Nonlinear Dynamics and Systems Theory, **20** (4) (2020) 451–460

General Simplex Method for Fully Fuzzy Linear Programming with the Piecewise Linear Fuzzy Number

M. Z. Tuffaha and M. H. Alrefaei *

Department of Mathematics and Statistics, Jordan University of Science and Technology, Irbid, Jordan

Received: October 25 2019; Revised: September 12, 2020

Abstract: In this paper, we consider the fully fuzzy linear programming problem at which all the attributes and variables of the problem are fuzzy numbers represented by a piecewise linear fuzzy number. This type of fuzzy numbers is used due to its importance as a generalization of some other types of fuzzy numbers. We propose a fuzzy version of the simplex method to solve the problem, which is shown to be a generalization of the conventional simplex method. We represent the simplex method in a tabulated form and discuss whether a final solution exists, the problem is infeasible or it is unbounded. Finally, it is shown that the proposed method is more realistic than some of the existing methods.

Keywords: *piecewise linear fuzzy number; linear programming; fully fuzzy linear programming; polygonal fuzzy number.*

Mathematics Subject Classification (2010): 90C70.

1 Introduction

Linear programming has been an important mathematical tool to solve real life problems for a long time. If some of the data in a linear programming problem are vague, i.e., not precise due to unclear boundaries, then these data are usually represented by fuzzy numbers. This fuzzy representation of the data gives a more realistic manipulation of the problem under consideration since many real life problems contain fuzzy expressions such as "approximately", "almost" or "about". Ignoring such expressions and representing the data as crisp (unfuzzy) numbers cost losing some information about the resources, costs or variables. Many applications of fuzziness can be found in different mathematical fields [1, 9]. The literature is rich of applications of fuzzy linear programming problems see, for example, [2,4,6]. A more general case is to have a fully fuzzy linear programming

* Corresponding author: mailto:alrefaei@just.edu.jo

(FFLP) problem, where all the attributes and the variables in the problem are fuzzy [5,14].

Different solution approaches for FFLP problems can be found in the literature. For instance, some authors convert the problem into one or more crisp linear programming (CLP) problems with one or more objectives using some ranking method to get rid of the fuzziness, then solve the new problems using the known methods for solving conventional single or multi-objective linear programming problems [7,14]. On the other hand, some researchers prefer to solve the problem directly without converting it into another type of problem. This is usually done using a fuzzy version of the simplex method [5,11].

Many types of fuzzy numbers were used in the literature to represent this fuzziness, namely, the triangular fuzzy number which was used by Ozkok et al. [14], the trapezoidal fuzzy number which was considered by Das et al. [7] and by Kumar & Kaur [12], and other types of fuzzy numbers used widely in the literature. A more general type of fuzzy numbers is the n-polygonal fuzzy number, which generalizes the triangular and the trapezoidal fuzzy numbers. It has been gaining a great interest recently, especially in neural networks [13,17].

In a previous work [15], Tuffaha and Alrefaie studied the piecewise linear fuzzy number of order n (PLFN-n), which is an n-polygonal fuzzy number with equidistant knots. Convenient arithmetic operations were proposed on the PLFN-n in [15] and shown to satisfy the most important properties such as commutativity, associativity, having an identity and preserving the ranking value. Moreover, the operations were shown to give a generalization of the conventional binary operations on the real numbers. Later on, Tuffaha and Alrefaei [16] showed extra properties for the arithmetic operations. These new definitions were used for solving a fully fuzzy transportation problem (FFTP) [3]. In this paper, we consider a fully fuzzy linear programming problem and represent the fuzziness by the PLFN-n for the first time. A generalization of the known simplex method is proposed using the arithmetic operations given in [15].

The paper is organized as follows. In Section 2, we give some preliminaries needed throughout the paper. After that, in Section 3, the fully fuzzy linear programming problem is constructed and the solution method is proposed. The advantages of the proposed method are shown in Section 4, and some concluding remarks are given in Section 5.

2 Preliminaries

In this section, we present the definition of PLFN-n and the binary operations on PLFN-n's. We also give some definitions to clarify some concepts related to the PLFN-n's, such as the fuzzy matrices and the maximum and minimum of a set of fuzzy values or fuzzy-valued functions.

The following definitions are about the piecewise linear fuzzy number that is presented in [15].

Definition 2.1 A fuzzy set $\tilde{A}$ is called a **Piecewise Linear Fuzzy Number of Order** n (PLFN-n) if its membership function is given by

$$f_{\tilde{A}}(x) = \begin{cases} \frac{1}{n}[\frac{x-p_i}{p_{i+1}-p_i}] + \frac{i}{n} & ; p_i \leq x \leq p_{i+1}, \quad i = 0,..,n-1 \\ 1 & ; p_n \leq x \leq q_0 \\ \frac{-1}{n}[\frac{x-q_i}{q_{i+1}-q_i}] + \frac{n-i}{n} & ; q_i \leq x \leq q_{i+1}, \quad i = 0,..,n-1 \\ 0 & otherwise. \end{cases}$$

PLFN-n is represented by its knots: $(p_0, p_1, .., p_n; q_0, q_1, .., q_n)$. The family of all PLFN-n's is denoted by $\mathcal{PL}_n$. Moreover, a crisp (unfuzzy) real number c can be represented in the PLFN-n form as $c = (c, c, .., c; c, c, .., c)$.

