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The Qualitative Analysis of an n-Dimensional Nonlinear Dynamical System Arising From the Modeling of Multilayer Scales on Pure Metals

R. L. Baker *

Department of Mathematics, University of Iowa,
14 MacLean Hall, Iowa City, IA 52246, USA

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Abstract: A metal oxide is a compound containing oxygen and metal. Certain pure metals can form different oxides, and oxidation of such metals produces a multilayer oxide scale on the metal. In one of their publications, F. Gesmundo and F. Viani qualitatively analyzed the parabolic growth of three-layer oxide scales on those metals which can form three oxides. They obtained a non-linear three-dimensional dynamical system as a model for the growth of such scales. In the present paper we generalize this dynamical system of Gesmundo and Viani to n-dimensions; we then qualitatively analyze this n-dimensional dynamical system.

Keywords: *differential equations; dynamical systems; nonlinear dynamical systems; cooperative dynamical systems.*

Mathematics Subject Classification (2010): 34C35, 70K05.

1 Introduction

A metal oxide is a compound containing oxygen and metal. For instance, common rust is caused by the oxidation of metal. Certain pure metals can form different oxides, and oxidation of such metals produces a multilayer oxide scale on the metal, where the oxide layer containing the highest concentration of metal is in contact with the surface of the metal, while the oxide layer containing the highest concentration of oxygen is in contact with the gas or oxygen to which the surface of the metal is exposed. In paper [4], F. Gesmundo and F. Viani analyzed the parabolic growth of three-layer oxide scales on

* Corresponding author: mailto:baker@math.uiowa.edu

those metals which can form three oxides. They obtained the following non-linear three-dimensional dynamical system as a model for the growth of such scales:

$$\begin{aligned} \dot{q}_1 &= m_1 \frac{K_1}{2q_1} - \frac{m_1 - 1}{m_1} \frac{K_2}{2q_2}, \\ \dot{q}_2 &= -m_1 \frac{K_1}{2q_1} + \left(\frac{m_1 - 1}{m_1} + \frac{m_2}{m_1} \right) \frac{K_2}{2q_2} - \frac{m_2 - 1}{m_2} \frac{K_3}{2q_3}, \\ \dot{q}_3 &= -\frac{m_2}{m_1} \frac{K_2}{2q_2} + \frac{K_3}{2q_3}. \end{aligned} \tag{1}$$

Here $K_i > 0$ $(i = 1, 2, 3)$ are rate constants, m_1, m_2 are parameters, $q_i > 0$ is the weight of oxygen contained in oxide i per unit area, and $\dot{q}_i$ $(i = 1, 2, 3)$ is the derivative of q_i with respect to time, t.

In the present paper we present the following n-dimensional generalization of the 3-dimensional system (1). This n-dimensional dynamical system models the parabolic growth of n-oxide scales on pure metals

$$\begin{aligned} \dot{q}_1 &= m_1 \frac{K_1}{2q_1} - \frac{m_1 - 1}{m_1} \frac{K_2}{2q_2}, \\ &\dots \\ \dot{q}_i &= -\frac{m_{i-1}}{m_{i-2}} \frac{K_{i-1}}{2q_{i-1}} + \left(\frac{m_{i-1} - 1}{m_{i-1}} + \frac{m_i}{m_{i-1}} \right) \frac{K_i}{2q_i} - \frac{m_i - 1}{m_i} \frac{K_{i+1}}{2q_{i+1}}, \quad 1 < i < n, \\ &\dots \\ \dot{q}_n &= -\frac{m_{n-1}}{m_{n-2}} \frac{K_{n-1}}{2q_{n-1}} + \frac{K_n}{2q_n}. \end{aligned} \tag{2}$$

Here, for $i = 1, ..., n$, $K_i > 0$ are rate constants, m_i are parameters (with $m_0 = 1$), and q_i is the weight of oxygen contained in oxide i per unit area.

Theorem 1.1 below is the main result of the present paper; this theorem provides a qualitative analysis of the n-dimensional system (2).

