Nonlinear Dynamics and Systems Theory, **20** (5) (2020) 465–478

On Approximate Controllability of Impulsive Fractional Semilinear Systems with Deviated Argument in Hilbert Spaces

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Received: November 30, 2018; Revised: November 14, 2020

Abstract: In this paper we apply a fixed-point theorem to study the existence and uniqueness of a mild solution and the approximate controllability of a fractional order impulsive differential equation with deviated argument in Hilbert spaces. An example is provided to show the effectiveness of the theory.

Keywords: *controllability; differential equations with impulses; deviated arguments; fractional derivatives and integrals; semigroup theory; fixed-point theorems.*

Mathematics Subject Classification (2010): 93B05, 34A37, 34K30, 26A33, 47H10.

1 Introduction

Differential equations with deviated arguments have received considerable attention in recent years due to their ability to generalize differential equations that show an unknown quantity and their derivatives in different values of their arguments. It is an ideal model for the study of automatic control theory, self-oscillating systems theory, long-term planning problems in economics, etc. For more details about differential equations with deviated arguments, we refer to the papers [8, 11, 15] and the references therein.

Interestingly, in this paper, we will enhance the study of differential equations with deviated arguments by fractional calculus, which, in turn, is currently attracting considerable interest from researchers, due to its wide range of applications in various scientific and technological fields such as thermal engineering, electromagnetism, control, robotics,

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viscoelasticity, edge detection, signal processing, and many other physical and biological processes. Fractional differential equations have also been applied in the modeling of many physical and engineering problems. For more details, the reader is kindly requested to go through [12, 14, 17] and the references therein.

On the other hand, impulsive differential equations have become the target of several authors, mainly because of their ability to model processes that undergo sudden changes of their states. They appear in nano-electronics, population dynamics, heat propagation, electromagnetic wave radiation, control theory and pharmacology. See [1, 3, 4, 10, 18–20] and the references therein.

Controllability of linear and nonlinear systems for various type of differential equations and inclusions was studied, on a large scale, using fixed point and semigroup theories, for more details the reader is kindly requested to go through [2, 5, 7, 9, 13] in order to know more details about these results. However, the controllability of fractional impulsive systems with deviated arguments requires a lot of attention since it has not yet received a careful study, and many aspects of this field have not been discovered yet.

In this paper, we consider the approximate controllability of fractional impulsive differential equations with deviated argument of the form

$$\begin{cases} {}^{C}D^{\alpha}x(t) = Ax(t) + Bu(t) + f(t, x(t), x(\varphi(x(t), t))), \quad t \in [0, b],\ 0 < \alpha < 1, \\ \Delta x(t_k) = I_k(x(t_k)),\ k = \overline{1, m},\ 0 < t_1 < t_2 < \cdots < t_m < b, \\ x(0) = x_0, \end{cases} \tag{1}$$

where the state function $\varphi(\cdot)$ takes values in a Hilbert space E. ${}^{C}D^{\alpha}$ is the Caputo fractional derivative of order α. The control function $u(\cdot)$ is given in $L^2([0, b], U)$, where U is a Hilbert space. B is a bounded linear operator from U into E. The linear operator A generates a strongly continuous semigroup $(T(t))_{t\geq 0}$ on E. f and φ are suitably defined functions satisfying certain conditions to be specified in Section 3. $I_k \in C(E, E)$, $k = 1, 2, \cdots, m$, and $\Delta x(t_k) = x(t_k^+) - x(t_k^-) = x(t_k^+) - x(t_k)$.

This paper is organized as follows. In Section 2, we introduce some notations and necessary preliminaries. In Section 3 we prove the existence of mild solutions for control systems and we establish its approximate controllability. In Section 4, an example is given to illustrate our results.

2 Preliminaries

Let $J = [0, b], 0 < t_1 < t_2 < \cdots < t_m < b$, and $J' = [0, b] \backslash \{t_1, t_2, \cdots, t_m\} \subset J$. $(E, \|\cdot\|)$ is a Hilbert space and $C(J, E)$ is the Hilbert space of all E-valued continuous functions from J into E,

$$PC(J, E) = \left\{x : [0, b] \to E;\ x \in C(J', E), x(t_k^+) \text{ and } x(t_k^-) \text{ exist},\ x(t_k^-) = x(t_k), 1 \leq k \leq m\right\},$$

$PC(J, E)$ is a Banach space with norm $\|x\| = \sup_{t\in J} \|x(t)\|$,

$$D = C_L(J, E) = \left\{x \in PC(J, E) : \|x(t) - x(s)\| \leq L|t - s|,\ \forall t, s \in J'\right\}, \tag{2}$$

where L is a positive constant. It is clear that D is a Banach space with the sup-norm $\|x\| = \sup_{t\in J} \|x(t)\|$.

Definition 2.1 [17] The fractional (arbitrary) order integral of the function $f \in L^1([a,b],\mathbb{R}_+)$ of order $\alpha \in \mathbb{R}_+$ is defined by

$$I_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(s)}{(t-s)^{1-\alpha}} ds,$$

where Γ is the Gamma function, when $a = 0$, we write $I_a^\alpha f(t) = I^\alpha f(t)$.