Definition 2.2 Let $(p_0, p_1, .., p_n; q_0, q_1, .., q_n)$ be a PLFN-n. Then its ranking value is given by

$$\Re(\tilde{P}) = \frac{1}{4n}[p_0 + 2p_1 + 2p_2 + ... + 2p_{n-1} + p_n + q_0 + 2q_1 + 2q_2 + ... + 2q_{n-1} + q_n].$$

Definition 2.3 Let $\tilde{P} = (p_0, p_1, .., p_n; q_0, q_1, .., q_n), \tilde{Q} = (r_0, r_1, .., r_n; s_0, s_1, .., s_n) \in \mathcal{PL}_n$. The **addition** of $\tilde{P}$ and $\tilde{Q}$ is defined as follows:

$$\tilde{P} \oplus \tilde{Q} = (p_0 + r_0, p_1 + r_1, .., p_n + r_n; q_0 + s_0, q_1 + s_1, .., q_n + s_n).$$

Moreover, the **multiplication** of $\tilde{P}$ and $\tilde{Q}$ is $\tilde{P} \otimes \tilde{Q} = (t_0, t_1, .., t_n; u_0, u_1, .., u_n)$, where

$$\begin{aligned}
u_n &= \frac{1}{4n}[I + \sum_{i=1}^{n}(2i-1)X_i + 2nX_{n+1} + \sum_{i=1}^{n}(2(n+i)-1)X_{n+1+i}] \\
u_{i-1} &= u_i - X_{n+1+i},\ for\ i = n, n-1, .., 1 \\
t_n &= u_0 - X_{n+1} \\
t_{i-1} &= t_i - X_i,\ for\ i = n, n-1, .., 1, \\
and\ \ I &= \frac{1}{4n}[(p_0 + 2p_1 + .. + 2p_{n-1} + p_n + q_0 + 2q_1 + .. + 2q_{n-1} + q_n)* \\
&\qquad (r_0 + 2r_1 + .. + 2r_{n-1} + r_n + s_0 + 2s_1 + .. + 2s_{n-1} + s_n)] \\
X_i &= (p_i - p_{i-1}) + (r_i - r_{i-1}),\ for\ i = n, n-1, .., 1 \\
X_{n+1} &= (q_0 - p_n) + (s_0 - r_n) \\
X_{n+1+i} &= (q_i - q_{i-1}) + (s_i - s_{i-1}),\ for\ i = n, n-1, .., 1.
\end{aligned}$$

This definition guarantees the preservation of the most common properties of PLFN-n's.

Definition 2.4 Let $\tilde{P} = (p_0, p_1, .., p_n; q_0, q_1, .., q_n) \in \mathcal{PL}_n$. If $\Re(\tilde{P}) \neq 0$, then the **multiplicative inverse** of $\tilde{P}$, in the sense that $\Re(\tilde{P} \otimes \tilde{P}^{-1}) = 1$, is defined to be $\tilde{P}^{-1} = (t_0, t_1, .., t_n; u_0, u_1, .., u_n)$, where

$$\begin{aligned}
t_0 &= \frac{1}{\Re(\tilde{P})} + \frac{1}{4n}(p_0 + 2p_1 + .. + 2p_{n-1} + p_n + q_0 + 2q_1 + .. + 2q_{n-1} - (4n-1)q_n) \\
t_i &= t_{i-1} + (q_{n-i+1} - q_{n-i})\ for\ all\ i = 1, .., n \\
u_0 &= t_n + (q_0 - p_n) \\
u_i &= u_{i-1} + (p_{n-i+1} - p_{n-i})\ for\ all\ i = 1, .., n.
\end{aligned}$$

The following definitions are about the equalities, inequalities and matrices with PLFN-n from [16].

Definition 2.5 Let $\tilde{a}$ and $\tilde{b}$ be two PLFN-n's. Then

- $\tilde{a}$ and $\tilde{b}$ are **equivalent**, denoted $\tilde{a} \approx \tilde{b}$, if $\Re(\tilde{a}) = \Re(\tilde{b})$.

- $\tilde{a}$ is **greater** than $\tilde{b}$, denoted $\tilde{a} \succeq \tilde{b}$, if $\Re(\tilde{a}) \geq \Re(\tilde{b})$.
- $\tilde{a}$ is **smaller** than $\tilde{b}$, denoted $\tilde{a} \preceq \tilde{b}$, if $\Re(\tilde{a}) \leq \Re(\tilde{b})$.

Definition 2.6

- A matrix whose entries are PLFN-n's is called a *piecewise linear fuzzy matrix* $\tilde{\mathbf{M}}$.
- The set of all piecewise linear fuzzy matrices is denoted by $\mathcal{M}(\mathcal{PL}_n)$.
- The addition and multiplication of piecewise linear fuzzy matrices are similar to those of real matrices, but using the binary operations given in Definition 2.3 on $\mathcal{PL}_n$.