Theorem 1.1 *Assume that in the dynamical system (2), we have $n \geq 3$, and $m_i > 1$, $i = 1, ..., n$. Then every solution $\mathbf{p}[0, a] \to [0, a] \to \mathbf{R}^n_{++}$, $0 < a < +\infty$, of (2) extends uniquely to a solution $\mathbf{p} : [0, +\infty) \to \mathbf{R}^n_{++}$ such that $\lim_{t \to +\infty} p_i(t) = +\infty$, $i = 1, ..., n$, and this solution is eventually monotone strictly increasing on $[0 + \infty)$. Moreover, the system (2) has a unique parabolic solution $q_i(t) = c_i \sqrt{t}$, $c_i > 0$, $i = 1, ..., n$, $0 < t < +\infty$. Finally, if $\mathbf{p} : [0, +\infty) \to \mathbf{R}^n_{++}$ is any other solution of (2), then*

$$\lim_{t \to +\infty} \|\mathbf{p}(t) - \mathbf{q}(t)\| = 0.$$

2 Preliminaries

In [1], [2], and [3], the present author (*et al.*) studied the following n-dimensional non-linear dynamical system, of which (1) and (2) are special cases:

$$\dot{q}_i = -\sum_{j=1}^{n} \frac{a_{ij}}{q_j}, \quad q_i(t) > 0, \quad i = 1, \dots, n. \tag{3}$$

In [3], we established the following key result ([3], Corollary I).

Theorem 2.1 *Assume that the $n \times n$ matrix $A = (a_{ij})$ in (3) satisfies the following four conditions:*

(i) $\det A \neq 0$ *and* $a_{ij} \geq 0$, *for* $i \neq j$*;*

(ii) *A is irreducible;*

(iii) *for all* $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbf{R}^n_+$, *if* $x_i \sum_{j=1}^n a_{ij}x_j = 0$ *for* $i = 1, \ldots, n$, *then* $\mathbf{x} = 0$*;*

(iv) *every real eigenvalue of A is negative.*

Then every solution of (3) of the form

$$\mathbf{q} = (q_1, \ldots, q_n) : [0, a] \to \mathbf{R}^n_{++}, \quad 0 < a < +\infty,$$

extends uniquely to a solution

$$\mathbf{q} : [0, +\infty) \to \mathbf{R}^n_{++},$$

such that

$$\lim_{t \to +\infty} q_i(t) = \infty, \quad i = 1, \ldots, n.$$

Moreover, if $\mathbf{r}(t)$, $t \in [0, +\infty)$, *is any other solution of (3) in* $\mathbf{R}^n_{++}$, *then we have*

$$\lim_{t \to +\infty} \| \mathbf{q}(t) - \mathbf{r}(t) \| = 0.$$

Finally, if the matrix A is tridiagonal, then any solution solution, $\mathbf{q}(t)$, $t \in [0, +\infty)$ *of (3) in* $\mathbf{R}^n_{++}$ *is eventually monotonically increasing on* $[0, +\infty)$.

Definition 2.1 Let $A = (a_{ij})$ be the tridiagonal matrix whose entries are defined as follows, where the index i has range $1 < i < n$,

$$a_{11} = -\frac{m_1 K_1}{2}, \qquad a_{12} = \frac{m_1 - 1}{m_1}\frac{K_2}{2};$$
$$a_{i,i-1} = \frac{m_{i-1}}{m_{i-1}}\frac{K_{i-1}}{2}, \qquad a_{i,i} = -\left(\frac{m_{i-1} - 1}{m_{i-1}} + \frac{m_i}{m_{i-1}}\right)\frac{K_i}{2}, \quad a_{i,i+1} = \frac{m_i - 1}{m_i}\frac{K_{i+1}}{2};$$
$$a_{n,n-1} = \frac{m_{n-1}}{m_{n-2}}\frac{K_{n-1}}{2}, \quad a_{n,n} = -\frac{K_n}{2}.$$

Theorem 2.2 *Let* $(x_1, x_2, \cdots, x_n) \in \mathbf{R}_+$ *be arbitrary. Let* $1 < j < n$, *and assume that the following equations hold:*

$$0 = \frac{m_1 K_1}{2}x_1 - \frac{m_1 - 1}{m_1}\frac{K_2}{2}x_2, \tag{4}$$
$$0 = -\frac{m_1 K_1}{2}x_1 + \left(\frac{m_1 - 1}{m_1} + \frac{m_2}{m_1}\right)\frac{K_2}{2}x_2 - \frac{m_2 - 1}{m_2}\frac{K_3}{2}x_3,$$
$$\ldots$$
$$0 = -\frac{m_{j-1}}{m_{j-2}}\frac{K_{j-1}}{2}x_j + \left(\frac{m_{j-1} - 1}{m_{j-1}i} + \frac{m_j}{m_{j-1}}\right)\frac{K_j}{2}x_j - \frac{m_j - 1}{m_j}\frac{K_{j+1}}{2}x_{j+1}.$$

Then we must have

$$0 = \frac{m_j}{m_{j-1}}\frac{K_j}{2}x_j - \frac{m_j - 1}{m_j}\frac{K_{j+1}}{2}x_{j+1}. \tag{5}$$

