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Uniqueness of Solution to the von Karman Equations with Free Boundary Conditions

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Abstract: The purpose of this paper is to give some theoretical results, under weaker hypotheses imposed on the external, internal, linear potential loads and three measurable portions with non null area of the boundary of the shallow shell, for the local existence and uniqueness of solution to the stationary von Karman equations, with free-type boundary conditions of the elastic shallow shell. Finally, in some theoretical results, we describe an iterative method for constructing a unique weak solution for the problem.

Keywords: *static von Karman equations; free-type boundary; elastic shallow shell.*

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1 Introduction

In nonlinear three-dimensional elasticity theory, the stationary von Karman equations are two dimensional equations for the nonlinearly elastic shallow shell. The mathematical model is a modeling of the physical situation of buckling phenomenon of the elastic shallow shell, which is perturbed by the external and internal forces and potentially non conservative loads $L(.)$ applied to the system, see [3]. In case of free-type and mixed homogenous boundary conditions, we know the static von Karman equations for vertical displacement u of the middle surface of the reference configuration of the shell from a plane, and the Airy stress function ϕ has the form, see, for instance, [3].

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Find $(u,\phi) \in \left(H_0^2(\omega))^2\right)$ such that

$$(\mathbb{P}) \begin{cases} \Delta^2 u - [\phi + F_0, u + \theta] + L(u) = p(x) & in \quad \omega, \\ \Delta^2 \phi + [u, u + 2\theta] = 0 & on \quad \omega, \\ u = \partial_\nu u = 0 & on \quad \Gamma_0, \\ u = 0, \quad \Delta u + (1-\mu) B_1 u = 0 & on \quad \Gamma_1, \\ \Delta u + (1-\mu) B_1 u = 0\,,\ \partial_\nu(\Delta u) + (1-\mu) B_2 u - \vartheta u = 0 & on \quad \Gamma_2, \\ \phi = 0, \partial_\nu \phi = 0 & on \quad \Gamma. \end{cases}$$

Here ω is the middle surface of the initial configuration of the shell, the parameter μ is the Poisson ratio, $\vartheta \geq 0$ is a positive reel and $[u, v]$ is a von Karman bracket defined by [15]

$$[\phi, u] = \partial_{11}\phi\partial_{22}u + \partial_{11}u\partial_{22}\phi - 2\partial_{12}\phi\partial_{12}u. \tag{1}$$

The shell is subjected to the internal force F_0, which is a given function determined by the in-plane mechanical loads, and the shell is subjected also to the external force p, and $\theta(x, y)$, see [3, 7], is a mapping measuring the deviation of the middle surface of the reference configuration of the shell from a plane. This function determines the initial form of the shell and the case $\theta = 0$ corresponds to the plate theory.

In [3], I.Chueshov and I.Lasiecka studied the stationary and dynamic von Karman equations and established different theoretical results for generalized, strong and weak solutions under weaker hypotheses imposed at different loads, namely, for free-type boundary conditions the authors take the assumption $F_0 \in H^{\frac{5}{2}+\epsilon}(\omega)$, by using the theory of nonlinear semi-group. To justify the uniqueness, the authors used the limit definition of generalized solution along weak continuity of the nonlinear terms involving the Airy stress function and knowing the Lipschitz continuity of von Karman bracket with the Airy stress function. Moreover, in [4], P.G. Ciarlet and L. Gratie justified the generalized von Karman equations by means of a formal asymptotic analysis and established the existence of the system.

The aim of this paper is to find a condition verified by the internal and external loads, the linear bounded operator L and, also, three measurable portions Γ_0, Γ_1, Γ_2 with non null area of the boundary $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2$ of the shallow shell. Moreover, in this paper, for justifying some theoretical results, we take only the following weak assumption $F_0 \in H^2(\omega)$.

This paper will be organized as follows. After this introduction, Section 2 contains some basic results and tools that will be needed later. Section 3 is devoted to the description of the mathematical structure of the model under consideration by using an iterative method for establishing the existence and uniqueness of the weak solution associated to the static von Karman equations.

2 Preliminary Results and Needed Tools

In this paper, ω denotes a nonempty connected and bounded open domain in IR^2, with its boundary $\Gamma = \partial\omega$ of C^∞ -regularity. We assume that in this section $\Gamma =$

$\Gamma_0 \cup \Gamma_1 \cup \Gamma_2$, where Γ_0 , Γ_1 and Γ_2 are three measurable portions of Γ with non null area and $\Gamma_0 \cap \Gamma_1 \cap \Gamma_2 = \emptyset$.

Let us consider the following problem [3]. Find $(u, \phi) \in H^2(\omega) \times H_0^2(\omega)$ such that

$$(\mathbb{P}) \begin{cases} \Delta^2 u - [\phi + F_0, u + \theta] + L(u) = p(x) & in \quad \omega, \\ \Delta^2 \phi + [u, u + 2\theta] = 0 & on \quad \omega, \\ u = \partial_\nu u = 0 & on \quad \Gamma_0, \\ u = 0, \quad \Delta u + (1-\mu) B_1 u = 0 & on \quad \Gamma_1, \\ \Delta u + (1-\mu) B_1 u = 0 \, , \, \partial_\nu(\Delta u) + (1-\mu) B_2 u - \vartheta u = 0 & on \quad \Gamma_2, \\ \phi = 0, \partial_\nu \phi = 0 & on \quad \Gamma, \end{cases}$$

where $[u, v]$ is defined in (1) and

$$B_1 u = 2n_1 n_2 \partial_{12} u - n_1^2 \partial_{11} u - n_2^2 \partial_{22} u,$$

$$B_2 u = \partial_\tau \left[(n_1^2 - n_2^2) \partial_{12} u + n_1 n_2 (\partial_{22} u - \partial_{11} u) \right]$$

with $n = (n_1, n_2)$ being the outer normal to Γ and $\tau = (-n_2, n_1)$ being the unit tangent vector along Γ.