Definition 2.7 Let $\tilde{\mathbf{A}} = [\tilde{a}_{ij}]_{m\times k}, \tilde{\mathbf{B}} = [\tilde{b}_{ij}]_{m\times k} \in \mathcal{M}(\mathcal{PL}_n)$. Then

1. $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{B}}$ are *equal* or *equivalent*, written $\tilde{\mathbf{A}} = \tilde{\mathbf{B}}$ or $\tilde{\mathbf{A}} \approx \tilde{\mathbf{B}}$, if their corresponding entries are equal or equivalent, respectively.
2. A set of rows of $\tilde{\mathbf{A}}$, $\{\tilde{\mathbf{a}}_{i_1}, \tilde{\mathbf{a}}_{i_2}, ..., \tilde{\mathbf{a}}_{i_p}\}$, are *linearly independent* if the equation $(c_1 \otimes \tilde{\mathbf{a}}_{i_1}) \oplus (c_2 \otimes \tilde{\mathbf{a}}_{i_2}) \oplus ... \oplus (c_p \otimes \tilde{\mathbf{a}}_{i_p}) \approx \tilde{\mathbf{0}}$ with $c_1, c_2, ..., c_p \in \mathbb{R}$ can only be satisfied by $c_i = 0$ for all $i = 1, .., p$.
3. The *rank* of $\tilde{\mathbf{A}}$ is the maximal number of linearly independent rows of $\tilde{\mathbf{A}}$.
4. If $m = k$, then $\tilde{\mathbf{A}}$ is a *square fuzzy matrix*, and we define the *determinant* of $\tilde{\mathbf{A}}$, denoted $det(\tilde{\mathbf{A}})$, to be a PLFN-n computed in a similar way to how we compute the determinant of a real square matrix, but using the binary operations given in Definition 2.3 on $\mathcal{PL}_n$. Furthermore, if $det(\tilde{\mathbf{A}}) \not\approx 0$, then the *inverse matrix* $\tilde{\mathbf{A}}^{-1}$ can also be found by similar techniques to finding the inverse matrix of a real matrix, but here $\tilde{\mathbf{A}} \otimes \tilde{\mathbf{A}}^{-1} \approx \tilde{\mathbf{I}}$, where the square matrix $\tilde{\mathbf{I}}$ is a *fuzzy identity matrix* in $\mathcal{M}(\mathcal{PL}_n)$ whose entries are equivalent to zero, except for the entries in the main diagonal which are equivalent to one.

It is known that a linear programming problem seeks the maximum or minimum of a function subject to given constraints. The following definitions present the definition of maximum or minimum of a set of fuzzy values or fuzzy-valued functions.

Definition 2.8 Let I be an arbitrary index set, and let $\tilde{S} = \{\tilde{a}_i : i \in I\}$ be a set of PLFN-n's. We define the *maximum* and *minimum fuzzy value* of the elements of $\tilde{S}$, denoted $max(\tilde{S})$ and $min(\tilde{S})$, to be the elements of $\tilde{S}$ with the maximum and minimum ranking values, respectively. In other words, if $h_1 = max\{\Re(\tilde{a}_k) : \tilde{a}_k \in \tilde{S}\}$ and $h_2 = min\{\Re(\tilde{a}_k) : \tilde{a}_k \in \tilde{S}\}$, then

$$max(\tilde{S}) = \{\tilde{a}_i \in \tilde{S} : \Re(\tilde{a}_i) = h_1\}, \quad min(\tilde{S}) = \{\tilde{a}_i \in \tilde{S} : \Re(\tilde{a}_i) = h_2\}.$$

Note that $max(\tilde{S})$ and $min(\tilde{S})$ may have more than one element of $\tilde{S}$ if it contains more than one PLFN-n with the maximum or minimum ranking value.

Definition 2.9 Let $\tilde{f} : (\mathcal{PL}_n)^k \to \mathcal{PL}_n$, where $k \in \mathbb{N}$, be a fuzzy-valued function. Then the *maximum* and *minimum* of $\tilde{f}$ are defined by

$$max(\tilde{f}) = max\{\tilde{f}(\tilde{\mathbf{x}}) : \tilde{\mathbf{x}} \in (\mathcal{PL}_n)^k\}, \quad min(\tilde{f}) = min\{\tilde{f}(\tilde{\mathbf{x}}) : \tilde{\mathbf{x}} \in (\mathcal{PL}_n)^k\}.$$

Remark 2.1 Note that $max(\tilde{f}) = -min(-\tilde{f})$.

3 Fully Fuzzy Linear Programming with the Piecewise Linear Fuzzy Number

In this section, we construct the FFLP problem with PLFN-n and propose a solution method. We generalize the simplex algorithm in order to solve a FFLP problem with PLFN-n's of the form

$$\begin{aligned} min \quad & \tilde{z} = \tilde{\mathbf{c}}\tilde{\mathbf{x}} \\ s.t. \quad & \tilde{\mathbf{A}}\tilde{\mathbf{x}} \approx \tilde{\mathbf{b}} \\ & \tilde{\mathbf{x}} \succeq \tilde{\mathbf{0}}, \end{aligned} \tag{1}$$

where $\tilde{\mathbf{c}} = [\tilde{c}_j]_{1\times l}$, $\tilde{\mathbf{x}} = [\tilde{x}_j]_{l\times 1}$, $\tilde{\mathbf{A}} = [\tilde{a}_{ij}]_{m\times l}$, $\tilde{\mathbf{b}} = [\tilde{b}_i]_{m\times 1}$ are fuzzy matrices with PLFN-n's. Moreover, $\tilde{b}_i \succeq \tilde{0}$ for all $i = 1, .., m$, and the matrix $\tilde{\mathbf{A}}$ is with rank m.