Note that in terms of the matrix A, the system (4) can be written as follows:

$$-\sum_{k=1}^{n} a_{ik}x_k = 0, \quad i = 1, \cdots, j. \tag{6}$$

Proof. It is easy to prove this by using mathematical induction on $2 \le j \le n-1$. $\square$

Theorem 2.3 *Let $(x_1, x_2, \cdots, x_n) \in \mathbf{R}_+$ be arbitrary. Let $1 < i < j < n$, and assume that the following system of equations is satisfied:*

$$\begin{aligned}
0 &= -\frac{m_{i-1}}{m_{i-1}}\frac{K_{i-1}}{2}x_{i-1} + \left(\frac{m_{i-1}-1}{m_{i-1}} + \frac{m_i}{m_{i-1}}\right)\frac{K_i}{2}x_i - \frac{m_i - 1}{m_i}\frac{K_{i+1}}{2}x_{i+1}, \\
&\cdots \\
0 &= -\frac{m_{j-1}}{m_{j-2}}\frac{K_{j-1}}{2}x_{j-1} + \left(\frac{m_{j-1}-1}{m_{j-1}} + \frac{m_j}{m_{j-1}}\right)\frac{K_j}{2}x_j - \frac{m_j - 1}{m_j}\frac{K_{j+1}}{2}x_{j+1}.
\end{aligned} \tag{7}$$

Then we must have

$$0 = -\frac{m_{i-1}}{m_{i-2}}\frac{K_{i-1}}{2}x_{i-1} + \frac{m_{i-1}-1}{m_{i-1}}\frac{K_i}{2}x_i + \frac{m_j}{m_{j-1}}\frac{K_j}{2}x_j - \frac{m_j - 1}{m_j}\frac{K_{j+1}}{2}x_{j+1}. \tag{8}$$

Note that Equation (7) is equivalent to the following system:

$$-\sum_{k=1}^{n} a_{pk}x_k = 0, \quad p = i, \cdots, j. \tag{9}$$

Proof. This theorem is easily proved using induction on $3 \le j < n$. $\square$

Theorem 2.4 *For all $(x_1, \cdots, x_n) \in \mathbf{R}^n_+$, if*

$$x_i \sum_{j=1}^{n} a_{ij}x_j = 0, \ \textit{for } i =, \cdots, n,$$

then $x_i = 0$, for $i = 1, \cdots, n$.

Proof. Assume that

$$x_i \sum_{j=1}^{n} a_{ij}x_j = 0, \ \text{for } i = 1, \cdots, n. \tag{10}$$

One of the following cases must hold.—We will show that only **Case 1** does not lead to a contradiction.
Case 1: In this case, $x_i = 0$, for $i = 1, \cdots, n$.
Case 2: In this case,

$$x_i \neq 0, \ \text{for } i = 1, \cdots, n.$$

Then (10) implies that

$$-\sum_{j=1}^{n} a_{ij}x_j = 0, \ \text{for } i = 1, \cdots, n-1. \tag{11}$$

Hence, by taking $j = n - 1$ in Theorem 2.2, we see that (9) implies

$$0 = \frac{m_{n-1}}{m_{n-2}} \frac{K_{n-1}}{2} x_{n-1} - \frac{m_{n-1} - 1}{m_{n-1}} \frac{K_n}{2} x_n.$$

But (10) also implies that

$$0 = \frac{m_{n-1}}{m_{n-2}} \frac{K_{n-1}}{2} x_{n-1} - \frac{K_n}{2} x_n.$$

Adding together these last two equations produces

$$0 = -\frac{1}{m_{n-1}} \frac{K_n}{2} x_n.$$

This contradicts the assumption that $x_n \neq 0$.
Case 3: In this case, for some $1 \leq j < n$, we have

$$x_1, \cdots, x_j \neq 0; \quad x_{j+1}, \cdots, x_n = 0.$$

Hence (10) implies that

$$-\sum_{k=1}^{n} a_{ik} x_k = 0, \quad i = 1, \cdots, j. \tag{12}$$

If $j = 1$, then (12) is equivalent to

$$0 = \frac{m_1 K_1}{2} x_1 - \frac{m_1 - 1}{m_1} \frac{K_2}{2} x_2.$$

But if $j = 1$, then $x_2 = 0$, so we get $0 = -\frac{m_1 K_1}{2} x_1$, which contradicts $x_1 \neq 0$. Hence in (12) we may assume that $1 < j < n$. Then Theorem 2.2 implies that

$$0 = \frac{m_j}{m_{j-1}} \frac{K_j}{2} x_j - \frac{m_j - 1}{m_j} \frac{K_{j+1}}{2} x_{j+1}.$$