Let $p \geq 1$ and $m \in IN^*$, we denote

$$|u|_p = (\int_\omega |u|^p)^{1/p}, \quad \|u\| = \sum_{\alpha,\beta=1,2} |\partial_{\alpha\beta} u|_2 \quad \text{and} \quad \|u\|_0^2 = \int_\omega (\Delta u)^2$$

and $\|u\|_{m,\omega}$ is the classical norm in $H^m(\omega)$. For the sake of simplicity, we define

$$\mathbb{V} = \left\{ u \in H^2(\omega) / u = \partial_\nu u = 0 \; on \, \Gamma_0 \; and \; u = 0 \; on \; \Gamma_1 \right\},$$

which is a subspace of $H^2(\omega)$, and

$$a_0(u, v) = \int_\omega (\Delta u \Delta v - (1-\mu) \, [u, v]). \tag{2}$$

The following result is of interest.

Proposition 2.1 *Let Γ_0 and Γ_1 be two portions of Γ, if we do not choose the next two portions Γ_0 or Γ_1 of Γ in a linear segment, then the semi norm $\|.\|$ is a norm in $\mathbb{V}$ equivalent to the usual norm of $H^2(\omega)$.*

Proof. To establish that the semi-norm $\|.\|$ is a norm in the subspace $\mathbb{V}$, we show the following result:

$$\forall u \in \mathbb{V}; \quad \|u\| = \sum_{\alpha,\beta=1,2} |\partial_{\alpha\beta} u|_2 = 0 \;\Rightarrow\; u = 0.$$

Then, for $\forall u \in \mathbb{V}$, we have

$$\|u\| = 0 \;\Rightarrow\; \forall \alpha, \beta = 1, 2, \;\; \partial_{\alpha\beta} u = 0.$$

Now, by using a classical result from distribution theory [5], and since the set ω is connected, with $\forall \alpha, \beta = 1, 2,\ \ \partial_{\alpha\beta} u = 0$, we have that

$$\forall (x, y) \in \overline{\omega},\ \ \exists (a, b, c) \in IR^3 \ \text{ such that } \ u(x, y) = ax + by + c.$$

If Γ_0 or Γ_1 is not in a linear segment, then

$$u_{\Gamma_0} = (ax + by + c)|_{\Gamma_0} = 0 \ \text{ and } \ u_{\Gamma_1} = (ax + by + c)|_{\Gamma_1} = 0,$$

this implies that

$$\Gamma_0 \subset \left\{(x, y) \in IR^2 / ax + by + c = 0\right\}$$

or

$$\Gamma_1 \subset \left\{(x, y) \in IR^2 / ax + by + c = 0\right\},$$

that contradicts the assumption that one of two portions Γ_0 or Γ_1 is not in a linear segment, and we conclude that $a = b = c = 0$.

Now, if we have that two portions Γ_0 and Γ_1 are in linear segments, then

$$\Gamma_0 \subset \left\{(x, y) \in IR^2 / ax + by + c = 0\right\} \ \text{ and } \Gamma_1 \subset \left\{(x, y) \in IR^2 / ax + by + c = 0\right\}.$$

Since Γ_0 and Γ_1 are not in the identical linear segment, we deduce that

$$a = b = c = 0 \ \Rightarrow \ u = 0.$$

Finally, the semi-norm $\|.\|$ is a norm in $\mathbb{V}$.

Now we show that the subspace $\mathbb{V}$ is a Banach space in $H^2(\omega)$. Let $(u_n)_{n \geq 0}$ be the sequence elements in the space $\mathbb{V}$ such that $(u_n)_{n \geq 0}$ converge to u in $H^2(\omega)$.

Since the operator "trace" and ∂_ν are continuous, we have the sequences $(u_n)_{|_{\Gamma_0}}$, $(u_n)_{|_{\Gamma_1}}$ and $\partial_\nu (u_n)_{|_{\Gamma_0}}$ converge to $u_{|_{\Gamma_0}}$, $u_{|_{\Gamma_1}}$ and $\partial_\nu u_{|_{\Gamma_0}}$, then $u_{|_{\Gamma_0}} = u_{|_{\Gamma_1}} = 0$ and $\partial_\nu u_{|_{\Gamma_0}} = 0$. Hence $u \in \mathbb{V}$, then $\mathbb{V}$ is a closed subspace in $H^2(\omega)$.

Moreover, we prove that the norm $\|.\|$ in the space $\mathbb{V}$ is equivalent to the usual norm of $H^2(\omega)$.

