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Asymptotic Stability of Some Class of Affine Nonlinear Control Systems through Partial Feedback Linearization

Firman *, Syamsuddin Toaha, Muh Nur

Department of Mathematics, Faculty of Mathematics and Natural Science, Hasanuddin University, Indonesia

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Abstract: The problem of asymptotic stability for some class of nonlinear control systems, where the relative degree of the system is well defined, with relative degree being 1 and $n-1$, n is the dimension of the system, is addressed in this paper. To solve the problem, we will design an input control. For the design of input controls, the system will be transformed through partial feedback linearization such that the zero dynamic of the system with respect to a new state is asymptotically stable and the new state is a linear combination of state variables.

Keywords: *relative degree of system; partial feedback linearization; zero dynamic; asymptotic stability.*

Mathematics Subject Classification (2010): 93C10, 93D20.

1 Introduction

In the analysis for nonlinear control systems, there is no general method which can be applied to any nonlinear control system in designing the control input for solving the stability problems. Therefore, in general, the researchers describe some particular nonlinear classes only. Recently, stability problems for nonlinear control systems have been intensively investigated. J. Naiborhu and K. Shimizu [1] proposed a dynamic feedback control for the asymptotic stability of a nonlinear class, where its unforced dynamic is asymptotically stable. In 2004, P. Chen et al. [2] and L. Diao et al. [3] introduced the problem of stability through system transformation, where the transformation of the system is made through dynamic feedback. One of popular methods for solving stability

* Corresponding author: mailto:firman.math11@gmail.com

problems is the input-output linearization method. Some researchers studied the stability problems of a nonlinear control system using the input-output linearization method, for example, Ricardo Marino and Patrizio Tomei [4] discussed the stability of a lower triangular nonlinear control system. Its stability control was the dynamic feedback of order $n + 2(r - 1)$ (n is the system order, r is the relative degree). Results on stabilization of nonlinear lower triangular systems with uncertainties in the output feedback form have been presented in [5] and [6]. In [7], J. Naiborhu et al. discussed the asymptotic stability problem for a nonlinear class, where its control's design used exact linearization. Furthermore, in [8], the authors have addressed the problem of stabilization for a class of nonlinear control systems with the relative degree of the system being not well defined. Then, in [9], the authors have introduced the problem of stabilization for a class of nonlinear systems with uncertainty.

In this paper, we will investigate asymptotic stability of some class of affine nonlinear control systems, with the relative degrees of the system being 1 and $n-1$. For the design of input controls, the system will be transformed through partial feedback linearization.

2 Problem Formulation

Consider the affine nonlinear control system

$$\dot{x}(t) = f_1(x(t)) + f_2(x(t))u, \tag{1}$$

where $x(t) \in \mathcal{R}^n$, $u(t) \in \mathcal{R}$. $f_1 : D \to \mathcal{R}^n$, $f(\vec{0}) = \vec{0}$ and $f_2 : D \to \mathcal{R}^n$ are sufficiently smooth in a domain $D \subset \mathcal{R}^n$.

Let a state $y(t) = f_3(x(t))$, $f_3 : D \to \mathcal{R}$ be sufficiently smooth in a domain $D \subset \mathcal{R}^n$, $f_3(\vec{0}) = 0$.

According to [10], if we have

$$y^{(\rho)}(t) = L^{\rho}_{f_1} f_3(x) + L_{f_2} L^{\rho-1}_{f_1} f_3(x)u, \tag{2}$$