By assumption, $x_{j+1} = 0$, hence we have

$$0 = \frac{m_j}{m_{j-1}} \frac{K_j}{2} x_j,$$

i.e., $x_j = 0$. This contradiction shows that **Case 3** can not hold.
Case 4: In this case, for some $1 < i \leq n$, we have

$$x_1, \cdots, x_{i-1} = 0; x_i, \cdots, x_n \neq 0.$$

Then (10) implies that

$$-\sum_{k=1}^{n} a_{pk} x_k = 0, \quad p = i, \cdots, n. \tag{13}$$

First, assume that $i = n$ or $i = n - 1$. If $i = n$, then $x_{n-1} = 0$, and (13) implies that

$$0 = \frac{m_{n-1}}{m_{n-2}} \frac{K_{n-1}}{2} x_{n-1} - \frac{K_n}{2} x_n.$$

This leads to the contradiction that $-\frac{K_n}{2}x_n = 0$. If $i = n-1$, then (11) implies that

$$0 = -\frac{m_{n-2}}{m_{n-3}}\frac{K_{n-2}}{2}x_{n-2} + \left(\frac{m_{n-2}-1}{m_{n-2}} + \frac{m_{n-1}}{m_{n-2}}\right)\frac{K_{n-1}}{2}x_{n-1} - \frac{m_{n-1}-1}{m_{n-1}}\frac{K_n}{2}x_n,$$
$$0 = -\frac{m_{n-1}}{m_{n-2}}\frac{K_{n-1}}{2}x_{n-1} + \frac{K_n}{2}x_n.$$

Adding together these two equations, and taking into consideration that $x_{n-2} = 0$, we see that

$$0 = \frac{m_{n-1}}{m_{n-2}}\frac{K_{n-1}}{2}x_{n-1} + \frac{1}{m_{n-1}}\frac{K_n}{2}x_n.$$

This contradicts the assumption that $x_{n-1}, x_n \neq 0$.—Thus, we may assume that $1 < i < n-1$. Then (13) implies

$$\sum_{k=1}^{n} a_{pk}x_k = 0, \quad p = i, \cdots, n-1. \tag{14}$$

Now, (14) allows us to apply Theorem 2.3 to the case where $1 < i < j = n-1 < n$, producing

$$0 = -\frac{m_{i-1}}{m_{i-2}}\frac{K_{i-1}}{2}x_{i-1} + \frac{m_{i-1}-1}{m_{i-1}}\frac{K_i}{2}x_i + \frac{m_{n-1}}{m_{n-2}}\frac{K_{n-1}}{2}x_{n-1} - \frac{m_{n-1}-1}{m_{n-1}}\frac{K_n}{2}x_n.$$

But (14) also implies that

$$0 = -\frac{m_{n-1}}{m_{n-2}}\frac{K_{n-1}}{2}x_{n-1} + \frac{K_n}{2}x_n.$$

Adding together these last two equations, we obtain

$$0 = \frac{m_{i-1}-1}{m_{i-1}}\frac{K_i}{2}x_i + \frac{1}{m_{n-1}}\frac{K_n}{2}x_n.$$

Hence, $x_i, x_n = 0$. This contradicts our assumption that $x_i, \cdots, x_n \neq 0$. Therefore, **Case 4** can not hold.

Case 5: In this case, there exist $1 < i < j < n$ such that

$$x_{i-1} = 0; x_i, \cdots, x_j \neq 0; x_{j+1} = 0.$$

Then (10) implies that

$$-\sum_{k=1}^{n} a_{pk}x_k = 0, \quad p = i, \cdots, j. \tag{15}$$

Because (15) implies (9), we may invoke Theorem 2.3, obtaining

$$0 = -\frac{m_{i-1}}{m_{i-2}}\frac{K_{i-1}}{2}x_{i-1} + \frac{m_{i-1}-1}{m_{i-1}} + \frac{m_j}{m_{j-1}}\frac{K_j}{2}x_j - \frac{m_j-1}{m_j}\frac{K_{j+1}}{2}x_{j+1}.$$

Because $x_{i-1} = x_{j+1}$, this equation implies that

$$0 = \frac{m_{i-1}-1}{m_{i-1}}\frac{K_i}{2}x_i + \frac{m_j}{m_{j-1}}\frac{K_j}{2}x_j.$$

But then we have the contradiction that $x_i = x_j = 0$. This shows that **Case 5** can not hold.