3.1 Basic feasible solutions

After possibly rearranging the columns $\tilde{\mathbf{a}}_{\mathbf{j}}$ of $\tilde{\mathbf{A}}$, let $\tilde{\mathbf{A}} = [\tilde{\mathbf{B}}\tilde{\mathbf{N}}]$, where $\tilde{\mathbf{B}}$ is an $m \times m$ invertible matrix consisting of m columns of $\tilde{\mathbf{a}}_{\mathbf{j}}$, and $\tilde{\mathbf{N}}$ is an $m \times (l-m)$ matrix with the rest of the columns. Then the constrains can be written as $[\tilde{\mathbf{B}}\ \ \tilde{\mathbf{N}}]\tilde{\mathbf{x}} \approx \tilde{\mathbf{b}}$. The variables vector can then be split as follows: $[\tilde{\mathbf{B}}\ \ \tilde{\mathbf{N}}]\begin{bmatrix}\tilde{\mathbf{x}}_B\\ \tilde{\mathbf{x}}_N\end{bmatrix} \approx \tilde{\mathbf{b}}$, which gives $\tilde{\mathbf{B}}\tilde{\mathbf{x}}_B \oplus \tilde{\mathbf{N}}\tilde{\mathbf{x}}_N \approx \tilde{\mathbf{b}}$ or

$$\tilde{\mathbf{x}}_B \oplus \tilde{\mathbf{B}}^{-1}\tilde{\mathbf{N}}\tilde{\mathbf{x}}_N \approx \tilde{\mathbf{B}}^{-1}\tilde{\mathbf{b}}. \tag{2}$$

One solution is $\tilde{\mathbf{x}} = \begin{bmatrix}\tilde{\mathbf{x}}_B\\ \tilde{\mathbf{x}}_N\end{bmatrix} = \begin{bmatrix}\tilde{\mathbf{B}}^{-1}\tilde{\mathbf{b}}\\ \tilde{\mathbf{0}}\end{bmatrix}$, which is called a *basic solution*. $\tilde{\mathbf{B}}$ is called the *basis*, and the components of $\tilde{\mathbf{x}}_B$ are called the *basic variables*. If $\tilde{\mathbf{x}}_B \succeq \tilde{\mathbf{0}}$, then $\tilde{\mathbf{x}}$ is called a *basic feasible solution (b.f.s.)*.

3.2 The fuzzy simplex method

Assume problem (1) has a basic feasible solution $\tilde{\mathbf{x}}' = \begin{bmatrix}\tilde{\mathbf{x}}_B\\ \tilde{\mathbf{x}}_N\end{bmatrix} = \begin{bmatrix}\tilde{\mathbf{B}}^{-1}\tilde{\mathbf{b}}\\ \tilde{\mathbf{0}}\end{bmatrix}$, whose objective value is given by

$$\tilde{z}_0 = \tilde{\mathbf{c}}\begin{bmatrix}\tilde{\mathbf{B}}^{-1}\tilde{\mathbf{b}},\\ \tilde{\mathbf{0}}\end{bmatrix} = [\tilde{\mathbf{c}}_B\ \ \ \tilde{\mathbf{c}}_N]\begin{bmatrix}\tilde{\mathbf{B}}^{-1}\tilde{\mathbf{b}}\\ \tilde{\mathbf{0}}\end{bmatrix} = \tilde{\mathbf{c}}_B\tilde{\mathbf{B}}^{-1}\tilde{\mathbf{b}}. \tag{3}$$

The objective function in augmented form is

$$\tilde{z} = \begin{bmatrix}\tilde{\mathbf{c}}_B & \tilde{\mathbf{c}}_N\end{bmatrix}\begin{bmatrix}\tilde{\mathbf{x}}_B\\ \tilde{\mathbf{x}}_N\end{bmatrix} = \tilde{\mathbf{c}}_B\tilde{\mathbf{x}}_B \oplus \tilde{\mathbf{c}}_N\tilde{\mathbf{x}}_N. \tag{4}$$

From (2), we have

$$\tilde{\mathbf{x}}_B \approx \tilde{\mathbf{B}}^{-1}\tilde{\mathbf{b}} \ominus \tilde{\mathbf{B}}^{-1}\tilde{\mathbf{N}}\tilde{\mathbf{x}}_N. \tag{5}$$

Substituting (5) in (4) and simplifying give

$$\tilde{z} \oplus (\tilde{\mathbf{c}}_B\tilde{\mathbf{B}}^{-1}\tilde{\mathbf{N}} \ominus \tilde{\mathbf{c}}_N)\tilde{\mathbf{x}}_N \approx \tilde{\mathbf{c}}_B\tilde{\mathbf{B}}^{-1}\tilde{\mathbf{b}}.$$

