of papers we showed the advantages of transforming the standard integral equation

$$x(t) = a(t) - \int_0^t A(t-s)f(s,x(s))ds \tag{7}$$

using a variation of parameters formula of Miller [15, pp. 191-192] into

$$x(t) = z(t) + \int_0^t R(t-s)\left[x(s) - \frac{f(s,x(s))}{J}\right]ds, \tag{8}$$

with

$$z(t) = a(t) - \int_0^t R(t-s)a(s)ds. \tag{9}$$

Here are the steps. Start with (7) and $a(t)$ continuous on $[0,\infty)$ while A satisfies (A1)–(A3) and J is an arbitrary positive constant at this point, but later is chosen precisely as previously discussed. We then have

$$\begin{aligned} x(t) &= a(t) - \int_0^t A(t-s)[Jx(s) - Jx(s) + f(s,x(s))]ds \\ &= a(t) - \int_0^t JA(t-s)x(s)ds + \int_0^t JA(t-s)\left[x(s) - \frac{f(s,x(s))}{J}\right]ds. \end{aligned}$$

The linear part is

$$z(t) = a(t) - \int_0^t JA(t-s)z(s)ds \tag{10}$$

and the resolvent equation is

$$R(t) = JA(t) - \int_0^t JA(t-s)R(s)ds \tag{11}$$

so that by the linear variation-of-parameters formula we have

$$z(t) = a(t) - \int_0^t R(t-s)a(s)ds \tag{12}$$

and by the non-linear variation of parameters formula [15, pp. 191-193]

$$x(t) = z(t) + \int_0^t R(t-s)\left[x(s) - \frac{f(s,x(s))}{J}\right]ds.$$

We will always write this as

$$x(t) = a(t) - \int_0^t R(t-s)a(s)ds + \int_0^t R(t-s)\left[x(s) - \frac{f(s,x(s))}{J}\right]ds \tag{13}$$

which has a reversible mapping into (7).

The transformation from (7) to (13) was first given in [4] for a Caputo equation in which case there are few difficulties. Further discussion of the transformation is found in [2] which allows $a(t)$ to be singular. In that reference the reader can follow from (2) on p. 249 to its transformed form on p. 263.

We cannot overemphasize the role of J, as we will see later.

5 Fixed Point Regions

The compactness required in so many fixed point theorems eludes us and we stop for a moment of thought. If we just work a bit harder we may let the mapping map G into G^o, the interior of G. Perhaps that will be enough to provide for us some usable information. This is where we pick up from earlier papers ([5, p. 297], [6, p. 344], [7, p. 7]) in which we showed that under our specified mapping condition the set G can be shown to

contain entirely every possible fixed point of (1).

Instead of fixed point theory, we are now studying what can be derived from **fixed point regions G**. Our next task is to show how this concept relates to standard fixed point theory and for this we begin with Schauder's and Schaefer's fixed point theorems. We will find that the new concept is a simple counterpart of the combination of both of those theorems. Here is the statement as given in [7].

Theorem 5.1 *Let $T > 0$ and $(\mathcal{B}, \|\cdot\|)$ be the Banach space of continuous functions $\phi : [0,T] \to \Re$ with the supremum norm and let P be a Volterra operator mapping $\mathcal{B} \to \mathcal{B}$ which is continuous. Let $r > 0$ and G be the closed ball of center zero and radius r in $\mathcal{B}$:*

$$G := \{\phi \in \mathcal{B} : \|\phi\| \le r\}.$$

Suppose that $P : G \to G^o$ and that P has the property that if $\phi \in \mathcal{B}$ and if $(P\phi)(0) = \phi(0)$, then $|\phi(0)| < r$. If $\phi \in \mathcal{B}$ is a fixed point of P, then ϕ resides in G^o.

Here is a typical example of quadratic type.

Example 5.1 Consider the scalar equation

$$x(t) = a(t) + \lambda(Arctanx(t)) \int_0^t R(t-s)[x(s) - \sin x(s))]ds.$$

Choose

$$0 < \lambda < 1/\pi, \|a\| \le \pi/4, \quad G = \{\phi : [0,\infty) \to \Re, \|\phi\| \le \pi/2\}.$$

Note that for $\|x\| \le \pi/2$ then $\left|1 - \frac{\sin x}{x}\right| \le 1$.