The inequality $\|u\| \leq \|u\|_{2,\omega}$ clearly holds. But if we suppose that the other inequality is false, then there exists a sequence (u_n) in $\mathbb{V}$, such that

$$\forall n \in IN,\ \ \|u_n\|_{2,\omega} = 1 \text{ and } \lim_{n \to +\infty} \|u_n\| = 0. \tag{3}$$

For more detail, see [5].

So, u_n is bounded in the space $H^2(\omega)$. We use the compact injection $H^2(\omega) \hookrightarrow_c L^2(\omega)$, then there exists a subsequence u_m such that, with (3), we have u_m converges in the space $L^2(\omega)$ and also u_m converges to 0, with the norm $\|.\|$ in the space $\mathbb{V}$.

Finally, we conclude that u_m is a Cauchy sequence with the norm $(|.|_2^2 + \|.\|^2)^{1/2}$. In [5], the norm $(|.|_2^2 + \|.\|^2)^{1/2}$ is equivalent to the usual norm of $H^2(\omega)$, this implies that u_m converges to u in $\mathbb{V}$, therefore the limit u satisfies

$$\|u\| = \lim_{m \to +\infty} \|u_m\| = 0 \ \Rightarrow \ u = 0,$$

but this result contradicts the equality $\forall m \in IN$, $\|u_m\|_{2,\omega} = 1$ and the desired result is obtained.

Remark 2.1 *The norm* $\|.\|$ *is equivalent to the norm* $\|.\|_0$ *in the space* $\mathbb{V}$.

Proof. By the analogous method as in Proposition 2.1, we prove that

$$\forall u \in \mathbb{V},\ \ \exists \alpha > 0,\ \beta > 0;\ \ \alpha \|u\|_0 \leq \|u\|_{2,\omega} \leq \beta \|u\|_0 \,,$$

and, with the result of Proposition 2.1, we have

$$\forall u \in \mathbb{V},\ \ \exists \alpha_1 > 0,\ \beta_1 > 0;\ \ \alpha_1 \|u\| \leq \|u\|_{2,\omega} \leq \beta_1 \|u\| \,,$$

then

$$\forall u \in \mathbb{V}, \quad \frac{\alpha_1}{\beta} \|u\| \leq \|u\|_0 \leq \frac{\beta_1}{\alpha} \|u\| \,.$$

Finally, the desired result is verified.

We recall the following results, see [1, 3, 8, 10, 11] for instance.

Theorem 2.1 *Let* $u \in H^4(\omega)$, $v \in H^2(\omega)$ *and* $\mu \in IR$, *we have that, with* (2),

$$\int_\omega \Delta^2 uv = a_0(u,v) + \int_\Gamma \left[(\partial_\nu \Delta u + (1-\mu)B_2 u)v - (\Delta u + (1-\mu)B_1 u)\partial_\nu v\right].$$

Lemma 2.1 *The space* $\mathbb{V} \cap H^4(\omega)$ *is dense in the space* $\mathbb{V}$ *for the induct norm of* $H^4(\omega)$ *and for every* u *and* v *in* $\mathbb{V}$ *the equality*

$$\int_\omega \Delta^2 uv = a_0(u,v) + \int_\Gamma \left[(\partial \nu \Delta u + (1-\mu)B_2 u)v - (\Delta u + (1-\mu)B_1 u)\partial_\nu v\right]$$

holds.

Theorem 2.2 *Let* $f \in L^1(\omega)$, *then the following problem*

$$\begin{cases} \Delta^2 v = f & in \quad \omega, \\ v = 0 & on \quad \Gamma, \\ \partial_\nu v = 0 & on \quad \Gamma \end{cases}$$

has one and only one solution v *in* $H_0^2(\omega)$ *satisfying the relation*

$$\|v\|_0 \leq c_0 \, |f|_1 \,,$$

where $c_0 \succ 0$ *is a constant which depends only on* $mes(\omega)$.

We are now in a position to state the following result.

Theorem 2.3 *Let* $f \in L^1(\omega)$, *the following problem*

$$(\mathbb{Q}) \begin{cases} \Delta^2 u = f & in \quad \omega, \\ u = \partial_\nu u = 0 & on \quad \Gamma_0, \\ u = 0, \ \ \Delta u + (1-\mu)B_1 u = 0 & on \quad \Gamma_1, \\ \Delta u + (1-\mu)B_1 u = 0 & on \quad \Gamma_2, \\ \partial_\nu(\Delta u) + (1-\mu)B_2 u - \vartheta u = 0 & on \quad \Gamma_2, \end{cases}$$

has one and only one solution in $\mathbb{V}$ *such that*

$$\|u\| \leq c_{00}\, |f|_1\,,$$

where $c_{00} \succ 0$ *is a constant which depends only on* $mes(\omega)$.