Definition 2.2 [17] For a function f given on the interval $[a,b]$, the Riemann-Liouville fractional-order derivative of order α of f is defined by

$$^{(R-L)}D_a^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1} f(s)\, ds,$$

here $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α, when $a = 0$, $D_a^\alpha f(t) = D^\alpha f(t)$.

Definition 2.3 [17] For a function f given on the interval $[a,b]$, the Caputo fractional-order derivative of order α of f is defined by

$$^C D_a^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds,$$

where $n = [\alpha] + 1$.

Definition 2.4 [16] A one parameter family $(T(t))_{t \geq 0}$ of bounded linear operators from E into E is a semi group of bounded linear operators on E if

(1) $T(0) = I$ (I is the identity operator in E).

(2) $T(t+s) = T(t)T(s)$, for every $t \geq 0$, $s \geq 0$ (the semigroup property).

A semigroup of bounded linear operators $T(t)$ is uniformly continuous if

$$\lim_{t \to 0} \|T(t) - I\| = 0.$$

The linear operator A defined by

$$D(A) = \left\{ x \in E : \lim_{t \to 0} \frac{T(t)x - x}{t} \ exists \right\}$$

and

$$Ax = \lim_{t \to 0} \frac{T(t)x - x}{t} = \left.\frac{d^+ T(t)x}{dt}\right|_{t=0}, \quad for \quad x \in D(A),$$

is the infinitesimal generator of the semigroup $T(t)$, $D(A)$ is the domain of A.

Definition 2.5 A mild solution of problem (1) is defined as a function $x(.) \in D$ that satisfies:

(i) $x(0) = x_0$.

(ii) $\Delta x(t_k) = I_k(x(t_k)),\ k = 1, 2, \cdots, m$.

(iii) The restriction of $x(t)$ to the interval J' is continuous and the following integral equation is satisfied:

$$\begin{aligned}
x(t) &= T(t)x_0 + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} T(t-s) B u_x(s) ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} T(t-s) B u_x(s) ds \\
&+ \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} T(t-s) f(s, x(s), x(\varphi(x(s), s))) ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} T(t-s) f(s, x(s), x(\varphi(x(s), s))) ds \\
&+ \sum_{0<t_k<t} T(t-t_k) I_k(x(t_k)), \quad 0 \le s < t < b, \text{ and } t \ne t_k.
\end{aligned} \tag{3}$$

Let $x_b(x_0; u)$ be the state value of (1) at terminal time b corresponding to the control u and the initial value x_0. Introduce the set

$$\mathfrak{R}(b; x_0) = \{x_b(x_0; u) : u \in L^2(J, U)\}$$

which is called the reachable set of system (1) at terminal time b, its closure in E is denoted by $\overline{\mathfrak{R}(b, x_0)}$.

Definition 2.6 The system (1) is said to be approximately controllable on J if $\overline{\mathfrak{R}(b; x_0)} = E$, that is, given an arbitrary $\epsilon > 0$, it is possible to steer from the point x_0 to within a distance ϵ from all point in the state space E at time b.

Consider the linear fractional differential system

$$\begin{cases} {}^C D^\alpha x(t) = Ax(t) + Bu(t), \quad t \in J = [0, b], \ 0 < \alpha < 1, \\ x(0) = x_0, \end{cases} \tag{4}$$

is approximately controllable. It is convenient at this position to introduce the controllability operator associated with (4), thus

$$\Gamma_0^b x = \frac{1}{\Gamma(\alpha)} \int_0^b (b-s)^{\alpha-1} T(b-s) B B^* T^*(b-s) x ds, \text{ for } x \in E.$$

For $\lambda > 0$, we consider the relevant operator $\mathcal{R}(\lambda; \Gamma_0^b) = (\lambda I + \Gamma_0^b)^{-1}$. It is convenient at this point to define the operators

$$\Gamma_{t_k}^b = \frac{1}{\Gamma(\alpha)} \int_{t_k}^b (b-s)^{\alpha-1} T(b-s) B B^* T^*(b-s) ds,$$

$$\Gamma_{t_{k-1}}^{t_k} = \frac{1}{\Gamma(\alpha)} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} T(t_k - s) B B^* T^*(t_k - s) ds,$$

$$\mathcal{R}(\lambda; \Gamma_{t_{k-1}}^{t_k}) = (\lambda I + \Gamma_{t_{k-1}}^{t_k})^{-1}, \text{for } \lambda > 0, k = 1, \cdots, m,$$

where B^* denotes the adjoint of B and $T^*(t)$ is the adjoint of $T(t)$. It is straightforward that the operator Γ_0^b is a linear bounded operator.