Under these conditions the natural mapping P satisfies

$$\begin{aligned}|(P\phi)(t)| &\le \|a\| + (\lambda\pi/2) \int_0^t R(t-s)|\phi(s)| \left|1 - \frac{\sin\phi(s)}{\phi(s)}\right| ds \\ &\le \|a\| + \lambda(\pi/2)(\pi/2) \int_0^t R(t-s)ds \\ &< \pi/4 + \lambda(\pi^2/4),\end{aligned}$$

so

$$\|P\phi\| \le \pi/4 + \lambda(\pi^2/4) < (\pi/4) + (\pi/4) = \pi/2.$$

Hence $P : G \to G^o$.

Now, we may go through the proof and see that all is unchanged if $\lambda Arctanx(t)$ is replaced by $\lambda\pi/2$.

Theorem 5.2 *(Schauder [20, p. 25]) Let G be a non-empty convex subset of a normed space $\mathcal{B}$. Let P be a continuous mapping of G into a compact set $K \subset G$. Then P has a fixed point.*

Schauder's theorem fails in our problem here because the map generated by our equation is not compact. Schauder's theorem fails us in another way also. Even if there is a fixed point in G his theorem offers us no way to tell if there are other fixed points outside of G. This can be a disaster. Generally, we select G for two reasons. First, we find through trial and error that we can show that P maps G into G. But there is a far more serious reason. In a given real-world problem there are points in the space which are favorable to our project and so we select G to contain them. On the other hand there may be points which are a total disaster for our project and we cannot move forward with the project until we know that they cannot be selected. Here, our method helps in two ways. Even if Schauder's theorem applies we are well advised to proceed with the mapping and be sure that G contains all possible fixed points. We can make choices to put all fixed points in a predetermined ball.

Theorem 5.3 *(Schaefer [20, p. 29]) Let $\mathcal{B}$ be a normed space, P, a continuous mapping of $\mathcal{B} \to \mathcal{B}$ which is compact on each bounded subset K of $\mathcal{B}$. Then either*
(i) the equation $x = \lambda P x$ has a solution for $\lambda = 1$, or
(ii) the set of all such solutions x (if any), for $0 < \lambda < 1$, is unbounded.

Not only is this theorem replete with demands for compactnes, but there is not a hint of where the fixed point might lie or how many there might be. Our method seems to offer no help for Schaefer's, but it does cast a light to show a somewhat equal worth. In principle, would we prefer to know where all possible solutions lie, or would we prefer to know that there is at least one lying somewhere? We are arguing that both conclusions can be targets of equally high value.

6 Mapping Bounded Sets into Bounded Sets

The perfect choice of J

So many fixed point theorems begin with the assumption that there is a closed bounded convex nonempty set G and a mapping of G into G. That part is stated so smoothly that it is reasonable to get the impression that it is a simple condition and that the real challenge consists of all the added conditions needed to ensure the existence of a fixed point. Several decades of study reveal the very opposite for many of us. We feel that the upcoming work which shows a simple way of getting the required G in this context is, perhaps, the main contribution of this paper.

In the transformation of (7) to (13) we will see that J has entered and we would have no idea how to select it. It will play a major role in our work. In case $f(t, x)$ depends only on x, there is a definite path to finding G so that the natural mapping, P, of $G \to G^o$ will contain the smallest possible fixed point region. It is successful if

$$\lim_{|x|\downarrow 0} \frac{f(x)}{x} = J \tag{14}$$

exists as a finite positive number. To explain the working here, we are asking that $xf(x) \geq 0$ so that

$$\frac{xf(x)}{x^2} \geq 0$$

which implies that if the limit exists, then it is ≥ 0. In particular, then the limit as $|x| \downarrow 0$ satisfies

$$\lim_{|x|\downarrow 0} \frac{f(x)}{Jx} = 1. \tag{15}$$

From this we see that the limit as $|x| \downarrow 0$ satisfies

$$\lim_{|x|\downarrow 0} \left[1 - \frac{f(x)}{Jx}\right] = 0 \tag{16}$$

and that is the perfect choice for J. It will allow the integrand to completely control the magnitude of the natural mapping P of G into G, making the fixed point region as small as possible. The only magnitude it cannot control is $\|a\|$ since by Theorem 5.1 $a(t)$ is always part of the fixed point region. In other words:

Remark 6.1 Equation (16) is exactly the property which helps us to find G and to make G map into G^o when f and g are bounded, f is independent of t, and (14) holds.