Proof. By virtue of Lemma 2.1, for all $(u,v) \in \mathbb{V}^2$ we have

$$\int_\omega \Delta^2 uv = a_0(u,v) + \int_\Gamma [(\partial_\nu \Delta u + (1-\mu)B_2 u)v - (\Delta u + (1-\mu)B_1 u)\partial_\nu v]\,. \tag{4}$$

Since $v \in \mathbb{V}$, we have

$$v_{|_{\Gamma_0}} = 0,\ v_{|_{\Gamma_1}} = 0,\ \partial_\nu v_{|_{\Gamma_0}} = 0,\ (\partial\nu\Delta u + (1-\mu)B_2 u)_{|_{\Gamma_2}} = 0,$$

and

$$(\Delta u + (1-\mu)B_1 u)_{|_{\Gamma_1\cup\Gamma_2}} = 0,\ \int_{\Gamma_2} (\partial_\nu \Delta u + (1-\mu)B_2 u)v = \vartheta \int_{\Gamma_2} uv,$$

hence, with (4) we deduce that

$$a_0(u,v) + \vartheta \int_{\Gamma_2} uv = \int_\omega fv = l(v).$$

The mapping $a_0(.,.)$ is a bilinear, symmetric and continuous in the Hilbert space $\mathbb{V}$. Moreover, the linear operator $l(.)$ is also continuous.
So

$$\forall u \in \mathbb{V},\ \ a_0(u,u) = \int_\omega (\Delta u)^2 - (1-\mu)\int_\omega [u,u] = \|u\|_0^2 - (1-\mu)\int_\omega [u,u]$$

and

$$\int_\omega [u,u] = \int_\omega \left(2\partial_{11}u\partial_{22}u - 2(\partial_{12}u)^2\right) \leq \int_\omega 2\partial_{11}u\partial_{22}u.$$

Moreover,

$$\int_\omega (\Delta u)^2 = \int_\omega (\partial_{11}u + \partial_{22}u)^2 = \int_\omega (\partial_{11}u)^2 + (\partial_{22}u)^2 + 2\int_\omega (\partial_{11}u\partial_{22}u).$$

It follows that

$$\int_\omega [u,u] \leq \|u\|_0^2\,,$$

this implies that

$$a_0(u,u) = \int_\omega (\Delta u)^2 - (1-\mu)\int_\omega [u,u] \geq \|u\|_0^2 - (1-\mu)\,\|u\|_0^2 = \mu\,\|u\|_0^2\,.$$

Using Remark 2.1, we have

$$\exists \alpha > 0, \quad a_0(u,u) \geq \alpha\,\|u\|^2\,.$$

Then the map $a_0(.,.)$ is coercive.

It turns out that, by the Lax-Milgramme theorem, the following problem

$$\forall v \in \mathbb{V}, \quad a_0(u,v) + \vartheta \int_{\Gamma_2} uv = \int_\omega fv = l(v)$$

has one and only one solution in $\mathbb{V}$.

To prove completely the theorem we show that

$$\|u\| \leq c_0 \, |f|_1 \, .$$

Since u is a solution of the following problem

$$\forall v \in \mathbb{V}, \quad a_0(u,v) + \vartheta \int_{\Gamma_2} uv = \int_\omega fv = l(v),$$

and using the injection $H^2(\omega) \hookrightarrow C(\overline{\omega})$, we have

$$\forall u \in \mathbb{V}, \;\; \exists \beta > 0 \;\text{ such that }\; \|u\|_\infty \leq \beta \, \|u\| \, ,$$

with $a_0(u,u) + \vartheta \int_{\Gamma_1} u^2$ being coercive, then there exists $\alpha \succ 0$ such that

$$\alpha \, \|u\|^2 \leq a_0(u,u) + \vartheta \int_{\Gamma_2} u^2 = \int_\omega fu \leq \|u\|_\infty \, |f|_1 \leq \beta \, \|u\| \, |f|_1 \, .$$

Finally,

$$\|u\| \leq c_{00} \, |f|_1$$

with $c_{00} = \frac{\beta}{\alpha}$.

Now, let us put

$$F_1(u,\phi) = [\phi + F_0, u + \theta] - L(u). \tag{5}$$

Before giving our main result, we now state the following results.

Proposition 2.2 *Let $(u,v) \in (H_0^2(\omega))^2$, $\theta \in H^2(\omega)$ and $F_0 \in H^2(\omega)$ be with small norms. Let $\phi, \varphi \in H_0^2(\omega)$ be the solutions of the following two problems:*

$$\Delta^2 \phi = -\,[u,u] \quad \text{and} \;\; \Delta^2 \varphi = -\,[v,v] \, .$$

Then the following estimations

$$\Big| \, [u,\phi] - [v,\varphi] \, \Big|_2 \leq c_1 \|u - v\|$$

and

$$|F_1(u,\phi) - F_1(v,\varphi)|_1 \leq c_1 \, \|u - v\|$$

hold for some $0 < c_1 < 1$.

Proof. Following [3] and Proposition 2.1 we have

$$\Big| \, [u,\phi] - [v,\varphi] \, \Big|_2 \leq k\big(\, \|u\|^2 + \|v\|^2 \, \big) \, \|u - v\|$$

for some $k > 0$. Let $c > 0$ be small enough so that $\|u\| \leq c$ and $\|v\| \leq c$. We have

$$\Big| \, [u,\phi] - [v,\varphi] \, \Big|_2 \leq 2kc^2 \, \|u - v\| \, ,$$

and

$$\Big| \, [u,\phi] - [v,\varphi] \, \Big|_1 \leq k_1 \Big| \, [u,\phi] - [v,\varphi] \, \Big|_2 \leq 2kk_1 c^2 \, \|u - v\| \, .$$