with $L_{f_2} L^k_{f_1} f_3(x) = 0, k = 0, 1, 2, \cdots, \rho - 2$, $L_{f_2} L^{\rho-1}_{f_1} f_3(x) \neq 0$, for all $x \in D_0$, then the relative degree of the system with respect to the state y is ρ, $1 < \rho < n$, in a region $D_0 \subset D$, where $L_{f_2} L^k_{f_1} f_3(x) = \frac{\partial\left(L^k_{f_1} f_3\right)}{\partial x} f_2(x)$, $L^k_{f_1} f_3(x) = L_{f_1} L^{k-1}_{f_1} f_3(x) = \frac{\partial\left(L^{k-1}_{f_1} f_3(x)\right)}{\partial x} f_1(x)$, $\cdots$, $L^2_{f_1} f_3(x) = \frac{\partial\left(L_{f_1} f_3(x)\right)}{\partial x} f_1(x)$, $L_{f_1} f_3(x) = \frac{\partial f_3(x)}{\partial x} f_1(x)$, $L^0_{f_1} f_3(x) = f_3(x)$.

Let the relative degree of the system (1) with respect to the state y be ρ. By partial feedback linearization, the system (1) with respect to the state y can be transformed to

$$\begin{aligned} \dot{z}_k &= z_{k+1}, k = 1, 2, \cdots, \rho - 1, && (3)\\ \dot{z}_\rho &= f(z) + g(z)u, && (4)\\ \dot{z}_{\rho+1} &= q_{\rho+1}(z), && (5)\\ &\vdots \\ \dot{z}_n &= q_n(z) && (6) \end{aligned}$$

with the Jacobian matrix of $z(x)$ being nonsingular at a point x_0, $z = (z_1, z_2, \cdots, z_n)$, $y = z_1$.

Consider the system (5)-(6). If $z_1 = 0$ for all t, then the system (5)–(6) is said to be zero dynamic with respect to the state $y = z_1$.

The following theorem describes the asymptotic stability of the nonlinear system (see [11]).

Theorem 2.1 *Consider a system*

$$\dot{\chi} = f(\chi, \nu), \tag{7}$$
$$\dot{\nu} = g(\nu) \tag{8}$$

and suppose that $\dot{\nu} = g(\nu)$ has an asymptotically stable equilibrium at $\nu = 0$. If $\dot{\chi} = f(\chi, 0)$ has an asymptotically stable equilibrium at $\chi = 0$, then the system (7)-(8) has an asymptotically stable equilibrium at $(\chi, \nu) = (0, 0)$.

In order to use Theorem 1, we need to choose the state variable such that the zero dynamic of the system (1) is asymptotically stable. In this paper, we present asymptotic stability of some class of affine nonlinear control systems, where the relative degree of the system (1) is well defined, with the relative degree being 1 and $n-1$, n is the dimension of the system. For a class of affine nonlinear control systems with the relative degree of the system being 1, the input control design is needed so that the zero dynamic of the system is asymptotically stable. Next, for a class of affine nonlinear control systems with the relative degree of the system being $n-1$, to achieve the stability, we will design an input control when the zero dynamic of the system is asymptotically stable.

3 Main Results

First, we will investigate the asymptotic stability for an affine nonlinear control sytem in the following form:

$$\dot{x} = Ax + \tau u, \quad x(t) \in \mathcal{R}^n, \quad \tau \in \mathcal{R}^n, \quad u(t) \in \mathcal{R}, \tag{9}$$

with

$$A = \begin{pmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ 1 & 0 & \dots & 0 \end{pmatrix}$$

and $\tau_1 = \tau_2 = \dots = 0$, $\tau_{n-1} = -\alpha_1, \alpha_1 > 0$, $\tau_n = \alpha_2 + \phi(x_1), \alpha_2 > 0$, $\phi(0) = 0$.

The following theorem states that the new state variable of the system (9) can be chosen such that the zero dynamic of the system (9) is asymptotically stable.