Denote $\tilde{\mathbf{z}}_N = \tilde{\mathbf{c}}_B\tilde{\mathbf{B}}^{-1}\tilde{\mathbf{N}}$, then

$$\tilde{z} \oplus (\tilde{\mathbf{z}}_N \ominus \tilde{\mathbf{c}}_N)\tilde{\mathbf{x}}_N \approx \tilde{\mathbf{c}}_B\tilde{\mathbf{B}}^{-1}\tilde{\mathbf{b}}. \tag{6}$$

From equations (6) and (2), and putting $\underline{\tilde{\mathbf{b}}} = \tilde{\mathbf{B}}^{-1}\tilde{\mathbf{b}}$, the current b.f.s. can be represented in tabular form as

	$\tilde{\mathbf{x}}_B$	$\tilde{\mathbf{x}}_N$	RHS
$\tilde{z}$	$\tilde{\mathbf{0}}$	$\tilde{\mathbf{z}}_N \ominus \tilde{\mathbf{c}}_N$	$\tilde{\mathbf{c}}_B\tilde{\mathbf{B}}^{-1}\tilde{\mathbf{b}}$
$\tilde{\mathbf{x}}_B$	$\tilde{\mathbf{I}}$	$\tilde{\mathbf{B}}^{-1}\tilde{\mathbf{N}}$	$\underline{\tilde{\mathbf{b}}}$

We assume the absence of *degeneracy*, i.e., we consider that $\underline{\tilde{b}} \succ 0$. The case of degeneracy, where $\underline{\tilde{b}}$ has zero values, is known to cause some problems and needs a special discussion that will be cited later.

Let J be the current set of indices of the non-basic variables, then $\tilde{z}_j \ominus \tilde{c}_j$, where $j \in J$ are the elements of $\tilde{\mathbf{z}}_N \ominus \tilde{\mathbf{c}}_N$. Now, from equation (6) we have

$$\tilde{z} \approx \tilde{z}_0 \ominus \sum_{j \in J} (\tilde{z}_j \ominus \tilde{c}_j)\tilde{x}_j. \tag{7}$$

If $\tilde{z}_j \ominus \tilde{c}_j \prec 0$ for all $j \in J$, then the current solution cannot be improved anymore, and it is optimal. On the other hand, if $\tilde{z}_j \ominus \tilde{c}_j \preceq 0$ for all $j \in J$, and $\tilde{z}_k \ominus \tilde{c}_k \approx 0$ for some $k \in J$, then increasing the value of $\tilde{x}_k$ does not affect the objective value, which means that we have alternative optimal solutions with the same objective value. However, such case is not treated differently than the previous case in this paper. In other words, even if we have alternative optimal solutions, we will take only one of them into consideration.

Finally, if there exists $\tilde{z}_k \ominus \tilde{c}_k \succ 0$ for some $k \in J$, then $\tilde{x}_k$ enters the basis because this improves the objective value and one basic solution leaves the basis. To determine the leaving variable, we use the ratio test in order to maintain feasibility, i.e., keep all basic variables non negative. In order to maintain the nonnegativity of the variables, $\tilde{x}_k$ is increased until the first point at which some basic variable $\tilde{x}_{B_r}$ drops to zero. In fact, we can increase $\tilde{x}_k$ until

$$\tilde{x}_k = \frac{\underline{\tilde{b}}_r}{\tilde{y}_{rk}} = min\left\{ \frac{\underline{\tilde{b}}_i}{\tilde{y}_{ik}};\ \tilde{y}_{ik} \succ 0,\ i = 1, .., m \right\}, \tag{8}$$

and then $\tilde{x}_{B_r}$ leaves the basis and we call it the *blocking variable*, and (8) is called the *minimum ratio*. In fact, the only purpose of finding the minimum ratio is to determine the blocking variable. However, we can use the ranking function to facilitate the calculations, and the following *ranked minimum ratio* is enough to achieve the purpose:

$$\frac{\Re[\underline{\tilde{b}}_r]}{\Re[\tilde{y}_{rk}]} = min\left\{ \frac{\Re[\underline{\tilde{b}}_i]}{\Re[\tilde{y}_{ik}]};\ \tilde{y}_{ik} \succ 0,\ i = 1, .., m \right\}. \tag{9}$$

In tabular format, we can change the basis using the elementary row operations, which are known to maintain an equivalent problem, such that $\tilde{x}_k$ enters the basis and $\tilde{x}_{B_r}$ leaves it.

If $\tilde{\mathbf{y}}_k \preceq 0$, i.e., the ranking values of all its elements are less than or equal to zero. Then there is no blocking variable, and the value of $\tilde{x}_k$ can be increased indefinitely giving always a better objective value without violating any of the constraints. Thus, the problem is *unbounded* and the vector $\tilde{\mathbf{d}} = \begin{bmatrix} -\tilde{\mathbf{y}}_k \\ \tilde{\mathbf{e}}_k \end{bmatrix}$ is the *direction of unboundedness.* To illustrate, we give two examples, one has optimal solution and the other is unbounded.