Moreover, we have

$$\begin{aligned} \left|\,[u-v,F_0\,]\right|_1 &\leq (\textstyle\int_\omega |\partial_{11}(u-v)|\,|\partial_{22}F_0|) + (\textstyle\int_\omega |\partial_{22}(u-v)|\,|\partial_{11}F_0|) \\ &+2(\textstyle\int_\omega |\partial_{12}(u-v)|\,|\partial_{12}F_0|) \\ &\leq \|\partial_{22}F_0\|_2\,|\partial_{11}(u-v)|_2 + \|\partial_{11}F_0\|_2\,|\partial_{22}(u-v)|_2 \\ &+2\,\|\partial_{12}F_0\|_2\,|\partial_{12}(u-v)|_2 \\ &\leq 4c_2\,\|F_0\|_{2,\omega}\,\|u-v\|\,. \end{aligned}$$

Using the similar proof for the next inequality, with Proposition 2.1 and Theorem 2.2 we have

$$\begin{aligned} \left|\,[\phi-\varphi,\theta\,]\right|_1 &\leq 4c_2\,\|\theta\|_{2,\omega}\,\|\phi-\varphi\| \leq 4c_0c_2\,\|\theta\|_{2,\omega}\,|[u,u]-[v,v]|_1 \\ &\leq 4c_0c_2\,\|\theta\|_{2,\omega}\Big(\,|[u,u-v]|_1 + |[v,u-v]|_1\Big) \\ &\leq 16c_0c_2\,\|\theta\|_{2,\omega}\,(\|u\|+\|v\|)\,\|u-v\| \\ &\leq 32c_0c_2c\,\|\theta\|_{2,\omega}\,\|u-v\|\,, \end{aligned}$$

and so, with $c_3 = 3\max(4c_2c_0, 32cc_0c_2, 1)$

$$\begin{aligned} |F_1(u,\phi)-F_1(v,\varphi)|_1 &\leq \Big|\;[\phi+F_0,u+\theta\,]-\,[\varphi+F_0,v+\theta\,]\Big|_1 + |L(u-v)|_1 \\ &\leq \Big|\;[\phi,u\,]-\,[\varphi,v\,]\Big|_1 + \Big|\;[F_0,u-v\,]\Big|_1 + \Big|\;[\theta,\phi-\varphi\,]\Big|_1 \\ &+\|L\|\,\|u-v\| \\ &\leq \big(2kk_1c^2 + 4c_2c_0\,\|F_0\|_{2,\omega} + 32cc_0c_2\,\|\theta\|_{2,\omega} + \|L\|\,\big)\,\|u-v\| \\ &\leq \big(2kk_1c^2 + c_3(\|F_0\|_{2,\omega}+\|\theta\|_{2,\omega}+\|L\|)\big)\,\|u-v\|\,. \end{aligned}$$

If we choose

$$\|\theta\|_{2,\omega}+\|F_0\|_{2,\omega}+\|L\| < \frac{1}{c_3} \quad \textit{and} \quad 0<c<\sqrt{\frac{1-c_3(\|F_0\|_{2,\omega}+\|\theta\|_{2,\omega}+\|L\|)}{2c_0kk_1}},$$

we have

$$0 < c_1 = 2kk_1c^2 + c_3(\|F_0\|_{2,\omega}+\|\theta\|_{2,\omega}+\|L\|) < 1,$$

we then conclude the proof.

Remark 2.2 In the next result, from the mechanical point of view, our weaker assumptions concerning F_0 in $H_0^2(\omega)$ mean that no external stresses are applied to the shell [3]. F_0 is a given function determined by mechanical loads. For some of the results, less regularity on F_0 is required. For example, to prove the uniqueness of weak solution to the dynamic problem with free boundary conditions and, also, to the thermoelastic plates, some authors take F_0 in $H_0^{3+\epsilon}(\omega)$.

3 Iterative Approach: the Main Results

We will study the problem $(\mathbb{P})$ by considering the following iterative problem.

Let $n \geq 1$, $0 \neq u_0 \in \mathbb{V}$ be given. We first find $\phi_n \in H_0^2(\omega)$ as a solution of the equation $\Delta^2\phi_n = -[u_{n-1}, u_{n-1} + 2\theta]$ and u_n as a solution of the following problem:

$$(\mathbb{P}_n)\begin{cases} \Delta^2 u_n = F_1(u_{n-1}, \phi_n) + p & in \quad \omega, \\ u_n = \partial_\nu u_n = 0 & on \quad \Gamma_0, \\ u_n = 0, \ \Delta u_n + (1-\mu)B_1 u_n = 0 & on \quad \Gamma_1, \\ \Delta u_n + (1-\mu)B_1 u_n = 0\,,\ \partial_\nu(\Delta u_n) + (1-\mu)B_2 u_n - \vartheta u_n = 0 & on \quad \Gamma_2, \end{cases}$$

where F_1 is defined by (5).

We are now in a position to state our main result of this section.

Theorem 3.1 *Let $p \in L^2(\omega)$. If $|p|_2$, $\|\theta\|_{2,\omega}$, $\|L\|$ and $\|F_0\|_{2,\omega}$ are small, then the problem $(\mathbb{P})$ has one and only one solution $(u, \phi) \in \mathbb{V} \times H_0^2(\omega)$.*

Proof. We divide it into three steps.