Case 6: In this final case, there exists $1 < i < n$ such that

$$x_{i-1} = 0; x_i \neq 0; x_{i+1} = 0.$$

It then follows from (10) that

$$0 = -\frac{m_{i-1}}{m_{i-2}}\frac{K_{i-1}}{2}x_{i-1} + \left(\frac{m_{i-1}-1}{m_{i-1}} + \frac{m_i}{m_{i-1}}\right)\frac{K_i}{2}x_i + \frac{m_i - 1}{m_i}\frac{K_{i+1}}{2}x_{i+1}.$$

Because $x_{i-1} = x_{i+1} = 0$, we obtain

$$0 = \left(\frac{m_{i-1}-1}{m_{i-1}} + \frac{m_i}{m_{i-1}}\right)\frac{K_i}{2}x_i.$$

That is, $x_i = 0$. This contradiction shows that **Case 6** can not hold.

Because the above cases are the only possible cases consistent with assumption (10), we conclude that **Case 1** must hold. This completes the proof of the theorem. □

Definition 2.2 Let $B = (b_{ij})$ be the tridiagonal matrix whose entries are defined as follows, where the index i has range $1 < i < n$,

$$\begin{aligned} &b_{11} = -m_1, && b_{12} = \frac{m_1 - 1}{m_1}; && \\ &b_{i,i-1} = \frac{m_{i-1}}{m_{i-1}}, && b_{i,i} = -\left(\frac{m_{i-1}-1}{m_{i-1}} + \frac{m_i}{m_{i-1}}\right), && b_{i,i+1} = \frac{m_i - 1}{m_i}\frac{K_{i+1}}{2}; \\ &b_{n,n-1} = \frac{m_{n-1}}{m_{n-2}}, && b_{n,n} = -1. && \end{aligned}$$

Note that

$$A = B \text{ diag}\left(\frac{K_1}{2}, \cdots, \frac{K_n}{2}\right).$$

Define $P = (p_{ij})$ to be following matrix:

$$P = \begin{pmatrix} \frac{1}{2} & \frac{m_1-1}{2m_1} & \frac{(m_1-1)(m_2-1)}{2m_1m_2} & \cdots & \frac{(m_1-1)(m_2-1)\cdots(m_n-1)}{2m_1m_2\cdots m_n} \\ \frac{m_1}{2} & \frac{m_1}{2} & \frac{m_1(m_2-1)}{2m_2} & \cdots & \frac{m_1(m_2-1)\cdots(m_n-1)}{2m_2m_3\cdots m_n} \\ \frac{m_2}{2} & \frac{m_2}{2} & \frac{m_2}{2} & \cdots & \frac{m_2(m_3-1)\cdots(m_n-1)}{2m_3\cdots m_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{m_{n-1}}{2} & \frac{m_{n-1}}{2} & \frac{m_{n-1}}{2} & \cdots & \frac{m_{n-1}(m_n-1)}{2m_n} \\ \frac{m_n}{2} & \frac{m_n}{2} & \frac{m_n}{2} & \cdots & \frac{m_n}{2} \end{pmatrix}$$

Observe that the entries of P are given by

$$P_{ij} = \begin{cases} \frac{m_{i-1}(m_i-1)\cdots(m_{j-1}-1)}{2m_i\cdots m_{j-1}}, & \text{if } 1 \le i < j \le n; \\ \frac{m_{i-1}}{2}, & \text{if } 1 \le j \le i \le n. \end{cases}$$

Theorem 2.5 *The matrices B and P satisfy*

$$BP = -\frac{I}{2}.$$

Hence B is invertible, with inverse

$$B^{-1} = -2P.$$

Proof. We must prove that for all $1 \le i \le j \le n$,

$$(BP)_{ij} = -\frac{1}{2}\delta_{ij}.$$

The following cases exhaust all possible cases for the pair $1 \le i \le j \le n$.
Case 1: In this case, $i = j = 1$. We then have

$$\begin{aligned}(BP)_{11} &= b_{11}P_{11} + b_{12}P_{21}\\ &= -\frac{m_1}{2} + \frac{m_1 - 1}{m_1}\frac{m_1}{2}\\ &= -\frac{1}{2}.\end{aligned}$$

Case 2: In this case, $i = j = n$. Then

$$\begin{aligned}(BP)_{nn} &= b_{n,n-1}P_{n-1,n} + b_{nn}P_{nn}\\ &= \frac{m_{n-1}}{m_{n-2}}\frac{m_{n-2}(m_{n-1} - 1)}{m_{n-1}} - \frac{m_{n-1}}{2}\\ &= \frac{m_{n-1} - 1}{2} - \frac{m_{n-1}}{2}\\ &= -\frac{1}{2}.\end{aligned}$$