Theorem 3.1 *Suppose the nonlinear system as in equation (9). Then there exists a state variable $y = a_0 x_1 + a_1 x_2 + \dots + a_{n-2} x_{n-1} + a_{n-1} x_n$ such that the relative degree of the system (9) with respect to the state y is 1. Furthermore, the zero dynamic for the system (9) with respect to the state y is asymptotically stable.*

Proof. Let $y = a_0 x_1 + a_1 x_2 + a_2 x_3 + \ldots + a_{n-2} x_{n-1} + a_{n-1} x_n$. We have

$$\dot{y} = a_0 x_2 + a_1 x_3 + a_2 x_4 + \ldots + a_{n-2} x_n + a_{n-1} x_1 + (a_{n-1}(\alpha_2 + \phi(x_1)) - a_{n-2}\alpha_1)\, u.$$

Thus, the relative degree of the system (9) with respect to the state y is 1, with $a_{n-1}(\alpha_2 + \phi(x_1)) - a_{n-2}\alpha_1 \neq 0$.

Furthermore, the partial feedback linearization is

$$
\begin{aligned}
\dot{y} &= v, & (10)\\
\dot{\eta_1} &= \eta_2, & (11)\\
\dot{\eta_2} &= \eta_3, & (12)\\
&\vdots & (13)\\
\dot{\eta}_{n-2} &= \eta_{n-1}, & (14)\\
\dot{\eta}_{n-1} &= x_n - \alpha_1 u, & (15)
\end{aligned}
$$

where $\eta_1 = x_1$, $\eta_2 = x_2$, ..., $\eta_{n-1} = x_{n-1}$.

We will choose the input control u such that the zero dynamic is asymptotically stable. Next, we choose $v = -k_1 a_0 x_1 - k_2 a_1 x_2 - k_3 a_2 x_3 - \ldots - k_n a_{n-1} x_n$, with $a_{n-1} = 1$. We have

$$
\begin{aligned}
u &= \frac{a_0 x_1 \left(-k_1 - \frac{-1}{a_0}\right) + a_1 x_2 \left(-k_2 - \frac{-a_0}{a_1}\right) + \cdots + a_{n-1} x_n \left(-k_n - a_{n-2}\right)}{\alpha_2 + \phi(x_1) - \alpha_1}\\
&= \frac{-\beta(a_0 x_1 + a_1 x_2 + a_2 x_3 + \cdots + x_n)}{\alpha_2 + \phi(x_1) - a_{n-2}\alpha_1}, \qquad (16)
\end{aligned}
$$

where $\beta = k_1 + \frac{1}{a_0} = k_2 + \frac{a_0}{a_1} = k_3 + \frac{a_1}{a_2} = \ldots = k_n + a_{n-2}$.

Next, if $y(t) = 0$, $\forall t$, then the zero dynamic of the system (9) with respect to the state y is

$$
\begin{aligned}
\dot{\eta_1} &= \eta_2, & (17)\\
\dot{\eta_2} &= \eta_3, & (18)\\
&\vdots & (19)\\
\dot{\eta}_{n-2} &= \eta_{n-1}, & (20)\\
\dot{\eta}_{n-1} &= -a_0\eta_1 - a_1\eta_2 - a_2\eta_3 - \cdots - a_{n-2}\eta_{n-1}. & (21)
\end{aligned}
$$

From Eqs. (17)–(21), it can be written

$$\dot{\eta} = \mathcal{A}_1 \eta,$$

where $\eta = (\eta_1, \eta_2, \ldots, \eta_{n-1})^T$ and

$$
\mathcal{A}_1 = \begin{pmatrix} 0 & 1 & \ldots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \ldots & 1\\ -a_0 & -a_1 & \ldots & -a_{n-2} \end{pmatrix}.
$$

If we choose $a_0, a_1, \ldots, a_{n-2}$ such that all the root of the characteristic polynomial of the matrix $\mathcal{A}_1$ $p(\lambda) = a_0 + a_1\lambda + \cdots + a_{n-2}\lambda^{n-2} + \lambda^{n-1}$ have negative real part, then the zero dynamic of system (9) with respect to the state y is asymptotically stable. Hence, there exits the state variable $y = a_0 x_1 + a_1 x_2 + \cdots + a_{n-1} x_n$ such that the zero dynamic of the system (9) with respect to the state y is asymptotically stable.