Example 3.1 Consider the following FFLP problem:

$$\begin{aligned} min\ \ & (1,2,3;3,4,5)\otimes\tilde{x}_1\oplus(-7,-4.5,-3;-1,1,2)\otimes\tilde{x}_2\\ s.t.\ \ & (-4,-2,-1;2,3,5)\otimes\tilde{x}_1\oplus(2,3,5;6,6.5,8)\otimes\tilde{x}_2\preceq(-1,0,1;1,3.5,4)\\ & (1,3,4;4,5,7)\otimes\tilde{x}_1\oplus(-3,-2.5,-2;-1,0,3)\otimes\tilde{x}_2\preceq(5,6,7;9,10,11)\\ & \tilde{x}_1,\tilde{x}_2\succeq\tilde{0}. \end{aligned} \tag{10}$$

Adding the slack variables gives the following first simplex table:

	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{y}_1$	$\tilde{y}_2$	RHS
$\tilde{z}$	$(-5,-4,-3;-3,-2,-1)$	$(-2,-1,1;3,4.5,7)$	0	0	0
RV	-3	2			
$\tilde{y}_1$	$(-4,-2,-1;2,3,5)$	$(2,3,5;6,6.5,8)$	1	0	$(-1,0,1;1,3.5,4)$
RV	0.5	$\rhd 5 \lhd$			1.5
$\tilde{y}_2$	$(1,3,4;4,5,7)$	$(-3,-2.5,-2;-1,0,3)$	0	1	$(5,6,7;9,10,11)$
RV	4	-1			8

where the ranking value (RV) of each fuzzy number is written below it.
$\tilde{z}_k \ominus \tilde{c}_k = max\{(-5,-4,-3;-3,-2,-1),(-2,-1,1;3,4.5,7),0,0\} = (-2,-1,1;3,4.5,7) \succ 0$, thus the current solution is not optimal. From the ranked minimum ratio test (9), we find
$\frac{\Re[\tilde{b}_r]}{\Re[\tilde{y}_{rk}]} = \Re[(-1,0,1;1,3.5,4)]\otimes\Re[(2,3,5;6,6.5,8)^{-1}] = 1.5*0.2 = 0.3$, so we pivot at $(2,3,5;6,6.5,8)$ by performing the elementary row operations

$$\begin{aligned} R_1 \leftarrow & (2,3,5;6,6.5,8)^{-1}\otimes R_1\\ R_0 \leftarrow & (-(-2,-1,1;3,4.5,7)\otimes R_1)\oplus R_0\\ R_2 \leftarrow & (-(-3,-2.5,-2;-1,0,3)\otimes R_1)\oplus R_2. \end{aligned}$$

This gives the second simplex table:

	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{y}_1$	$\tilde{y}_2$	RHS
$\tilde{z}$	$(-17.7,-10.7,-6.7;$ $-0.7,5.3,10.3)$	$(-15,-9,-3;$ $3,9,15)$	$(-8.4,-4.4,-2.4;$ $0.6,4.6,6.6)$	0	$(-11.1,-6.1,-3.1;$ $-0.1,6.4,8.9)$
RV	-3.2	0	-0.4		-0.6
$\tilde{x}_2$	$(-7.4,-3.9,-2.4;$ $1.6,4.6,7.6)$	$(-5,-2.5,0;$ $2,4.5,7)$	$(-2.8,-1.3,-0.8;$ $0.2,2.2,3.2)$	0	$(-5.2,-2.7,-1.2;$ $-0.2,4.3,5.8)$
RV	0.1	1	0.2		0.3
$\tilde{y}_2$	$(-10.4,-1.9,1.6;$ $6.6,11.1,16.6)$	$(-12,-6,-2;$ $2,6,12)$	$(-6.8,-2.3,-0.8;$ $1.2,3.7,5.2)$	1	$(-4.2,2.3,5.8;$ $9.8,15.8,18.8)$
RV	4.1	0	0.2		8.3

$\tilde{z}_k \ominus \tilde{c}_k = max\{(-17.7,-10.7,-6.7;-0.7,5.3,10.3),(-15,-9,-3;3,9,15),$ $(-8.4,-4.4,-2.4;0.6,4.6,6.6),0\} = 0$, thus the solution is optimal. The optimal solution for the problem is $\tilde{x}_1 = 0, \tilde{x}_2 = (-5.2,-2.7,-1.2;-0.2,4.3,5.8)$ with the fuzzy objective value $\tilde{z}^* = (-8.9,-6.4,0.1,3.1,61,11.1)$. Now, we solve the RLP problem for problem (10), which is

$$\begin{aligned} min\ \ & 3x_1-2x_2\\ s.t.\ \ & 0.5x_1+5x_2\leq 1.5\\ & 4x_1-x_2\leq 8\\ & x_1,x_2\geq 0. \end{aligned}$$

Its optimal solution is $x_1 = 0,\ x_2 = 0.3$ with the optimal objective value $z^* = -0.6$. As expected, we have $x_1 = \Re(\tilde{x}_1)$, $x_2 = \Re(\tilde{x}_2)$ and $z^* = \Re(\tilde{z}^*)$.