3 Main Result

This section deals with the existence and uniqueness of mild solutions and approximate controllability of the problem (1). Before stating and proving the main results, we introduce the following hypotheses:

$(H1)$ A generates a strongly continuous semigroup $(T(t))_{t\geq 0}$ in the Hilbert space E ($T(t)$ is compact for $t \geq 0$) and there exists a constant $\widehat{M} \geq 1$ such that
$$\|T(t)\| \leq \widehat{M} \quad \text{for every} \quad t \geq 0$$
has an inverse operator $T^{-1}(t)$ and there exists a positive constant $\widehat{M_1}$ such that
$$\|T^{-1}(t)\| \leq \widehat{M_1} \quad \text{for every} \quad t \geq 0.$$

$(H2)$ The nonlinear map $f : J \times E \times E \to E$ satisfies the Lipschitz condition such that there exist constants $M_1 = M_1(t, x, y, r) > 0$ and $M_2 = M_2(t, 0, x, r) > 0$, we have for all $x_i,\ y_i \in B_r,\ i = 1, 2.$
$$\|f(t, x_1, y_1) - f(t, x_2, y_2)\| \leq M_1 \left\{\|x_1 - x_2\| + \|y_1 - y_2\|\right\} \text{ for each } t \in J$$
and $\max\limits_{t\in J} \|f(t, 0, x(0))\| = M_2$.

$(H3)$ $\varphi : D \times \mathbb{R}_+ \to \mathbb{R}_+$ is globally continuous on $E \times \mathbb{R}_+$ and satisfies $\varphi(\cdot, 0) = 0$ and there exists a constant $L_\varphi = L_\varphi(x, t, r) > 0$ such that
$$|\varphi(x, t) - \varphi(y, s)| \leq L_\varphi \left\{\|x - y\| + |t - s|\right\}$$
for every $x, y \in B_r$, and $t, s \in J$.

$(H4)$ B is a bounded linear operator from U into E, such that $\|B\| = \widetilde{M}$, for a constant $\widetilde{M} > 0$.

$(H5)$ for each $0 \leq t < b$ and $t \neq t_k, k = 1, \cdots, m$ the operators $\lambda\mathcal{R}(\lambda; \Gamma_{t_k}^b) \to 0$ and $\lambda\mathcal{R}(\lambda; \Gamma_{t_{k-1}}^{t_k}) \to 0$ as $\lambda \to 0^+$ in the strong operator topology.

$(H6)$ There exist constants $d, L_k, \ell, d_k > 0$, $k = \overline{1, m}$, such that $\|I_k(\cdot)\| < d_k$, $\sum\limits_{k=1}^{m} d_k = d$, $\|I_k(x) - I_k(y)\| \leq L_k\|x - y\|$, for every $x, y \in E$, and $\sum\limits_{k=1}^{m} L_k = \ell$.

For brevity, let ω_1, ω_2 be the positive numbers
$$\frac{\widehat{M}^2}{\lambda}\widetilde{M}\left(\|\widehat{z}_b\| + \widehat{M}\|x_0\| + \frac{b^\alpha}{\Gamma(\alpha+1)}\widehat{M}(rM_1 + M_2) + \frac{b^{\alpha+1}}{\Gamma(\alpha+1)}\widehat{M}M_1LL_\varphi(1 + L)\right) = \omega_1,$$
$$\frac{\widehat{M}^2}{m\lambda}\widehat{M_1}\widetilde{M}\left(\|\widetilde{z}_b\| + m\frac{b^\alpha}{\Gamma(\alpha+1)}\widehat{M}(rM_1 + M_2) + m\frac{b^{\alpha+1}}{\Gamma(\alpha+1)}\widehat{M}M_1LL_\varphi(1 + L) + m\widehat{M}d\right) = \omega_2,$$
and put
$$\widehat{M}\|x_0\| + \frac{b^\alpha}{\Gamma(\alpha+1)}\widehat{M}\widetilde{M}(m\omega_2 + \omega_1) + m\widehat{M}d + (m+1)\frac{b^\alpha}{\Gamma(\alpha+1)}\widehat{M}\Big(rM_1 + M_2 + M_1LL_\varphi(1+L)\Big) \leq r, \tag{5}$$
where $r > 0$ is a constant.

We denote $B_r = \{x \in D;\ \|x(t)\| \leq r\}$.

Theorem 3.1 *Suppose* $(H1)-(H6)$ *and* (5) *hold, moreover, let*

$$\rho = \frac{b^\alpha}{\Gamma(\alpha+1)}\widehat{M}M_1(2+LL_\varphi)\Big(\frac{b^\alpha}{\lambda\Gamma(\alpha+1)}\widehat{M}^3\widetilde{M}^2+1\Big) + m\Big(\frac{b^\alpha}{\Gamma(\alpha+1)}\widehat{M}M_1(2+LL_\varphi)+\widehat{M}\ell\Big)\Big(\frac{b^\alpha}{\lambda\Gamma(\alpha+1)}\widehat{M}^3\widehat{M}_1\widetilde{M}^2+1\Big) \tag{6}$$

be such that $\rho\in(0,1)$. *Then the problem* (1) *is approximate controllable on* J.