Step 1: Let us consider the problem $(\mathbb{P}_n)$ with $u_0 \neq 0$. We will show that

$$\forall n \in IN, \ \ \|u_n\| \leq \|u_0\| \ \text{ and } \ \|\phi_{n+1}\| \leq \|u_0\|\,.$$

For $n = 0$, we have $\|u_0\| \leq \|u_0\|$. Otherwise, for ϕ_1 being the solution of the problem $\Delta^2\phi_1 = -[\,u_0, u_0 + 2\theta\,]$, Proposition 2.1 and Theorem 2.2 ensure that there exists $c_0 > 0$ such that

$$\|\phi_1\| \leq c_0 \left|[\,u_0, u_0 + 2\theta\,]\right|_1,$$

using the proof of Proposition 2.2 with

$$\|u_0\| < c, \quad 0 < c_0 k k_1 c < 1 \ \ and \ \ \|\theta\|_{2,\omega} \leq \frac{1 - c_0 k k_1 c}{8c_0},$$

we can deduce that

$$\|\phi_1\| \leq c_0 k k_1 \|u_0\|^2 + 8c_0 \|\theta\|_{2,\omega} \|u_0\| \leq (c_0 k k_1 c + 8c_0 \|\theta\|_{2,\omega}) \|u_0\| \leq \|u_0\|\,.$$

The desired inequalities are true for $n = 0$.

Suppose that

$$\forall k = 1, ..., n, \ \ \|u_k\| \leq \|u_0\| \ \text{ and } \ \|\phi_{k+1}\| \leq \|u_0\|\,.$$

Since u_{n+1} is a solution of the problem $(\mathbb{P}_{n+1})$, Theorem 2.3 yields that there exists $c_{00} \succ 0$, and by Proposition 2.2 we have that

$$\begin{aligned} \|u_{n+1}\| \leq \ & c_{00}(|F_1(u_n, \phi_{n+1})|_1 + |p|_1) \\ & \leq c_{00}(c_1 \|u_n\| + |p|_1) \\ & \leq c_{00}(c_1 \|u_0\| + |p|_1). \end{aligned}$$

If we choose $c \succ 0$ sufficiently small, such that

$$0 \prec c_1 \prec 1,\ \ 0 < c_{00}c_1 \prec 1 \text{ and } \ |p|_1 \leq \frac{(1-c_{00}c_1)}{c_{00}} \|u_0\|,$$

it follows that

$$\|u_{n+1}\| \leq \|u_0\|.$$

Moreover, for ϕ_{n+2} being the solution of the problem $\Delta^2\phi_{n+2} = -[\, u_{n+1}, u_{n+1} + 2\theta\,]$ and after the case $n = 0$, we have

$$\|u_0\| < c, \quad 0 < c_0kk_1c < 1 \ \ \textit{and} \ \ \|\theta\|_{2,\omega} \leq \frac{1-c_0kk_1c}{8c_0},$$

moreover, we can deduce that

$$\|\phi_{n+2}\| \leq c_0kk_1 \|u_{n+1}\|^2 + 8c_0 \|\theta\|_{2,\omega} \|u_{n+1}\| \leq (c_0kk_1c + 8c_0 \|\theta\|_{2,\omega}) \|u_{n+1}\| \leq \|u_0\|.$$

Hence,

$$\forall n \in IN, \ \ \|u_n\| \leq \|u_0\|, \ \text{ and } \ \|\phi_{n+1}\| \leq \|u_0\|.$$

Step 2: Let $m \prec n$, u_n (resp, u_m) be a solution of the problem $(\mathbb{P}_n)$ (resp, $(\mathbb{P}_m)$), then $u_n - u_m$ is a solution of the following problem :

$$\begin{cases} \Delta^2(u_n-u_m) = F_1(u_{n-1},\phi_{n-1}) - F_1(u_{m-1},\phi_{m-1}) & in \ \ \omega, \\ u_n - u_m = 0 \ \ \partial_\nu(u_n-u_m) = 0 & on \ \ \Gamma_0, \\ u_n - u_m = 0, \ \ \Delta(u_n-u_m) + (1-\mu)B_1(u_n-u_m) = 0 & on \ \ \Gamma_1, \\ \Delta(u_n-u_m) + (1-\mu)B_1(u_n-u_m) = 0 & on \ \ \Gamma_2, \\ \partial_\nu(\Delta(u_n-u_m) + (1-\mu)B_2(u_n-u_m) - \vartheta(u_n-u_m) = 0 & on \ \ \Gamma_2. \end{cases}$$

Using Theorem 2.3 again, we have

$$\begin{aligned} \|u_n - u_m\| &\leq c_{00} \left|F_1(u_{n-1},\phi_{n-1}) - F_1(u_{m-1},\phi_{m-1})\right|_1 \\ &\leq c_{00}c_1 \|u_{n-1} - u_{m-1}\| \\ &\leq (c_{00}c_1)^m \|u_{n-m+1} - u_0\| \\ &\leq (c_{00}c_1)^m \textstyle\sum_{k=0}^{n-m-1} (c_{00}c_1)^k \|u_1 - u_0\| \\ &\leq 2(c_{00}c_1)^m \textstyle\sum_{k=0}^{n-m-1} (c_{00}c_1)^k \|u_0\|. \end{aligned}$$

Moreover, for $\phi_n - \phi_m$ being the solution of the problem

$$\Delta^2(\phi_n - \phi_m) = -[\, u_{n+1}, u_{n+1} + 2\theta\,] + [\, u_{m+1}, u_{m+1} + 2\theta\,],$$