7 The Shrinking Functions

We have mentioned earlier that investigators started with the standard Volterra equation

$$x(t) = a(t) - \int_0^t A(t-s)f(s,x(s))ds$$

and studied the problems raised by addition of the term $g(t,x(t))$ in

$$x(t) = a(t) - g(t,x(t))\int_0^t A(t-s)f(s,x(s))ds,$$

which was actually an arbitrary choice. (Remember that J is chosen by (15) if possible.)

We choose instead to start with the equivalent transformed equation (13) and add in the $g(t,x(t))$ to it obtaining

$$x(t) = a(t) - \int_0^t R(t-s)a(s)ds + g(t,x(t))\int_0^t R(t-s)\left[x(s) - \frac{f(s,x(s))}{J}\right]ds \tag{17}$$

and noticing that $a(t) - \int_0^t R(t-s)a(s)ds$ is the forcing function. Refer back to Chandrasekhar's equation

$$x(t) = 1 + x(t)\int_0^t \phi(s)x(s)ds$$

and notice that $g(t,x) = x$ and that does not multiply the forcing function, namely 1. It multiplies the integral.

In Example 5.1 there appeared an example of

$$S(x) =: x - f(x),$$

namely,

$$x - \sin x,$$

from which we obtained

$$|x|\left|1-\frac{\sin x}{x}\right|$$

with

$$0 \leq \frac{\sin x}{x} < 1, \quad x \neq 0,$$

when we take

$$G = \{\phi : [0,\infty) \to \Re, \|\phi\| \leq \pi/2\}, \quad \|a\| \leq \pi/4.$$

Let us review Example 5.1 and the first line of the mapping P in the proof. We have

$$\begin{aligned}|(P\phi)(t)| &\leq \|a\| + (\lambda\pi/2)\int_0^t R(t-s)|\phi(s)|\left|1-\frac{\sin\phi(s)}{\phi(s)}\right|ds \\ &\leq \|a\| + \lambda(\pi/2)(\pi/2)\int_0^t R(t-s)ds \\ &< \pi/4 + \lambda(\pi/2)(\pi/2),\end{aligned}$$

so for $\lambda < \frac{1}{\pi}$

$$\|P\phi\| \leq \pi/4 + \lambda(\pi/2)(\pi/2) < \pi/4 + (1/\pi)(\pi^2/4) = \pi/2.$$

In that first line we see that the last term shrinks the entire remainder of the mapping (excluding a) yielding $P : G \to G^o$. Thus, $1 - \frac{f(x)}{x}$ is shrinking $|x|$ which is exactly what it must do if the mapping is to map G into G^o.

We will now work a problem to see how it unfolds, writing $a(t)$ instead of all of z. This will not distort the norms we later encounter. Before we start notice that if $xf(x) > 0$ for $x \neq 0$, then $f(0) = 0$ and if $f'(0)$ exists and it is positive, then by L'Hospital's rule

$$\lim_{x\to 0}\frac{f(x)}{x} = \frac{0}{0} = \frac{f'(0)}{1} > 0.$$

That is

$$\lim_{x\to 0}\frac{f(x)}{x} = J > 0$$

so

$$\lim_{x\to 0}\frac{f(x)}{Jx} = 1.$$

Example 7.1 Part I, f is not specified.