In this section, it will be shown that the system (1) is approximately controllable if for all $\lambda>0$, there exists a continuous function $x(\cdot)\in D$ such that

$$\begin{aligned} x(t) &= T(t)x_0 + \frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}T(t-s)Bu_x(s)ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}T(t-s)Bu_x(s)ds \\ &+ \frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}T(t-s)f(s,x(s),x(\varphi(x(s),s)))ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}T(t-s)f(s,x(s),x(\varphi(x(s),s)))ds \\ &+ \sum_{0<t_k<t}T(t-t_k)I_k(x(t_k)) = \widehat{z}(t)+\widetilde{z}(t). \end{aligned} \tag{7}$$

For $k=1,\cdots,m$, we put

$$\begin{aligned} \widehat{z}(t) &= T(t)x_0 + \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}T(t-s)Bu_x(s)ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}T(t-s)f(s,x(s),x(\varphi(x(s),s)))ds. \\ \widetilde{z}(t) &= \frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}T(t-s)Bu_x(s)ds \\ &+ \frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}T(t-s)f(s,x(s),x(\varphi(x(s),s)))ds \\ &+ \sum_{0<t_k<t}T(t-t_k)I_k(x(t_k)). \end{aligned}$$

Proof. Transform the problem (1) into a fixed-point problem. For $\lambda>0$, we define the operators $F_\lambda, G_\lambda : D\to D$ as $(F_\lambda x + G_\lambda x) = x(t)$, where

$$\begin{aligned} F_\lambda x(t) &= T(t)x_0 + \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}T(t-s)B\mathfrak{u}(s)ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}T(t-s)f(s,x(s),x(\varphi(x(s),s)))ds, \quad \text{for } k=1,\cdots,m. \end{aligned} \tag{8}$$

$$G_\lambda x(t) = \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} T(t-s) B \mathfrak{v}(s) ds$$
$$+ \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} T(t-s) f(s, x(s), x(\varphi(x(s), s))) ds$$
$$+ \sum_{0<t_k<t} T(t-t_k) I_k(x(t_k)), \text{ for } k = 1, \cdots, m. \tag{9}$$

We take the controls

$$\mathfrak{u}(t) = B^* T^*(b-t) \mathcal{R}(\lambda, \Gamma_{t_k}^b) p(x(\cdot)), \tag{10}$$
$$\mathfrak{v}(t) = \frac{1}{m} B^* T^*(t_k - t) T^{-1}(b - t_k) \mathcal{R}(\lambda, \Gamma_{t_{k-1}}^{t_k}) q(x(\cdot)), \tag{11}$$

where, for $k = 1, \cdots, m$

$$p(x(\cdot)) = \widehat{z}_b - T(b)x_0 - \frac{1}{\Gamma(\alpha)} \int_{t_k}^{b} (b-s)^{\alpha-1} T(b-s) f(s, x(s), x(\varphi(x(s), s))) ds, \tag{12}$$

$$q(x(\cdot)) = \widetilde{z}_b - \frac{1}{\Gamma(\alpha)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} T(b-s) f(s, x(s), x(\varphi(x(s), s))) ds$$
$$- \sum_{k=1}^{m} T(b - t_k) I_k(x(t_k)), \tag{13}$$

for any $\lambda > 0$, we shall show that $F_\lambda + G_\lambda$ has a fixed point on D, which is a mild solution of the system (1). Clearly, $(F_\lambda x + G_\lambda x)(b) = x_b = \widehat{z}_b + \widetilde{z}_b$.

From (10) and (11), we have

$$\|\mathfrak{u}(t)\| \le \frac{\widehat{M}^2}{\lambda} \widetilde{M} \|p(x(\cdot))\|; \ \|\mathfrak{v}(t)\| \le \frac{\widehat{M}^2}{m\lambda} \widehat{M}_1 \widetilde{M} \|q(x(\cdot))\|,$$

using $(H1) - (H6)$ and (2), we get

$$\|p(x(\cdot))\| \le \|\widehat{z}_b\| + \|T(b)\| \|x_0\| + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{b} (b-s)^{\alpha-1} \|T(b-s)\| \|f(s, x(s), x(\varphi(x(s), s)))\| ds$$
$$\le \|\widehat{z}_b\| + \widehat{M} \|x_0\| + \frac{\widehat{M}}{\Gamma(\alpha)} \int_{t_k}^{b} (b-s)^{\alpha-1} \Big\{ \|f(s, x(s), x(\varphi(x(s), s))) - f(s, 0, x(0))\|$$
$$+ \|f(s, 0, x(0))\| \Big\} ds$$
$$\le \|\widehat{z}_b\| + \widehat{M} \|x_0\| + \frac{\widehat{M}}{\Gamma(\alpha)} M_1 \int_{t_k}^{b} (b-s)^{\alpha-1} \Big\{ \|x(s)\|$$
$$+ \|x(\varphi(x(s), s)) - x(\varphi(x(0), 0))\| \Big\} ds + \frac{b^\alpha}{\Gamma(\alpha+1)} \widehat{M} M_2$$

$$\|p(x(\cdot))\| \leq \|\widehat{z}_b\| + \widehat{M}\|x_0\| + \frac{b^\alpha}{\Gamma(\alpha+1)}\widehat{M}(rM_1+M_2)$$