Proposition 3.1 *Consider system (9) with the state* $y = a_0x_1 + a_1x_2 + \cdots + a_{n-1}x_n$. *Chose* $a_{n-1} = 1$ *and* $a_0, a_1, \ldots, a_{n-2}$ *such that the polynomial*

$$p(\lambda) = a_0 + a_1\lambda + \ldots + a_{n-2}\lambda^{n-2} + \lambda^{n-1} \tag{22}$$

is Hurwitz. If we choose the new input v *as in Eq. (10) such that* $\dot{y} = v$ *has an asymptotically stable equilibrium at* $y = 0$*, then at using the input control in Eq.(16), the system (10)-(15) has an asymptotically stable equilibrium at* $(y, \eta) = (0, 0))$. *Furthermore, the system (9) has an asymptotically stable equilibrium at* $x = 0$.

Next, the affine nonlinear control system is considered in the following form:

$$\dot{x} = Mx + \tau u + \theta(x_1), x(t) \in \mathcal{R}^n, u(t) \in \mathcal{R}, \tag{23}$$

with $\theta(x_1) \in \mathcal{C}^\infty(\mathcal{R}^n)$, $\theta(0) = 0$, $\tau = (0, 0, \ldots, 0, \tau_{n-1}, \tau_n)^T$, $\tau_{n-1} \neq 0$,

$$\tau_{n-1} = -\tau_n,\ M = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix},\ \theta_1(x_1) + \theta_2(x_1) + \cdots + \theta_n(x_1) = 0.$$

Theorem 3.2 *Suppose the nonlinear system as in equation (23). Let the state variables* $y = \alpha x_1 + x_2 + \cdots + x_n$, $\alpha \neq 1$. *Then the relative degree of the system (23) with respect to the state* y *is* $n - 1$. *Furthermore, the zero dynamic for the system (23) with respect to the state* y *is asymptotically stable.*

Proof. Let $y = ax$, $a = (\alpha, 1, \ldots, 1)$, $x = (x_1, x_2, \ldots, x_n)^T$, $\alpha \neq 1$. We have

$$\begin{aligned}
\dot{y} &= a\dot{x} = aMx + a\theta(x_1), && (24)\\
\ddot{y} &= aM^2x + aM\theta(x_1) + a\frac{d\theta}{dt}, && (25)\\
&\vdots && (26)\\
y^{(n-2)} &= aM^{(n-2)}x + aM^{(n-3)}\theta(x_1) + \cdots + aM(\theta(x_1))^{(n-4)} + a(\theta(x_1)^{(n-3)}, && (27)\\
y^{(n-1)} &= aM^{(n-1)}x + aM^{(n-2)}\theta(x_1) + \cdots \\
&\quad + aM(\theta(x_1))^{(n-3)} + a(\theta(x_1)^{(n-2)} + b_n(1-\alpha)u. && (28)
\end{aligned}$$

Thus, the relative degree of the system (23) with respect to the state y is $n - 1$. Furthermore, the linearized input state for system (23) with respect to the state $y = z_1$ is

$$\begin{aligned}
\dot{z}_k &= z_{k+1}, k = 1, 2, \ldots, n-2, && (29)\\
\dot{z}_{n-1} &= f(z, \eta) + g(z, \eta)u, && (30)\\
\dot{\eta} &= \dot{x}_1 + \dot{x}_2 + \cdots + \dot{x}_n = \eta - x_1 && (31)
\end{aligned}$$

with $f(z, \eta) = aM^{(n-1)}x + aM^{(n-2)}\theta(x_1) + \cdots + aM(\theta(x_1))^{(n-3)} + a(\theta(x_1))^{(n-2)}$,
$g(z, \eta) = b_n(1 - \alpha)$, $\alpha \neq 1$, $z = (z_1, z_2, \ldots, z_{n-1})$.