Theorem 2.2 ensures that there exists $c_0 > 0$ such that

$$\begin{aligned}
\|\phi_n - \phi_m\| &\leq c_0 \left|[\, u_{n+1}, u_{n+1} + 2\theta \,] - [\, u_{m+1}, u_{m+1} + 2\theta \,]\right|_1 \\
&\leq c_0 \Big(\left|[u_{n+1}, u_{n+1} - u_{m+1}]\right|_1 \\
&+ \left|[u_{m+1}, u_{n+1} - u_{m+1}]\right|_1 + \left|[2\theta, u_{n+1} - u_{m+1}]\right|_1 \Big) \\
&\leq 8c_0 k k_1 c \|u_{n+1} - u_{m+1}\| + 8c_0 \|\theta\|_{2,\omega} \|u_{n+1} - u_{m+1}\| \\
&\leq (8c_0 k k_1 c + 8c_0 \|\theta\|_{2,\omega}) \|u_{n+1} - u_{m+1}\| .
\end{aligned}$$

Using the proof of Proposition 2.2 and Theorem 2.2 with

$$\|u_0\| < c, \quad 0 < 8c_0 k k_1 c < 1 \ \textit{and} \ \ \|\theta\|_{2,\omega} \leq \frac{1 - 8c_0 k k_1 c}{8c_0},$$

we can deduce that

$$\|\phi_n - \phi_m\| \leq (8c_0 k k_1 c + 8c_0 \|\theta\|_{2,\omega}) \|u_{n+1} - u_{m+1}\| .$$

This implies that the sequence $(u_n, \phi_n)_{n\geq 0}$ is a Cauchy sequence in $\mathbb{V} \times H_0^2(\omega)$, hence the sequence $(u_n, \phi_n)_{n\geq 0}$ converges to (u, ϕ) in $\mathbb{V} \times H_0^2(\omega)$ and, with Proposition 2.2, we deduce that $F_1(u_n, \phi_{n+1}) + p$ converges to $F_1(u, \phi) + p$ in $L^1(\omega)$.

Since the operator "trace" and the operator "∂_ν" are continuous, we have that $(u_n)_{|\Gamma_0}$, $(u_n)_{|\Gamma_1}$, $(u_n)_{|\Gamma_2}$ and $\partial_\nu (u_n)_{|\Gamma_0}$ converge to $u_{|\Gamma_0}$, $u_{|\Gamma_1}$, $u_{|\Gamma_2}$ and $\partial_\nu u_{|\Gamma_0}$ and ϕ_n converges to ϕ on Γ.

Finally, we have that $u_{|\Gamma_0} = u_{|\Gamma_1} = 0$, $\partial_\nu u_{|\Gamma_0} = 0$, $\phi_{|\Gamma} = 0$ and $\partial_\nu \phi_{|\Gamma} = 0$.

To conclude that u is a solution of the problem $(\mathbb{P})$, we show that u satisfies the following equality:

$$(\Delta u + (1-\mu) B_1 u)|_{\Gamma_1 \cup \Gamma_2} = 0 \ \text{ and } \ (\partial_\nu \Delta u + (1-\mu) B_2 u - \vartheta u)|_{\Gamma_2} = 0.$$

By Lemma 2.1 we have for all $v \in \mathbb{V}$

$$\int_\omega \Delta^2 (u_n - u) v = a_0(u_n - u, v) + \int_\Gamma \big(\partial_\nu \Delta(u_n - u) + (1-\mu) B_2 (u_n - u)\big) v$$

$$- \int_\Gamma \big(\Delta(u_n - u) + (1-\mu) B_1 (u_n - u)\big) \partial_\nu v.$$

But u_n is a solution of the problem $(\mathbb{P}_n)$, it follows that

$$(\Delta u_n + (1-\mu) B_1 u_n)_{|\Gamma_1 \cup \Gamma_2} = 0 \text{ and } (\partial_\nu \Delta u_n + (1-\mu) B_2 u_n)_{|\Gamma_2} - \vartheta (u_n)_{|\Gamma_2} = 0, \quad (6)$$

or $v \in \mathbb{V}$ implies that $v_{|\Gamma_1 \cup \Gamma_2} = 0$ and $\partial_\nu v_{|\Gamma_0} = 0$, then

$$\int_\Gamma \big(\Delta(u_n - u) + (1-\mu) B_1 (u_n - u)\big) \partial_\nu v = \int_{\Gamma_0 \cup \Gamma_1 \cup \Gamma_2} \big(\Delta(u_n - u) + (1-\mu) B_1 (u_n - u)\big) \partial_\nu v,$$

$$= \int_{\Gamma_1\cup\Gamma_2} \big(\Delta(u_n-u)+(1-\mu)B_1(u_n-u)\big)\partial_\nu v.$$

This, together with (6), yield

$$\int_{\Gamma_1\cup\Gamma_2} \big(\Delta(u_n-u)+(1-\mu)B_1(u_n-u)\big)\partial_\nu v = -\int_{\Gamma_1\cup\Gamma_2} \big(\Delta u+(1-\mu)B_1 u\big)\partial_\nu v,$$

hence

$$\int_{\Gamma} \big(\Delta(u_n-u)+(1-\mu)B_1(u_n-u)\big)\partial_\nu v = -\int_{\Gamma_1\cup\Gamma_2} \big(\Delta u+(1-\mu)B_1 u\big)\partial_\nu v.$$