$$+ \frac{\widehat{M}}{\Gamma(\alpha)}M_1 L \int_{t_k}^{b}(b-s)^{\alpha-1}|\varphi(x(s),s)-\varphi(x(0),0)|ds$$

$$\leq \|\widehat{z}_b\| + \widehat{M}\|x_0\| + \frac{b^\alpha}{\Gamma(\alpha+1)}\widehat{M}(rM_1+M_2)$$

$$+ \frac{\widehat{M}}{\Gamma(\alpha)}M_1 L L_\varphi \int_{t_k}^{b}(b-s)^{\alpha-1}\{\|x(s)-x(0)\|+|s|\}ds$$

$$\leq \|\widehat{z}_b\| + \widehat{M}\|x_0\| + \frac{b^\alpha}{\Gamma(\alpha+1)}\widehat{M}(rM_1+M_2) + \frac{b^{\alpha+1}}{\Gamma(\alpha+1)}\widehat{M}M_1 L L_\varphi(1+L),$$

in the same way for $t_k < b, k = 1, \cdots, m$, we get

$$\|q(x(\cdot))\| \leq \|\widetilde{z}_b\| + m\frac{b^\alpha}{\Gamma(\alpha+1)}\widehat{M}(rM_1+M_2) + m\frac{b^{\alpha+1}}{\Gamma(\alpha+1)}\widehat{M}M_1 L L_\varphi(1+L) + m\widehat{M}d.$$

Thus there exist positive numbers ω_1, ω_2 such that

$$\begin{aligned} \|\mathfrak{u}(t)\| &\leq \frac{\widehat{M}^2}{\lambda}\widetilde{M}\Bigg(\|\widehat{z}_b\| + \widehat{M}\|x_0\| + \frac{b^\alpha}{\Gamma(\alpha+1)}\widehat{M}(rM_1+M_2) \\ &\qquad + \frac{b^{\alpha+1}}{\Gamma(\alpha+1)}\widehat{M}M_1 L L_\varphi(1+L)\Bigg) = \omega_1, \\ \|\mathfrak{v}(t)\| &\leq \frac{\widehat{M}^2}{m\lambda}\widehat{M_1}\widetilde{M}\Bigg(\|\widetilde{z}_b\| + m\frac{b^\alpha}{\Gamma(\alpha+1)}\widehat{M}(rM_1+M_2) \\ &\qquad + m\frac{b^{\alpha+1}}{\Gamma(\alpha+1)}\widehat{M}M_1 L L_\varphi(1+L) + m\widehat{M}d\Bigg) = \omega_2. \end{aligned} \tag{14}$$

The proof will be given in two steps.

Step 1. $F_\lambda + G_\lambda$ maps B_r into itself.

Let $x \in B_r$. By (14), we have for each $t \in J$

$$\|(F_\lambda x)(t) + (G_\lambda x)(t)\| \leq \|T(t)\|\|x_0\| + \frac{1}{\Gamma(\alpha)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}\|T(t-s)\|\|B\|\|\mathfrak{v}(s)\|ds$$

$$+ \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}\|T(t-s)\|\|B\|\|\mathfrak{u}(s)\|ds$$

$$+ \frac{1}{\Gamma(\alpha)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}\|T(t-s)\|\|f(s,x(s),x(\varphi(x(s),s)))\|ds$$

$$+ \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}\|T(t-s)\|\|f(s,x(s),x(\varphi(x(s),s)))\|ds$$

$$+ \sum_{k=1}^{m}\|T(t-t_k)\|\|I_k(x(t_k))\|.$$

Using the same method to find $\|p(x(\cdot))\|, \|q(x(\cdot))\|$ and (14), we get

$$\begin{aligned}\|(F_\lambda x + G_\lambda x)(t)\| &\le \widehat{M}\|x_0\| + \frac{b^\alpha}{\Gamma(\alpha+1)}\widehat{M}\widetilde{M}(m\omega_2 + \omega_1) + m\widehat{M}d \\ &+ (m+1)\frac{b^\alpha}{\Gamma(\alpha+1)}\widehat{M}\Big(rM_1 + M_2 + M_1 L L_\varphi(1+L)\Big) \\ &\le r.\end{aligned}$$

Thus, $F_\lambda + G_\lambda$ maps B_r into itself.

Step 2. We shall show now that the operator $F_\lambda + G_\lambda$ is a contraction mapping. Let $x, y \in B_r$. By (10) and (11), for each $t \in J$, we have