We begin with

$$x(t) = a(t) + g(t,x(t))\int_0^t R(t-s)x(s)\left[1-\frac{f(x(s))}{x(s)}\right]ds$$

and assume that

$$\lim_{|x|\downarrow 0}\frac{f(x)}{Jx} = 1$$

so that if we define the function h to be zero at zero and by

$$h(x) = 1 - \frac{f(x)}{Jx}$$

then it is continuous. For a given $\epsilon > 0$ there is a $D > 0$ such that $|x - 0| < D$ implies $|h(x) - h(0)| < \epsilon$. But $h(0) = 0$, so taking $\epsilon = 1/2$ we can find $D > 0$ so that $|x| < D$ implies

$$0 \leq \left|1 - \frac{f(x)}{Jx}\right| < 1/2$$

which happens when $1/2 < f(x)/Jx < 3/2$. We cannot specify that in Part I because f is unknown. Assume again that $|g(t,x)| \leq 1$. We can now obtain G provided that

$$\|a\| < D/2.$$

Now, let

$$G = \{\phi : [0, \infty) \to \Re, \|\phi\| \leq D\}.$$

Then

$$\|P\phi\| \leq \|a\| + (1)(1)\|\phi\|(1/2) < (D/2) + (D/2) = D.$$

Thus $P : G \to G^0$.

In view of the continuity of g, ϕ (thus boundedness of ϕ), one may see that for any $\phi \in \mathcal{B}$ (the space of continuous functions on $[0,T]$ with $T > 0$ arbitrary), we have $|P(\phi)(0)| = |a(0)| < |D/2$, thus by Theorem 5.1 every possible fixed point resides entirely in G.

Continuing, notice that the same argument can be given as $D \to 0$ with the result that when $a(t) = 0$, then by inspection $x = 0$ is a solution and it is unique. This concludes Part I of this example. We now turn to

Part II in which we specify

$$f(x) = \frac{x}{1 + x^2}$$

which satisfies $xf(x) \geq 0$, f is bounded, and

$$\lim_{|x|\downarrow 0} \frac{f(x)}{x} = \lim_{|x|\downarrow 0} \frac{1}{1 + x^2} = 1 = J.$$

Given $\epsilon = 1/2$, we must determine D so that $0 \leq |x| \leq D$ implies

$$0 < 1 - \frac{f(x)}{x} < 1/2.$$

Write that as

$$0 < 1 - \frac{1}{1 + x^2} < 1/2.$$

Now $1 - \frac{1}{1+x^2}$ was zero at $x = 0$, so we let x increase to the value one making $\frac{1}{1+x^2} = 1/2$, telling us that $D = 1$ and

$$G = \{\phi : [0, \infty) \to \Re : \|\phi\| \leq 1\}.$$

This requires us to be able to choose $\|a\| < D/2 = 1/2$ since $\|\phi\| \leq 1$ implies that

$$|P\phi| \leq \|a\| + (1/2)D = \|a\| + (1/2) < 1$$

requires $\|a\| < 1/2$. This makes $P : G \to G^o$ and assures us that any fixed point lies entirely in G.

This concludes Part II and the example.

8 Another Perturbation

We are now concerned with a slightly different perturbation than the one studied in Section 7. Recall that we deal with the Volterra integral equation

$$x(t) = a(t) - \int_0^t A(t-s) f(s, x(s))\, ds, \quad t \geq 0, \tag{18}$$

with a, f continuous and A continuous on $(0, \infty)$ with

$$\int_0^\infty A(s)\, ds = \infty,$$

and satisfying specific assumptions yielding that

$$\int_0^\infty R(u)\, du = 1.$$

Equation (18) is often perturbed by multiplying the integral by the factor $g(t, x)$, i.e, as

$$x(t) = a(t) - g(t, x(t)) \int_0^t A(t-s) f(s, x(s)) ds. \tag{19}$$

Considering a properly chosen $J > 0$ we, again, transform equation (18) into the (equivalent) equation

$$x(t) = a(t) + \int_0^t R(t-s) \left[x(s) - a(s) - \frac{f(s, x(s))}{J} \right] ds, \tag{20}$$

where R is the resolvent kernel of A satisfying the equation

$$R(t) = JA(t) - \int_0^t JA(t-u) R(u)\, du. \tag{21}$$

Now we choose to multiply the whole integral in (20) by the factor $g(t, x)$, i.e., we consider the equation

$$x(t) = a(t) - g(t, x(t)) \int_0^t R(t-s) \left[a(s) - \left(x(s) - \frac{f(s, x(s))}{J} \right) \right] ds,$$

which we write as

$$\begin{aligned} x(t) &= a(t) - g(t, x(t)) \int_0^t R(t-s)\, a(s)\, ds \\ &\quad + g(t, x(t)) \int_0^t R(t-s) \left[x(s) - \frac{f(s, x(s))}{J} \right] ds. \end{aligned} \tag{22}$$

Comparing to the perturbation in Section 7, here multiplication by $g(t, x)$ includes the part of the integral containing the function a. Our view now is that we leave $x - a$ untouched and perturb the integral part of the equation (20), just as it is done in equation (19). It should be mentioned that while equations (18) and (20) share solutions, the perturbations (19) and (22) do not.