Case 3: In this case, $1 < i = j < n$. We then obtain

$$\begin{aligned}(BP)_{ii} &= b_{i,i-1}P_{i-1,i} + b_{ii}P_{ii} + b_{i,i+1}P_{i+1,i}\\ &= \frac{m_{i-1}}{m_{i-2}}\frac{m_{i-2}(m_{i-1} - 1)}{2m_{i-1}} - \left(\frac{m_{i-1} - 1}{m_{i-1}} + \frac{m_i}{m_{i-1}}\right)\frac{m_{i-1}}{2} + \frac{m_i - 1}{m_i}\frac{m_i}{2}\\ &= \frac{m_{i-1} - 1}{2} - \frac{m_{i-1} - 1 + m_i}{2} + \frac{m_i - 1}{2}\\ &= -\frac{1}{2}.\end{aligned}$$

Case 4: In this case, $1 < i + 1 < j \le n$. We obtain

$$\begin{aligned}(PB)_{ij} &= b_{i,i-1}P_{i-1,j} + b_{ii}P_{i,j} + b_{i,i+1}P_{i+1,j}\\ &= \frac{m_{i-1}}{m_{i-2}}\frac{m_{i-2}(m_{i-1} - 1)\cdots(m_{j-1} - 1)}{2m_{i-1}\cdots m_{j-1}} - \left(\frac{m_{i-1} - 1}{m_{i-1}} + \frac{m_i}{m_{i-1}}\right)\times\\ &\quad\times\frac{m_{i-1}(m_i - 1)\cdots(m_{j-1} - 1)}{2m_i\cdots m_{j-1}} + \frac{m_i(m_{i+1} - 1)\cdots(m_{j-1} - 1)}{2m_{i+1}\cdots m_{j-1}}\\ &= \frac{(m_{i-1} - 1)\cdots(m_{j-1} - 1)}{2m_i\cdots m_{j-1}} - ([m_{i-1} - 1] + m_i)\frac{(m_i - 1)\cdots(m_{j-1} - 1)}{2m_i\cdots m_{j-1}} +\\ &\quad+ \frac{(m_i - 1)(m_{i+1} - 1)\cdots(m_{j-1} - 1)}{2m_{i+1}\cdots m_{j-1}}\\ &= 0.\end{aligned}$$

Case 5: In this case, $1 < i < j = i+1 \leq n$. We then have

$$\begin{aligned}
(PB)_{ij} &= b_{i,i-1}P_{i-1,j} + b_{ii}P_{ij} + b_{i,i+1}P_{i+1,j} \\
&= b_{i,i-1}P_{i-1,i+1} + b_{ii}P_{i,i+1} + b_{i,i+1}P_{i+1,i+1} \\
&= \frac{m_i - 1}{m_{i-2}} \frac{m_{i-2}(m_{i-1}-1)(m_i-1)}{2m_{i-1}m_i} - \left(\frac{m_{i-1}-1}{m_{i-1}} + \frac{m_i}{m_{i-1}}\right)\frac{m_{i-1}(m_i-1)}{2m_i} \\
&\quad + \frac{m_i - 1}{m_i}\frac{m_i}{2} \\
&= \frac{(m_{i-1}-1)(m_i-1)}{2m_i} - \frac{(m_{i-1}-1)(m_i-1)}{2m_i} - m_i\frac{(m_i-1)}{2m_i} + \frac{(m_i-1)}{2} \\
&= 0.
\end{aligned}$$

Case 6: In this case, we have $i = 1 < 2 < j \leq n$. We see that

$$\begin{aligned}
(BP)_{ij} = (BP)_{1j} &= b_{11}P_{ij} + b_{12}P_{2j} \\
&= -m_1\frac{(m_1-1)\cdots(m_{j-1}-1)}{m_1\cdots m_{j-1}} + \frac{m_1-1}{m_1}\frac{m_1(m_2-1)\cdots(m_{j-1}-1)}{m_2\cdots m_{j-1}} \\
&= 0.
\end{aligned}$$

Case 7: In this case, $1 = i < 2 = j \leq n$. We then obtain

$$\begin{aligned}
(BP)_{ij} = (BP)_{12} &= b_{11}P_{12} + b_{12}P_{22} \\
&= -m_1\frac{(m_1-1)}{m_1} + \frac{m_1-1}{m_1}\frac{m_2}{2} \\
&= 0.
\end{aligned}$$

Case 8: In this case, $1 \leq j < i < n$. We then have

$$\begin{aligned}
(BP)_{ij} &= b_{i,i-1}P_{i-1,j} + b_{ii}P_{ij} + b_{i,i+1}P_{i+1,j} \\
&= \frac{m_{i-1}}{m_{i-2}}\frac{m_{i-2}}{2} - \left(\frac{m_{i-1}-1}{m_{i-1}} + \frac{m_i}{m_{i-1}}\right)\frac{m_{i-1}}{2} + \frac{m_i-1}{m_i}\frac{m_i}{2} \\
&= \frac{m_{i-1}}{2} - \frac{(m_{i-1}-1+m_i)}{2} + \frac{m_i-1}{2} \\
&= 0.
\end{aligned}$$