$$\begin{aligned}\|F_\lambda(x)(t) - F_\lambda(y)(t)\| &\le \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}\|T(t-s)\|\|B\|\|B^*\|\|T^*(b-t)\|\|\mathcal{R}(\lambda, \Gamma_{t_k}^b)\| \\ &\times\left(\frac{1}{\Gamma(\alpha)}\int_{t_k}^{b}(b-\tau)^{\alpha-1}\|T(b-\tau)\|\right. \\ &\times\left.\Big\{\|f(\tau, y(\tau), y(\varphi(y(\tau),\tau))) - f(\tau, x(\tau), x(\varphi(x(\tau),\tau)))\|\Big\}d\tau\right)ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}\|T(t-s)\|\|f(s, x(s), x(\varphi(x(s),s))) \\ &\qquad - f(s, y(s), y(\varphi(y(s),s)))\|\,ds \\ &\le \frac{b^\alpha}{\lambda\Gamma(\alpha+1)}\widehat{M}^4\widetilde{M}^2 M_1\Big(\frac{1}{\Gamma(\alpha)}\int_{t_k}^{b}(b-s)^{\alpha-1}\Big\{\|y(s)-x(s)\| \\ &\qquad + \|y(\varphi(y(s),s)) - x(\varphi(x(s),s))\|\Big\}ds\Big) \\ &+ \frac{\widehat{M}}{\Gamma(\alpha)}M_1\int_{t_k}^{t}(t-s)^{\alpha-1}\Big\{\|x(s)-y(s)\| \\ &\qquad + \|x(\varphi(x(s),s)) - y(\varphi(y(s),s))\|\Big\}ds \\ &\le \frac{b^\alpha}{\lambda\Gamma(\alpha+1)}\widehat{M}^4\widetilde{M}^2 M_1\Big(\frac{1}{\Gamma(\alpha)}\int_{t_k}^{b}(b-s)^{\alpha-1}\Big\{\|y(s)-x(s)\| \\ &+ \|y(\varphi(y(s),s)) - x(\varphi(y(s),s))\| \\ &\qquad + \|x(\varphi(y(s),s)) - x(\varphi(x(s),s))\|\Big\}ds\Big) \\ &+ \frac{\widehat{M}}{\Gamma(\alpha)}M_1\int_{t_k}^{t}(t-s)^{\alpha-1}\Big\{\|x(s)-y(s)\| \\ &+ \|x(\varphi(x(s),s)) - y(\varphi(x(s),s))\| \\ &\qquad + \|y(\varphi(x(s),s)) - y(\varphi(y(s),s))\|\Big\}ds\end{aligned}$$

$$
\begin{aligned}
\|F_\lambda(x)(t)-F_\lambda(y)(t)\| &\leq \frac{b^\alpha}{\lambda\Gamma(\alpha+1)}\widehat{M}^4\widetilde{M}^2 M_1\Big(\frac{1}{\Gamma(\alpha)}\int_{t_k}^{b}(b-s)^{\alpha-1}\Big\{\|y(s)-x(s)\| \\
&+\|y(\varphi(y(s),s))-x(\varphi(y(s),s))\|+L|\varphi(y(s),s)-\varphi(x(s),s)|\Big\}ds\Big) \\
&+\frac{\widehat{M}}{\Gamma(\alpha)}M_1\int_{t_k}^{t}(t-s)^{\alpha-1}\Big\{\|x(s)-y(s)\| \\
&+\|x(\varphi(x(s),s))-y(\varphi(x(s),s))\|+L|\varphi(x(s),s)-\varphi(y(s),s)|\Big\}ds \\
&\leq \frac{b^\alpha}{\lambda\Gamma(\alpha+1)}\widehat{M}^4\widetilde{M}^2 M_1\Big(\frac{1}{\Gamma(\alpha)}\int_{t_k}^{b}(b-s)^{\alpha-1}\Big\{\|y(s)-x(s)\| \\
&+\|y(\varphi(y(s),s))-x(\varphi(y(s),s))\|+LL_\varphi\|y(s)-x(s)\|\Big\}ds\Big) \\
&+\frac{\widehat{M}}{\Gamma(\alpha)}M_1\int_{t_k}^{t}(t-s)^{\alpha-1}\Big\{\|x(s)-y(s)\| \\
&+\|x(\varphi(x(s),s))-y(\varphi(x(s),s))\|+LL_\varphi\|x(s)-y(s)\|\Big\}ds \\
&\leq \frac{b^{2\alpha}}{\lambda(\Gamma(\alpha+1))^2}\widehat{M}^4\widetilde{M}^2 M_1(2+LL_\varphi)\|y-x\| \\
&+\frac{b^\alpha}{\Gamma(\alpha+1)}\widehat{M}M_1(2+LL_\varphi)\|x-y\| \\
&\leq \frac{b^\alpha}{\Gamma(\alpha+1)}\widehat{M}M_1(2+LL_\varphi)\Big(\frac{b^\alpha}{\lambda\Gamma(\alpha+1)}\widehat{M}^3\widetilde{M}^2+1\Big)\|x-y\|.
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
\|(G_\lambda x)(t)-(G_\lambda y)(t)\| &\leq \frac{1}{m\Gamma(\alpha)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}\|T(t-s)\|\|B\|\|B^*\|\|T^*(t_k-t)\| \\
&\times\|T^{-1}(b-t_k)\|\|\mathcal{R}(\lambda,\Gamma_{t_{k-1}}^{t_k})\|\Bigg(\frac{1}{\Gamma(\alpha)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}(t_k-\tau)^{\alpha-1}\|T(b-\tau)\| \\
&\times\|f(\tau,y(\tau),y(\varphi(y(\tau),\tau)))-f(\tau,x(\tau),x(\varphi(x(\tau),\tau)))\|d\tau \\
&+\sum_{k=1}^{m}\|T(b-t_k)\|\|I_k(y(t_k))-I_k(x(t_k))\|\Bigg)ds \\
&+\frac{1}{\Gamma(\alpha)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}\|T(t-s)\| \\
&\times\|f(s,x(s),x(\varphi(x(s),s)))-f(s,y(s),y(\varphi(x(s),s)))\|ds \\
&+\sum_{k=1}^{m}\|T(t-t_k)\|\|I_k(x(t_k))-I_k(y(t_k))\| \\
&\leq m\Big(\frac{b^\alpha}{\Gamma(\alpha+1)}\widehat{M}M_1(2+LL_\varphi)+\widehat{M}\ell\Big)\Big(\frac{b^\alpha}{\lambda\Gamma(\alpha+1)}\widehat{M}^3\widehat{M}_1\widetilde{M}^2+1\Big)\|x-y\|.
\end{aligned}
$$