As already mentioned, J may be any positive number, but the proper choice of $J > 0$ is crucial for our study. Any (arbitrary) choice of $J > 0$ leads to a unique kernel satisfying (21) as well as to the corresponding transformed equation (22) equivalent to the original equation (20). Clearly, there are so many equations to work with, and one may wonder which one might be a proper value of J that allows us to achieve our goal. In the two propositions that follow, we use different techniques to spot proper values of $J > 0$ allowing us to obtain the desired results.

We may now proceed to presenting conditions yielding fixed point regions for equation (22) when g, a are bounded. Note that f is not assumed to be bounded. Since we are interested in continuous solutions on $[0, \infty)$, due to the continuity of g one may see that for any continuous function ϕ, we have $|\mathcal{T}(\phi)(0)| = |a(0)|$ with $\mathcal{T}$ being the natural mapping defined by the right-hand side of the equation (22). It turns out that in order to obtain a fixed point region for the equation (22) it is sufficient to find a suitable $D > 0$ (with $|a(0)| < D$) so that the corresponding ball in the space $\mathcal{B}$ of bounded continuous functions with the usual sup-norm be mapped in its interior by $\mathcal{T}$. The Propositions below present sufficient conditions posed on g, a, f which yield the existence of such a D.

Proposition 8.1 *Let g, a be bounded by $\|g\|$ and $\|a\|$, respectively. Assume that there exist $m, M > 0$ with*

$$m \leq \frac{f\left(t,x\right)}{x} \leq M, \quad x \neq 0, \ \ t \geq 0, \tag{23}$$

and such that

$$\|g\| \left(1 - \frac{m}{M}\right) = k < 1. \tag{24}$$

Then the set $G := \{x\left(t\right), t \geq 0 : \|x\| \leq D\}$ with $D > 0$ satisfying

$$\frac{\|a\| \left(1 + \|g\|\right)}{D} + k = k_0 < 1, \tag{25}$$

is a fixed point region for the equation (22) with $J = M$.

Proof. Firstly, note that by (23) we have

$$m \leq \frac{f\left(t,x\right)}{x} \leq M \Longrightarrow 0 \leq 1 - \frac{f\left(t,x\right)}{Mx} \leq 1 - \frac{m}{M},$$

thus choosing $J = M$ it holds

$$0 \leq 1 - \frac{f\left(t,x\right)}{Jx} \leq 1 - \frac{m}{M}. \tag{26}$$

Then setting $\mathcal{T} : C\left([0,\infty),\mathbb{R}\right) \to C\left([0,\infty),\mathbb{R}\right)$ with

$$\begin{aligned} \mathcal{T}x\left(t\right) \quad : \quad &= a\left(t\right) - g\left(t,x\left(t\right)\right) \int_0^t R\left(t-s\right) a\left(s\right) ds \\ &+ g\left(t,x\left(t\right)\right) \int_0^t R\left(t-s\right) \left[x\left(s\right) - \frac{f\left(s,x\left(s\right)\right)}{J}\right] ds, \end{aligned}$$

and choosing D by (25), in view of (24) and (26) we have for $t \geq 0, \|x\| \leq D$

$$\begin{aligned}
|\mathcal{T}x(t)| &\leq \|a\| + |g(t, x(t))| \int_0^t R(t-s)|a(s)|\, ds \\
&\quad + |g(t, x(t))| \int_0^t R(t-s) \left| x(s) - \frac{f(s, x(s))}{J} \right| ds \\
&\leq \|a\| + \|g\| \|a\| \int_0^t R(s)\, ds \\
&\quad + \|g\| \int_0^t R(t-s)|x(s)| \left| 1 - \frac{f(s, x(s))}{Jx(s)} \right| ds \\
&< \|a\| + \|g\| \|a\| \cdot 1 + \|g\| \int_0^t R(t-s) \left(1 - \frac{m}{M}\right) \|x\|\, ds \\
&< \|a\|(1 + \|g\|) + kD \cdot 1 \\
&= D \left[\frac{\|a\|(1 + \|g\|)}{D} + k \right] = Dk_0,
\end{aligned}$$

so $\|\mathcal{T}x\| \leq Dk_0 = D_0 < D$, and $\mathcal{T}(G) \subset G^o \subset B(0; D)$.