Case 9: In this final case, $1 \leq j < i = n$. Consequently,

$$\begin{aligned}
(BP)_{ij} = (BP)_{nj} &= b_{n,n-1}P_{n-1,j} + bnnP_{nj} \\
&= \frac{m_{n-1}}{m_{n-2}}\frac{m_{n-2}}{2} - \frac{m_{n-1}}{2} \\
&= 0.
\end{aligned}$$

The proof of the theorem is now complete. □

Theorem 2.6 *Every real eigenvalue of the matrix A is negative.*

Proof. We apply Gershgoin's Circle Theorem to the transpose of A, concluding that the eigenvalues of A are contained in the union of the following closed disks in the complex plane:

D_1: center $= -\dfrac{m_1K_1}{2}$, radius $= \dfrac{m_1K_1}{2}$;
D_i: center $= -\left(\dfrac{m_{i-1}-1}{m_{i-i}} + \dfrac{m_i}{m_{i-1}}\right)\dfrac{K_i}{2}$, radius $= \left(\dfrac{m_{i-1}-1}{m_{i-i}} + \dfrac{m_i}{m_{i-1}}\right)\dfrac{K_i}{2}$, $1 < i < n$;
D_n: center $= -\dfrac{K_n}{2}$, radius $= \dfrac{m_{n-1}-1}{m_{n-1}}\dfrac{K_n}{2}$.

Because $m_i > 1$ for $i = 1 \cdots n$, these disks are all in the closed left half-plane, hence all eigenvalues have non-positive real parts. By Theorem 2.5 the matrix B is invertible, and hence A is invertible, because $A = B \text{ diag}\left(\dfrac{K_1}{2}, \cdots, \dfrac{K_n}{2}\right)$, with $K_i > 0$, $i = 1, \cdots, n$. Therefore, all real eigenvalues of A are negative. □

Theorem 2.7 *The matrix A satisfies conditions* (i)-(iv) *of Theorem* 2.1.

Proof. By Theorem 2.5, A is invertible, hence condition (i) of Theorem 2.1 is satisfied. Because A is tridiagonal and $a_{ij} \neq 0$ whenever $|i - j| = 1$, condition (ii) of Theorem 2.1 holds. By Theorem 2.4, condition (iii) of Theorem 2.1 is satisfied. Finally, Theorem 2.6 implies that condition (iv) of Theorem 2.1 holds. □

3 Proof of Theorem 1.1

In this section we prove Theorem 1.1. We first prove that under the hypothesis of Theorem 1.1, there exists a unique parabolic solution of (2). We prove the remainder of Theorem 1.1 by an application of Theorem 2.1.

The next theorem is a key to proving the existence of a parabolic solution of (2). The following notation is used. We denote by $\triangle^m$ the *standard m-simplex*, i.e., the set of all points $\mathbf{x} = (x_1, \cdots, x_{m+1}) \in \mathbf{R}_+^{m+1}$ such that $\sum\limits_{i=1}^{m+1} x_i = 1$; we denote the boundary of $\triangle^m$ by $\partial\triangle^m$. Let $e_1 = (1, 0, \cdots, 0), \cdots, e_{m+1} = (0, \cdots, 0, 1)$ be the standard basis for $\triangle^m$. For $1 \leq i, j \leq m+1$, we let $[e_i, e_j]$ be the boundary simplex determined by the pair e_i, e_j, that is, $[e_i, e_j]$ is the convex hull of the pair e_i, e_j. Observe that $\partial\triangle^m$ is the union of all the boundaries $[e_i, e_j]$, $i \neq j$, $1 \leq i, j \leq m+1$.

Theorem 3.1 *Let $f : \triangle^m \to \triangle^m$ be a continuous map which maps each vertex to itself and each edge into itself. Then $f(\triangle^m) = \triangle^m$.*

Proof. Standard theorems in algebraic topology show that any extension to the simplex, of a continuous map of the boundary of a simplex to itself having nonzero degree, must map onto the simplex. By looking at each edge it is easy to prove that the restriction of f to the boundary is a map of the boundary to itself which is homotopic to the identity; it is well known that this implies degree 1. Therefore f is onto. □

Theorem 3.2 *Define $f = (f_1, \cdots, f_n) : \triangle^{n-1} \to \triangle^{n-1}$ by*

$$f_i(\mathbf{x}) = \left(x_i \sum_{j=1}^{n} P_{ij}x_j\right) \Bigg/ \left(\sum_{k=1}^{n} x_k \sum_{j=1}^{n} P_{kj}x_j\right), \quad \mathbf{x} = (x_1, \cdots, x_n) \in \triangle^{n-1},\ i = 1, \cdots, n.$$

Then f maps $\triangle^{n-1}$ onto itself.