So, we write

$$
\begin{aligned}
\|(F_\lambda x + G_\lambda x)(t) - (F_\lambda + G_\lambda)(y)(t)\| &\leq \|F_\lambda(x)(t) + F_\lambda(y)(t)\| + \|G_\lambda(x)(t) - G_\lambda(y)(t)\| \\
&\leq \Bigg\{ \frac{b^\alpha}{\Gamma(\alpha+1)} \widehat{M} M_1 (2 + LL_\varphi) \Big(\frac{b^\alpha}{\lambda\Gamma(\alpha+1)} \widehat{M}^3 \widetilde{M}^2 + 1 \Big) \\
&+ m \Big(\frac{b^\alpha}{\Gamma(\alpha+1)} \widehat{M} M_1 (2 + LL_\varphi) + \widehat{M}\ell \Big) \\
&\times \Big(\frac{b^\alpha}{\lambda\Gamma(\alpha+1)} \widehat{M}^3 \widehat{M_1} \widetilde{M}^2 + 1 \Big) \Bigg\} \|x - y\|.
\end{aligned}
$$

From (6), we have

$$
\|(F_\lambda + G_\lambda)(x) - (F_\lambda + G_\lambda)(y)\| \leq \rho \|x - y\|.
$$

So, for $\lambda > 0$, we say the operator $F_\lambda + G_\lambda$ is a contraction mapping on B_r. Hence there exists a unique fixed point $x \in B_r$ such that $(F_\lambda x + G_\lambda x)(t) = x(t)$. The unique fixed point of $F_\lambda + G_\lambda$ is a mild solution of (1) on J, which satisfies $x(b) = x_b$. Hence, by the Banach contraction principle, the semilinear fractional system (1) is approximate controllable on J.

4 An Example

Throughout this section, we provide an illustrative example to demonstrate the effectiveness of the previously proven theoretical results using the heat equation, which is a parabolic partial differential equation, to describe the physical phenomenon of thermal conduction in a metal bar. Then, we consider an initial boundary value problem with time-fractional differential equation of the following form:

$$
\begin{cases}
\dfrac{\partial^\alpha \nu}{\partial t^\alpha}(t,\varepsilon) = \dfrac{\partial^2 \nu}{\partial \varepsilon^2}(t,\varepsilon) + \mu(t,\varepsilon) + \sin\big(|\nu(t,\varepsilon)|\big) + \big(1 + e^{(\nu(t,\varepsilon))}\big)^\beta, \ \beta \in \mathbb{R}, \\
\nu(t,0) = \nu(t,1) = 0, \quad t \in [0,b], \\
\nu(0,\varepsilon) = \nu_0(\varepsilon), \quad \varepsilon \in (0,1), \\
\Delta\nu(t_k)(\varepsilon) = \varepsilon\big(\big|\nu(t_k)(\varepsilon)\big| + e^{t_k}\big), \quad k = 1, \cdots, m,
\end{cases}
\tag{15}
$$

where $\alpha \in (0,1)$, and $\mu : J \times (0,1) \to (0,1)$ is the control function and it is continuous.

- $\nu(t,\varepsilon)$ is the temperature at any point ε and any time t.
- $Q(t,\varepsilon) = \sin\big(|\nu(t,\varepsilon)|\big) + \big(1 + e^{(\nu(t,\varepsilon))}\big)^\beta$ is the heat energy generated per unit volume per unit time.

  If $Q(t,\varepsilon) > 0$, then the heat energy is being added to the system at that location and time, and if $Q(t,\varepsilon) < 0$, then the heat energy is being removed from the system at that location and time.
- $\nu(t,0)$ and $\nu(t,1)$ are the temperatures at the ends of the bar. These are called the boundary conditions.

  To keep things simple, we will solve the IBVP (15) for the heat equation with $\nu(t,0) = \nu(t,1) = 0\,°C$. These are called the homogeneous boundary conditions.