Moreover,

$$\begin{aligned}\int_{\Gamma} \big(\partial_\nu\Delta(u_n-u)+(1-\mu)B_2(u_n-u)\big)v &= \int_{\Gamma_0\cup\Gamma_1\cup\Gamma_2} \big(\partial_\nu\Delta(u_n-u)+(1-\mu)B_2(u_n-u)\big)v,\\ &= \int_{\Gamma_2} \big(\partial_\nu\Delta(u_n-u)+(1-\mu)B_2(u_n-u)\big)v.\end{aligned}$$

In view of (6), we deduce that

$$\int_{\Gamma} \big(\partial_\nu\Delta(u_n-u)+(1-\mu)B_2(u_n-u)\big)v = \vartheta\int_{\Gamma_2} u_n v - \int_{\Gamma_2} \big(\partial_\nu\Delta u+(1-\mu)B_2 u\big)v.$$

It follows that

$$\begin{aligned}\int_{\omega} \Delta^2(u_n-u)v = a_0(u_n-u,v) + \vartheta\int_{\Gamma_2} u_n v - \int_{\Gamma_2} \big(\partial_\nu\Delta u+(1-\mu)B_2 u\big)v\\ + \int_{\Gamma_1\cup\Gamma_2} \big(\Delta u+(1-\mu)B_1 u\big)\partial_\nu v.\end{aligned}$$

Now, letting $n\to+\infty$ in the next equality, we deduce that

$$\forall v\in\mathbb{V},\ \ \vartheta\int_{\Gamma_2} uv - \int_{\Gamma_2} \big(\partial_\nu\Delta u+(1-\mu)B_2 u\big)v + \int_{\Gamma_1\cup\Gamma_2} \big(\Delta u+(1-\mu)B_1 u\big)\partial_\nu v = 0.$$

This equality implies that

$$\forall v\in H_0^1(\omega)\cap\mathbb{V},\quad \int_{\Gamma_1\cup\Gamma_2} \big(\Delta u+(1-\mu)B_1 u\big)\partial_\nu v = 0,$$

it turns out that

$$\Delta u+(1-\mu)B_1 u = 0,\ \text{ on }\ \Gamma_1\cup\Gamma_2.$$

And also, we deduct

$$\forall v\in\mathbb{V},\ \ \vartheta\int_{\Gamma_2} uv - \int_{\Gamma_2} \big(\partial_\nu\Delta u+(1-\mu)B_2 u\big)v = 0,$$

it follows that

$$\partial_\nu\Delta u+(1-\mu)B_2 u-\vartheta u = 0,\ \text{ on }\ \Gamma_2.$$

Finally, (u, ϕ) is a solution of the static von Karman equations in $\mathbb{V} \times H_0^2(\omega)$.

Step 3 : For the uniqueness, we suppose that the problem $(\mathbb{P})$ has two solutions (u_1, ϕ_1) and (u_2, ϕ_2) in $\mathbb{V} \times H_0^2(\omega)$ such that

$$\|u_1\| \leq c \text{ and } \|u_2\| \leq c,$$

where, c is sufficiently small. Since $u_1 - u_2$ is a solution of the following problem:

$$\begin{cases} \Delta^2(u_1 - u_2) = F_1(u_1, \phi_1) - F_1(u_2, \phi_2) & in \quad \omega, \\ u_1 - u_2 = 0, \quad \partial_\nu(u_1 - u_2) = 0 & on \quad \Gamma_0, \\ u_1 - u_2 = 0, \quad \Delta(u_1 - u_2) + (1-\mu)B_1(u_1 - u_2) = 0 & on \quad \Gamma_1, \\ \Delta(u_1 - u_2) + (1-\mu)B_1(u_1 - u_2) = 0 & on \quad \Gamma_2, \\ \partial_\nu(\Delta(u_1 - u_2) + (1-\mu)B_2(u_1 - u_2) - \vartheta(u_1 - u_2) = 0 & on \quad \Gamma_2. \end{cases}$$

Theorem 2.3 implies that there exists $c_{00} \succ 0$ such that

$$\|u_1 - u_2\| \quad \leq c_{00} \left|F_1(u_1, \phi_1) - F_1(u_2, \phi_2)\right|_1 \leq c_{00} c_1 \|u_1 - u_2\|,$$

c is small, thus $0 < c_{00} c_1 < 1$, then $u_1 = u_2$ and $\phi_1 = \phi_2$.

Lastly, the stationary von Karman equations have one and only one solution (u, ϕ) in the space $\mathbb{V} \times H_0^2(\omega)$.

Remark 3.1 In this section we described an iterative method for constructing a unique weak solution, this technique is a good tool for illustrating this weak solution from the numerical point of view.

4 Conclusion

In this paper, we described an iterative method for constructing a unique weak solution to the model with free boundary conditions of buckling and flexible phenomenon of small nonlinear vibrations of the homogenous, isotropic and elastic thin shells of uniform thickness. Our approach is a good tool for justifying the theoretical results under the following weak assumption $F_0 \in H^2(\omega)$. Similar study for the models of dynamic von Karman equations with and without rotational inertia and for free boundary conditions of the shell could be the purpose for future research.

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