Clearly, if x is a fixed point of $\mathcal{T}$, then by continuity we will have $|x(0)| = |a(0)| < D$, so, due to $\mathcal{T}(G) \subset G^o$, the solution x cannot leave G.

In the same direction we consider the equation

$$x(t) = a(t) + \int_0^t A(t-s) f(s, x(s))\, ds, \quad t \geq 0, \tag{27}$$

with f, a and A as before, but now we relax condition (23). Note that condition (23) includes the sign condition $xf(x) > 0,\ x \neq 0$. In fact, now we do not ask for any sign condition.

As before, we want to transform equation (27) using a properly chosen $J > 0$ and then perturb it to a quadratic equation by multiplying the integral by a bounded function $g(t, x)$. This time we assume that the bound of $|g|$ is less than 1 and choose $J > 0$ depending on the bound of g. It turns out that by asking that $\|g\| < 1$ we may avoid the left hand side assumption in (23) along with condition (24).

So we consider the perturbed transformed equation

$$\begin{aligned}
x(t) &= a(t) - g(t, x(t)) \int_0^t R(t-s) a(s)\, ds \qquad (28) \\
&\quad + g(t, x(t)) \int_0^t R(t-s) \left[x(s) + \frac{f(s, x(s))}{J} \right] ds,
\end{aligned}$$

where $J > 0$ is some properly chosen constant depending on the bounds of g and a and the behavior of $\frac{f(t,x)}{x}$ on $[0, \infty) \times \{0\}$.

Proposition 8.2 *Let g and a be bounded by $\|g\| < 1$ and $\|a\|$, respectively, and assume that*

$$|f(t, x)| \leq \psi(x), \quad (t, x) \in [0, \infty) \times \mathbb{R},$$

for a continuous function $\psi : \mathbb{R} \to [0, \infty)$. If there exist $K, \delta > 0$ with

$$\left| \frac{\psi(x)}{x} \right| \leq K, \quad 0 < |x| \leq \delta, \tag{29}$$

then there exists a bounded set G which is a fixed point region for the equation (28) with J satisfying (32) .

Proof. Since $\|g\| < 1$, we may consider a $D > 0$ such that

$$\frac{2\|a\|}{D} < 1 - \|g\| . \tag{30}$$

Due to continuity of $\frac{\psi(x)}{x}$, $|x| \geq \delta$ and (29), there exists an $M > 0$ with

$$\left|\frac{\psi(x)}{x}\right| \leq M, \quad |x| \leq D. \tag{31}$$

Now with this M in hand and in view of (30) and the assumption that $\|g\| < 1$ we may take a $J > 0$ such that

$$\frac{2\|a\|}{D} + \|g\| \left(1 + \frac{M}{J}\right) = k_1 < 1. \tag{32}$$

Taking into consideration (32) and (31) we have for $t \geq 0, \|x\| \leq D$

$$\begin{aligned}
|\mathcal{T}x(t)| &\leq \|a\| + |g(t, x(t))| \int_0^t R(t-s)|a(s)|\, ds \\
&\quad + |g(t, x(t))| \int_0^t R(t-s)\left|x(s) + \frac{f(s, x(s))}{J}\right| ds \\
&\leq \|a\| + \|g\| \int_0^t R(s)\|a\|\, ds \\
&\quad + \|g\| \int_0^t R(t-s)|x(s)|\left[1 + \frac{|f(s, x(s))|}{J|x(s)|}\right] ds \\
&\leq \|a\| + \|g\| \int_0^t R(s)\|a\|\, ds \\
&\quad + \|g\| \int_0^t R(t-s)|x(s)|\left[1 + \frac{1}{J}\left|\frac{\psi(x(s))}{x(s)}\right|\right] ds \\
&< \|a\| + \|a\| \cdot 1 + \|g\| \int_0^t R(t-s)\left(1 + \frac{M}{J}\right)\|x\|\, ds \\
&< 2\|a\| + \|g\|\left(1 + \frac{M}{J}\right) D \\
&= D\left[\frac{2\|a\|}{D} + \|g\|\left(1 + \frac{M}{J}\right)\right] = k_1 D,
\end{aligned}$$