- $\nu(0,\varepsilon)$ is the initial temperature distribution. This is called the initial condition.
- $\Delta\nu(t_k)(\varepsilon)$ is the sudden instantaneous perturbation in heat distribution. This is called the impulsive condition.

One end of the bar is assumed to be at $\varepsilon_0 = 0$ and the other is at $\varepsilon_1 = 1$ (a long metal bar of length $|\varepsilon_0 - \varepsilon_1| = 1$). The bar is much longer than it is thick, so we can treat the distribution of heat as a function of just t and ε. Assuming that the bar specific heat capacity is known, we will know how heat is distributed if we can find a function for the temperature $\nu(t,\varepsilon)$.

Now, we will satisfy the previous assumptions and theoretical results using the IBVP (15) and get the required controllability.

Set $E = L^2\big[(0,1)\big]$, and $A : D(A) \subset E \to E$ is an operator defined by

$$A\omega = \omega^{''}, \quad \omega \in D(A)$$

with the domain

$$D(A) = \{\omega \in E; \omega, \omega^{'} \text{ are absolutely continuous, } \omega^{''} \in E, \omega(0) = \omega(1) = 0\}.$$

Then

$$A\omega = \sum_{n=1}^{\infty} n^2 (\omega, \omega_n)\omega_n, \quad \omega \in D(A),$$

where $\omega_n(x) = \sqrt{2}\sin(nx), n \in \mathbb{N}$ is the orthogonal set of eigenvectors of A.

It is well known that A is a generator of an analytic semigroup $\big(T(t)\big)_{t\geq 0}$ in E which is given by

$$T(t)\omega = \sum_{n=1}^{\infty} e^{-n^2 t}(\omega, \omega_n)\omega_n, \quad \omega \in E,\ t > 0.$$

Further, for each $t \in J$, we have $T^*(t)x = T(t)x$, where $x \in E$.
Therefore, for $(t,\varepsilon) \in [0,b] \times (0,1)$, we have

$$\begin{aligned} x(t)(\varepsilon) &= \nu(t,\varepsilon), \\ f(t,\nu(t),\nu(a(\nu(t),t)))(\varepsilon) &= Q(t,\varepsilon), \\ I(\nu(t_k))(\varepsilon) &= \varepsilon\big(\big|\nu(t_k)(\varepsilon)\big| + e^{t_k}\big), \quad k = 1,2,\cdots,m, \\ Bu(t)(\varepsilon) &= \mu(t,\varepsilon). \end{aligned}$$

The system (15) is the abstract form of (1).

We define an infinite dimensional control space as

$$U = \{u : u = \sum_{n=2}^{\infty} u_n\omega_n, \sum_{n=2}^{\infty} |u_n|^2 < \infty\},$$

endowed with the norm $\|u\|_U = \big(\sum\limits_{n=2}^{\infty} |u_n|^2\big)^{1/2}$.

Let $B : U \to E$ and

$$Bu = 2u_2\omega_1 + \sum_{n=2}^{\infty} u_n\omega_n.$$

Then B is a bounded linear map and the adjoint is

$$B^*v = (2v_1 + v_2)\omega_2 + \sum_{n=3}^{\infty} v_n\omega_n.$$

Moreover,

$$B^*T^*(t)y = (2y_1e^{-t} + y_2e^{-4t})\omega_2 + \sum_{n=3}^{\infty} y_n e^{-n^2t}\omega_n$$

for $v = \sum\limits_{n=1}^{\infty} v_n\omega_n$ and $y = \sum\limits_{n=1}^{\infty} y_n\omega_n$. For $t \in J$, it can be shown that

$$\|B^*T^*(t)y\| = 0 \Rightarrow \|2y_1e^{-t} + y_2e^{-4t}\|^2 + \sum_{n=3}^{\infty} \|y_n e^{-n^2t}\|^2 = 0 \Rightarrow y = 0.$$

Therefore, by Theorem 4.1.7 [6], the linear system corresponding to (15) is approximately controllable. On the other hand, we have $\lambda\mathcal{R}(\lambda, \Pi_{t_k}^b) \to 0, \lambda\mathcal{R}(\lambda, \Pi_{t_{k-1}}^{t_k}) \to 0$, as $\lambda \to 0^+$, for $k = 1, \cdots, m$ in the strong operator topology, which is a necessary and sufficient condition for the linear system to be approximately controllable. Further, the conditions $(H1) - (H6)$ are satisfied. Hence, by Theorem 3.1, the IBVP (15) is approximate controllable on J.

5 Conclusions

This paper focuses on establishing the approximate controllability of an impulsive fractional semilinear system with deviated argument in Hilbert spaces through the application of one of the most important results of the analysis and it is considered the main source of the metric fixed point theory known as the "Banach Contraction Principle" that accompanied the formulation of a certain set of sufficient conditions. These ease the proof of the existence and uniqueness of the mild solution attached to the system under study.

In the future, we aim to expand this study by adapting some techniques used to other ideas and extracting new results that show the effectiveness of this study and its effect in the midst of scientific research. The closest result we would like to prove is the establishment of the approximate controllability of an impulsive stochastic differential system with deviated argument delay of fractional order.

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