Here is another example for an unbounded FFLP with PLFN-n's.

Example 3.2 Suppose we have the following simplex table that represents some step in the fuzzy simplex algorithm to solve some FFLP with PLFN-n's:

	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{x}_4$	RHS
$\tilde{z}$	$(-13,-8,-3,0;0,3,8,13)$	$(-13,-9,-3,0;0,4,8,13)$	$(-16,-9,-4,0;2,6,11,18)$	$(-17,-12,-7,-3;-1,3,8,13)$	$(-4,-1,1,4;4,7,9,12)$
RV	0	0	1	-2	-10
$\tilde{x}_2$	$(-8,-6,-3,-1;1,3,6,8)$	$(-6,-4,-2,0;2,4,6,8)$	$(-11,-8,-4,-2;0,2,6,9)$	$(-13,-6,-2,1;1,6,8,11)$	$(-4,-1,1,4;4,7,9,12)$
RV	0	1	-1	1	4
$\tilde{x}_1$	$(-2,-1,0,1;1,2,3,4)$	$(-3,-2,-1,0;0,1,2,3)$	$(-7,-5,-3,-2;-2,-1,1,3)$	$(-3,-1,0,1;1,2,3,5)$	$(-2,-1,0,2;2,4,5,6)$
RV	1	0	-2	1	2

It is clear that the variable $\tilde{x}_3$ needs to enter the basis. However,

$$\tilde{\mathbf{y}}_3 = \begin{bmatrix} \tilde{y}_{23} \\ \tilde{y}_{13} \end{bmatrix} = \begin{bmatrix} (-11,-8,-4,-2;0,2,6,9) \\ (-7,-5,-3,-2;-2,-1,1,3) \end{bmatrix} \preceq \tilde{\mathbf{0}}.$$

Therefore, the problem is unbounded with the direction of unboundedness:

$$\tilde{\mathbf{d}} = \begin{bmatrix} -\tilde{y}_{13} \\ -\tilde{y}_{23} \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} (-3,-1,1,2;2,3,5,7) \\ (-9,-6,-2,0,2,4,8,11) \\ (1,1,1,1;1,1,1,1) \\ (0,0,0,0;0,0,0,0) \end{bmatrix}.$$

4 Advantages of the Proposed Method

Applying the proposed method to a FFLP problem with PLFN-n's preserves the ranking values in each step of every iteration, which gives a fuzzy solution with ranking values equal to the solution of the corresponding RLP problem. This property makes the proposed method more realistic than the other methods existing in the literature that do not guarantee the ranking values are preserved throughout the solution. To illustrate, we implement the proposed method using the following example that was solved by Das et al. [7]. The results are compared to the results obtained via the Das method. The problem is a special case of the PLFN-n which is a trapezoidal fuzzy number that is considered as a flat PLFN-1.

Example 4.1

$$\begin{aligned} max\ \ & \tilde{z} = (7,10;14,17) \otimes \tilde{x}_1 \oplus (8,13;15,20) \otimes \tilde{x}_2 \\ s.t.\ \ & (11,13;15,17) \otimes \tilde{x}_1 \oplus (7,11;13,17) \otimes \tilde{x}_2 \preceq (94,100;102,108) \\ & (12,14;16,18) \otimes \tilde{x}_1 \oplus (8,12;14,18) \otimes \tilde{x}_2 \preceq (104,112;114,122) \\ & \tilde{x}_1, \tilde{x}_2 \succeq \tilde{0}. \end{aligned} \tag{11}$$

	Solution by Das	Solution by the proposed method	RS of Das	RS of the proposed method	Solution of the corresponding RLP
$\tilde{x}_1$	$(0,3;3,6)$	$(0,0;0,0)$	3	0	0
$\tilde{x}_2$	$(4.38,4.38;4.38,4.38)$	$(\frac{-43}{12},\frac{77}{12};\frac{125}{12},\frac{245}{12})$	4.38	$\frac{101}{12}$	$\frac{101}{12}$
$\tilde{z}^*$	$(71.94,86.94;$ $95.7,110.7)$	$(\frac{-815}{6},\frac{-725}{6};$ $\frac{-689}{6},\frac{-599}{6})$	91.32	117.8	117.8

It is clear that the solution by the proposed method coincide with the solution of the RLP problem, while the solution in [7] does not.

Similarly, if the problem is solved by some existing methods, the solution does not coinside with the solution of the RLP. For instance, Das et al. [7] have solved the previous problem using two methods proposed by Kumar & Kaur [12] and Ganesan & Veeramani [8]. The ranking values of the optimal objective values are 70.3 and 94, respectively. Both of these values do not coinside with the optimal objective value for the RLP problem.

Another advantage is that the proposed method is a generalization of the conventional simplex method. Suppose we have a crisp linear programming problem (P). Since every crisp real number "a" can be written in the form of a PLFN-n as $(a,a,..,a;a,a,..,a)$, then we can replace every crisp number in problem (P) by its PLFN-n form. This results in a FFLP problem, call it the *fuzzified problem*, that can be solved by the proposed method. However, the proposed method preserves the ranking values in each step and the arithmetic operations on the piecewise linear fuzzy number generalize the known operations on the crisp numbers. This means that the solution of the fuzzified problem by the proposed method is identical to the solution of the original crisp problem using the known simplex method. This means that the simplex method is generalized by the proposed fuzzy simplex method.