so, for any $k_2 \in (k_1, 1)$ it holds $\|\mathcal{T}x\| < k_2 D := D_1 < D$, $\|x\| \leq D$, which implies that for $G := \{x \in C([0, +\infty)) : \|x\| \leq D\}$ we have $\mathcal{T}(G) \subset B(0; D_1) \subset G^o$.

Clearly, if x is a fixed point of $\mathcal{T}$, then by continuity of x, a and f we find

$$\begin{aligned}
&\lim_{t \to 0+} |g(t, x(t))| \int_0^t R(t-s)\left|x(s) + \frac{f(s, x(s))}{J}\right| ds \\
\leq\ &\left|1 + \frac{M}{J}\right| \|g\| D \lim_{t \to 0+} \int_0^t R(t-s)\, ds = 0,
\end{aligned}$$

so

$$|x(0)| = \lim_{t\to 0+} |\mathcal{T}x(t)| = \left| a(0) + \lim_{t\to 0+} |g(t,x(t))| \int_0^t R(t-s)\,|a(s)|\,ds \right| = |a(0)| < \frac{D}{2}$$

and we may conclude that any fixed point x of $\mathcal{T}$ is a function starting at $x(0)$ with $|x(0)| = |a(0)| < D$ and due to $\mathcal{T}(G) \subset G^o$ it cannot leave G.

When the function g is bounded by a bound which is greater than or equal to one, then multiplying our equation by g, i.e.,

$$x(t) = a(t) + g(t,x(t)) \int_0^t A(t-s)\, f(s,x(s))\, ds$$

is equivalent to considering

$$x(t) = a(t) + \frac{g(t,x(t))}{\|g\|+1} \int_0^t A(t-s)\,(\|g\|+1)\, f(t,x(s))\, ds$$

or

$$x(t) = a(t) + g_0(t,x(t)) \int_0^t A(t-s)\, f_0(s,x(s))\, ds,$$

with $f_0(t,x) := (\|g\|+1)\, f(t,x)$ and $g_0(t,x) := \frac{g(t,x)}{\|g\|+1}$.

As the last equation may be seen as a perturbation (by $g_0(t,x(t))$) of the equation

$$x(t) = a(t) + \int_0^t A(t-s)\, f_0(s,x(s))\, ds, \tag{33}$$

alternatively, one may choose to perturb the transformed equation of (33) by multiplying by $g_0(t,x(t))$, thus considering

$$\begin{aligned} x(t) &= a(t) + g_0(t,x(t)) \int_0^t R(t-s)\, a(s)\, ds \\ &\quad + g_0(t,x(t)) \int_0^t R(t-s) \left[x(s) + \frac{f_0(s,x(s))}{J} \right] ds. \end{aligned}$$

Clearly, if f satisfies (29), then so does f_0 (with $K_0 = K(\|g\|+1)$), so the last Proposition is applicable and a fixed point region might be yielded.

Before closing the paper we cite three remarks. The first one concerns the assumptions on the kernel A which allow the kernel to have singularities as long as conditions (A1)-(A3) are satisfied. It is worth noticing that fractional kernels $(t-s)^q$ with $q \in (0,1)$ do satisfy these conditions, so our results do apply in this case. The second remark concerns L^1 kernels. As it has already been mentioned, in such a case the integral of the resolvent kernel is less than one. With this in hand one may see that the results in both Propositions of this section still hold while conditions (24) and $\|g\| < 1$ may be relaxed to

$$\|g\| \left(1 - \frac{m}{M}\right) = k \leq 1 \tag{34}$$

and $\|g\| \leq 1$, respectively, yet m is allowed to be zero. As a final remark, we would like to emphasize on the fact that under the conditions of this study, not only the fixed point regions "trap" all bounded (continuous) solutions of the perturbed equation, but they also yield that there do not exist any unbounded ones.

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