Furthermore, we will investigate the stability of the zero dynamic of the system (23) with respect to the state $y = z_1$. Consider

$$\eta\dot{\eta} \quad = \quad \eta(\eta - x_1) = \eta^2 - \eta(\frac{z_1 - \eta}{\alpha - 1}). \tag{32}$$

If $z_1 = 0$ and $0 < \alpha < 1$, then

$$\eta\dot{\eta} = \frac{\alpha\eta^2}{\alpha - 1} < 0. \tag{33}$$

Therefore, the zero dynamic of the system (23) with respect to the state $y = z_1$ is asymptotically stable.

Consider the system (29)-(31). If we choose the input control

$$u = \frac{1}{g(z,\eta)}\left(-f(z,\eta) - c_0 z_1 - c_1 z_2 - \ldots - c_{n-2} z_{n-1}\right), \tag{34}$$

then we have the normal form of the system (23) with respect to the state $y = z_1$

$$\dot{z} \quad = \quad \mathcal{B}z, \tag{35}$$

$$\dot{\eta} \quad = \quad q(z,\eta), \tag{36}$$

with

$$\mathcal{B} = \begin{pmatrix} 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \\ -c_0 & -c_1 & \ldots & -c_{n-2} \end{pmatrix},$$

$\dot{\eta} = q(z,\eta)$ in Eq.(31). In particulir, the matrix $\mathcal{B}$ has a characteristic polynomial

$$p(\lambda) = c_0 + c_1\lambda + \cdots + c_{n-2}\lambda^{n-2} + \lambda^{n-1}. \tag{37}$$

Because the zero dynamic of the system (23) with respect to the state $y = z_1$ has an asymptotically stable equilibrium at $\omega = 0$ and if we choose $c_0, c_1, \cdots, c_{n-2}$ such that all the roots of the polynomial $p(\lambda)$ have negative real part, then the system (29)-(31) has an asymptotically stable equilibrium at $(z,\eta) = (0,0)$.

Proposition 3.2 *Consider the zero dynamic of the system (23) with respect to the state $y = z_1$. Let all the roots of the polynomial in Eq.(37) have negative real part. Then, at using the input control in Eq.(34), the system (29)-(31) has an asymptotically stable equilibrium at $(z,\eta) = (0,0)$.*

Example 3.1 Suppose the nonlinear control system is

$$\dot{x_1} \quad = \quad x_2, \tag{38}$$

$$\dot{x_2} \quad = \quad x_3 - \alpha_1 u, \tag{39}$$

$$\dot{x_3} \quad = \quad x_1 + \left(\alpha_2 + x_1^2\right) u. \tag{40}$$

Let $y = c_0x_1 + c_1x_2 + x_3$. Then $\dot{y} = c_0x_2 + c_1x_3 + x_1 + (\alpha_2 + x_1^2 - c_1\alpha_1)u$.

Thus, if $\alpha_2 > c_1\alpha_1$, the relative degree is one for all x. Then, according to (16), the input control is

$$u(t) = \frac{-\beta\left(c_0x_1 + c_1x_2 + c_2x_3 + x_4\right)}{\alpha_2 + x_1^2 - c_1\alpha_1} \tag{41}$$

with $\beta = k_1 + \frac{1}{c_0} = k_2 + \frac{c_0}{c_1} = k_3 + c_1$. If we choose $k_1, k_2, k_3 > 0$ and c_0, c_1 such that all roots of the polynomial $p(\lambda) = c_0 + c_1\lambda + \lambda^2$ are Hurwitz, then the system (38)-(40) has an asymptotically stable equilibrium at $(x_1, x_2, x_3) = (0, 0, 0)$.

Simulation result is shown in Fig.1a for constants $k_1 = \frac{59}{6}$, $k_2 = \frac{44}{5}$,$k_3 = 5$, $c_0 = 6, c_1 = 5$. The initial value $x_1(0) = -1,5$, $x_2(0) = 2,5$, $x_3(0) = -3$.