Proof. First of all, observe that f sends $\triangle^{n-1}$ into itself, because the entries of P are positive. It is easy to check that f is continuous and maps each vertex of the simplex to itself and each edge of the simplex into itself. Therefore Theorem 3.1 implies f is onto. $\square$

Theorem 3.3 *There exists a unique parabolic solution of (2).*

Proof. Uniqueness: Let $\mathbf{q} = (q_1, \cdots, q_n)$ and $\mathbf{r} = (r_1, \cdots, r_n)$ be two parabolic solutions of (2), with $q_i(t) = c_i\sqrt{t}, r_i(t) = d_i\sqrt{t}, c_i > 0, d_i > 0, i = 1, \cdots, n$. By Theorem 2.7, the matrix A in (2) satisfies conditions (i)–(iv) of Theorem 2.1, hence by that theorem we have $\lim\limits_{t\to+\infty} |q_i(t) - r_i(t)| = 0$, for $i = 1, \cdots, n$. But $|q_i(t) - r_i(t)| = \sqrt{t}|c_i - d_i|$, $i = 1, \cdots, n$. We conclude that $c_i = d_i$, for $i = 1, \cdots, n$. Existence: Let K_i be as in (2) and define $\mathbf{y} \in \triangle^{n-1}$ by

$$y_i = K_i \Bigg/ \left(\sum_{j=1}^{n} K_j\right), \quad i = 1, \cdots, n.$$

Let $f : \triangle^{n-1} \to \triangle^{n-1}$ be defined as in Theorem 3.2; then by that theorem there exists a point $\mathbf{u} \in \triangle^{n-1}$, such that $\mathbf{y} = f(\mathbf{u})$. Define ζ, η by

$$\zeta = \left(\sum_{j=1}^{n} K_j\right)^{\frac{1}{2}}, \quad \eta = \left(\sum_{i=1}^{n} u_i \sum_{j=1}^{n} P_{ij} u_j\right)^{\frac{1}{2}}.$$

Let $\mathbf{c} = (\zeta/\eta)\mathbf{x}$. Then $\mathbf{y} = f(\mathbf{u})$ implies $K_i = c_i \sum\limits_{j=1}^{n} P_{ij} c_j$, for $i = 1, \cdots, n$. This last set of equations is equivalent to:

$$\frac{1}{2} c_i = -\sum_{j=1}^{n} a_{ij} \left(\frac{1}{c_j}\right), \quad i = 1, \cdots, n.$$

Define $\mathbf{q}(t), t \in (0, +\infty)$, by $q_i(t) = c_i\sqrt{t}, i = 1, \cdots, n$. The preceding equations imply that $\mathbf{q}(t)$ is a parabolic solution of (2). This proves the theorem. $\square$

3.1 Proof of Theorem 1.1

To prove Theorem 1.1, let $\mathbf{p}(t) = (p_1(t), \cdots, p_n(t))$, $t \in [0, a]$, $0 < a < +\infty$, be a solution of (2) in $\mathbf{R}^3_{++}$. By Theorem 2.7, the matrix A in (2) satisfies conditions (i)-(iv) of Theorem 2.1, hence, by that theorem, there exists a unique extension of $\mathbf{p}(t)$, $t \in [0, a]$, to a solution $\mathbf{p}(t) = (p_1(t), \cdots, p_n(t))$, $t \in [0, +\infty)$, of (2) in $\mathbf{R}^3_{++}$, such that $\lim\limits_{t\to+\infty} p_i(t) = +\infty$, $i = 1, \cdots, n$. Moreover, if $\mathbf{r}(t)$, $t \in [0, \infty)$ is any other solution of (2), then by Theorem 2.1, we have

$$\lim_{t\to\infty} ||\mathbf{p}(t) - \mathbf{r}(t)|| = 0.$$

Because the matrix A of the system (2) is tridiagonal, Theorem 2.1 implies that the extended solution $\mathbf{p}(t)$ is eventually monotone increasing on $[0, +\infty)$. By Theorem 3.3, there exists a unique parabolic solution $\mathbf{q}(t) = (c_1\sqrt{t}, \cdots, c_n\sqrt{t})$, $(c_1, \cdots, c_n) > 0$, $t \in (0, +\infty)$, of (2). This completes the proof of Theorem 1.1.

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