5 Conclusion

In this paper, a fully fuzzy linear programming problem with piecewise linear fuzzy numbers is constructed. A solution method depending on extending the simplex method is then proposed. The considered technique results in a generalization of the conventional simplex algorithm. When the proposed method is applied to crisp linear programming problems, it gives the same results as those obtained by using the classical simplex method.

Considering the PLFN-n to represent the fuzziness in the problem gives a wider range of problems that can be solved by the proposed method. The mostly used types of fuzzy numbers in the literature are the triangular, the trapezoidal and the hexagonal fuzzy numbers which are special cases of the PLFN-n.

Many applications of the constructed problem and the proposed solution method can be done, such as transportation or supply chain problems with fuzzy data.

Finding the initial basic feasible solution in the case of having constraints of the type " $\approx$ " or " $\succeq$ " is under study and will be presented in a future paper.

Acknowledgments

This work is supported by Jordan University of Science and Technology under Research Grant Number: 20170376.

References

[1] A. Abbadi, F. Hamidia, A. Morsli and A. Tlemcani. Optimal Voltage Controller using T-S Fuzzy Model for Multimachine Power Systems. *Nonlinear Dynamics and Systems Theory* **19** (2) (2019) 217–226.

[2] M. Alrefaei, A. Alawneh and M. Hassan. Fuzzy linear programming for supply chain management in steel industry. *Applied Mathematical Sciences* **8** (103) (2014) 5105–5114.

[3] M. Alrefaei, N. Ibrahim and M. Tuffaha. Solving Fully Fuzzy Transportation Problem with n-Polygonal Fuzzy Numbers. 1st International Conference on Mechanical, Aeronautical and Industrial Engineering Technologies (MechaniTek 2020) **1** (1) (2020) 69–74.

[4] A. Baykasoğlu and K. Subulan. An analysis of fully fuzzy linear programming with fuzzy decision variables through logistics network design problem. *Knowledge-Based Systems* **90** (2015) 165–184.

[5] T. Beaula and S. Rajalakshmi. Solving fully fuzzy linear programming problem using breaking points. *International Journal of Applied Operational Research* **2** (3) (2012) 11–20.

[6] B. Bilgen. Supply chain network modeling in a golf club industry via fuzzy linear programming approach. *Journal of Intelligent & Fuzzy Systems* **21** (4) (2010) 243–253.

[7] S. K. Das, T. Mandal and S. Edalatpanah. A mathematical model for solving fully fuzzy linear programming problem with trapezoidal fuzzy numbers. *Applied Intelligence* **46** (3) (2017) 509–519.

[8] K. Ganesan and P. Veeramani. Fuzzy linear programs with trapezoidal fuzzy numbers. *Annals of Operations Research* **143** (1) (2006) 305–315.

[9] F. Hamidia and A. Abbadi, A. Tlemcani and M.S. Boucherit. Dual Star Induction Motor Supplied with Double Photovoltaic Panels Based on Fuzzy Logic Type-2. *Nonlinear Dynamics and Systems Theory* **18** (4) (2018) 359–371.

[10] J. Kaur and A. Kumar. Mehar's method for solving fully fuzzy linear programming problems with l-r fuzzy parameters. *Applied Mathematical Modelling* **37** (12) (2013) 7142–7153.

[11] I. U. Khan, T. Ahmad and N. Maan. A simplified novel technique for solving fully fuzzy linear programming problems. *Journal of Optimization Theory and Applications* **159** (2) (2012) 536–546.

[12] A. Kumar and J. Kaur. A new method for solving fuzzy linear programs with trapezoidal fuzzy numbers. *Journal of fuzzy set valued analysis* **2011** (2011) 12 pages.

[13] X. Li and D. Li. The structure and realization of a polygonal fuzzy neural network. *International Journal of Machine Learning and Cybernetics* **7** (3) (2016) 373–389.

[14] B. A. Ozkok, I. Albayrak, H. G. Kocken and M. Ahlatcioglu. An approach for finding fuzzy optimal and approximate fuzzy optimal solution of fully fuzzy linear programming problems with mixed constraints. *Journal of Intelligent & Fuzzy Systems* **31** (1) (2016) 623–632.

[15] M. Z. Tuffaha and M. H. Alrefaei. Arithmetic operations on piecewise linear fuzzy number. *AIP Conference Proceedings* **1991** (1) (2018) 020024.

[16] M. Z. Tuffaha and M. H. Alrefaei. Properties of Binary Operations of n-Polygonal Fuzzy Numbers. *Advances in Intelligent Systems and Computing* **1111 AISC** (2020) 256–265.

[17] G. Wang and X. Li. Universal approximation of polygonal fuzzy neural networks in sense of k-integral norms. *Science China Information Sciences* **54** (11) (2011) 2307–2323.

[18] L. A. Zadeh. Fuzzy sets. *Information and control* **8**(3) (1965) 338–353.