Example 3.2 Suppose the nonlinear control system is

$$\begin{aligned} \dot{x_1} &= x_2 + x_1^2, && (42)\\ \dot{x_2} &= x_3 - u + x_1^2, && (43)\\ \dot{x_3} &= u - 2x_1^2. && (44) \end{aligned}$$

The relative degree of the system (42)-(44) with respect to the state $y = \alpha x_1 + x_2 + x_3$ is 2, $0 < \alpha < 1$. The system (42)-(44) can be transformed to

$$\begin{aligned} \dot{z_1} &= z_2, \\ \dot{z_2} &= f(z,\eta) + g(z,\eta)u, \\ \dot{\eta} &= \eta - \left(\frac{z_1 - \eta}{\alpha - 1}\right), \end{aligned} \tag{45}$$

with $y = z_1$, $f(z,\eta) = \alpha x_3 + (\alpha - 2)x_1^2 + 2(\alpha - 1)x_1x_2 + 2(\alpha - 1)x_1^3$, $g(z,\eta) = 1 - \alpha$.

Thus, the zero dynamic of the system (42)-(44) with respect to the state $y = z_1$ is asymptotically stable. Then, according to (34), the input control is

$$u = \frac{1}{1-\alpha}\left(-a(z,\eta) - c_0z_1 - c_1z_2\right). \tag{46}$$

If we choose c_0, c_1 such that all roots of the polynomial $p(\lambda) = c_0 + c_1\lambda + \lambda^2$ have negative real part, then the system (42)-(44) has an asymptotically stable equilibrium at $(x_1, x_2, x_3) = (0, 0, 0)$.

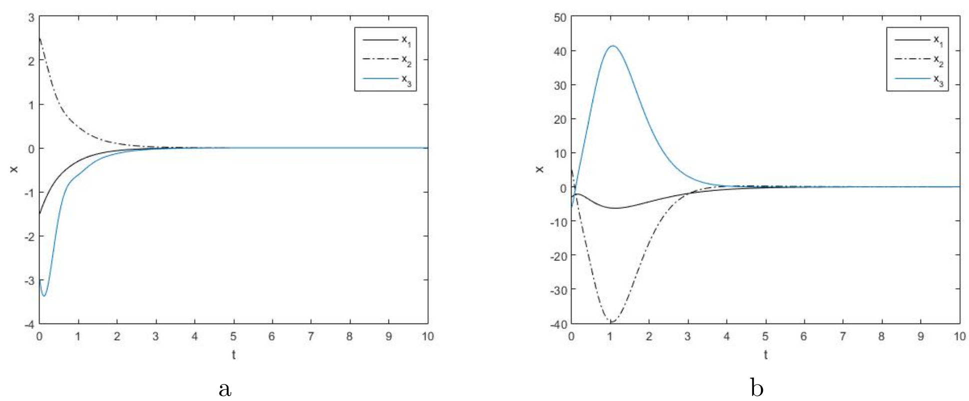

Figure 1: a) simulation result for Example 3.1, b) simulation result for Example 3.2.

Simulation result is shown in Fig.1b for constants $\alpha = 0,5$, $c_0 = 6, c_1 = 5$. The initial value $x_1(0) = -3$, $x_2(0) = 5$, $x_3(0) = -6$.

4 Conclusion

In this paper, we have investigated asymptotic stability of some class of affine nonlinear control systems through partial feedback liearization. For a class of affine nonlinear control systems with the relative degree of the system (9) with respect to the state y being 1, the input control is designed so that the zero dynamic of the system (9) is asymptotically stable. Next, for a class of affine nonlinear control systems with the relative degree of the system (23) with respect to the state y being $n-1$, the input control is designed when the zero dynamic of the system (9) is asymptotically stable, where the state y is the linear combination of the state variables.

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