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A Dynamic Contact Problem for Elasto-Viscoplastic Piezoelectric Materials with Normal Compliance, Normal Damped Response and Damage

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Abstract: This work studies a mathematical model involving a dynamic contact between two elasto-viscoplastic piezoelectric bodies with damage. The contact is modelled with a combination of a normal compliance and a normal damped response law associated with friction. We derive a variational formulation of the problem and we prove an existence and uniqueness result for the weak solution. The proof is based on the classical existence and uniqueness result for parabolic inequalities, differential equations and fixed-point arguments.

Keywords: *dynamic process; elastic-viscoplastic piezoelectric materials; damage; normal compliance; normal damped; fixed point.*

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1 Introduction

In this paper we study a contact problem which involves viscous friction of Tresca type described in [1]. A nonlinear elasto-viscoplastic constitutive law is used to model the piezoelectric material. The piezoelectricity can be described as follows: when mechanical pressure is applied to a certain class of crystalline materials (e.g., ceramics $BaTiO_3$, $BiFeO_3$), the crystalline structure produces a voltage proportional to the pressure. Conversely, when an electric field is applied, the structure changes its shape producing dimensional modifications in the material. Different models have been developed to describe the interaction between the electrical and mechanical fields, see, for example, [5, 17] and the

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references therein. For contact problems involving elasto-piezoelectric materials see [15]. Different models of viscoelastic piezoelectric problems have been studied in [3, 14, 20], contact problems for electro-elasto-viscoplastic materials were studied in [5, 11].

The damage is an extremely important topic in engineering since it affects directly the useful life of the designed structure or component. There exists a very large engineering literature on it. Models taking into account the influence of the internal damage of the material on the contact process have been investigated mathematically. General models for damage were derived in [6] from the virtual power principle. The models of mechanical damage, which were derived from thermodynamical considerations and the principle of virtual work, can be found in [8]. The new idea of [7] was the introduction of the damage function $\beta^\ell = \beta^\ell(x,t)$, which is the ratio between the elastic moduli of the damage and damage-free materials. In an isotropic and homogeneous elastic material, let E_Y^ℓ be the Young modulus of the original material and E_{eff}^ℓ be the current modulus, then the damage function is defined by $\beta^\ell = E_{eff}^\ell / E_Y^\ell$. Clearly, it follows from this definition that the damage function β^ℓ is restricted to have values between zero and one. When $\beta^\ell = 1$, there is no damage in the material, when $\beta^\ell = 0$, the material is completely damaged, when $0 < \beta^\ell < 1$, there is partial damage and the system has a reduced load carrying capacity. Contact problems with damage have been investigated in [12]. The differential inclusion used for the evolution of the damage field is

$$\dot{\boldsymbol{\beta}}^\ell - \kappa^\ell \Delta \boldsymbol{\beta}^\ell + \partial_{k^\ell}(\boldsymbol{\beta}^\ell) \ni \boldsymbol{S}^\ell(\boldsymbol{\sigma}^\ell - \mathcal{A}^\ell \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell) - (\mathcal{E}^\ell)^* \nabla \varphi^\ell(s), \, \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell), \boldsymbol{\beta}^\ell),$$

where K^ℓ denotes the set of admissible damage functions defined by

$$K^\ell = \{\xi \in V^\ell; \; 0 \le \xi \le 1, \text{ a.e. in } \Omega^\ell\}, \tag{1}$$

κ^ℓ is a positive coefficient, $\partial \varphi_{K^\ell}$ represents the subdifferential of the indicator function of the set K^ℓ, and S^ℓ is a given constitutive function which describes the sources of the damage in the system. The paper is structured as follows. In Section 2, we present the physical setting and describe the mechanical problem. In Section 3, we introduce some notation, list the assumptions on the problems data, and derive the variational formulation of the model. In Section 4, we state our main existence and uniqueness result, Theorem 4.1. The proof of the theorem is based on the arguments of nonlinear evolution equations with monotone operators, a classical existence and uniqueness result for parabolic inequalities and fixed-point arguments.

2 The Model

We describe the model for the process and we present its variational formulation. We consider the following physical setting. Let us consider two electro-elastic-viscoplastics bodies, occupying two bounded domains Ω^1, Ω^2 of the space $\mathbb{R}^d (d = 2, 3)$. For each domain Ω^ℓ, the boundary Γ^ℓ is assumed to be Lipschitz continuous, and is partitioned into three disjoint measurable parts Γ_1^ℓ, Γ_2^ℓ and Γ_3^ℓ, on one hand, and into two measurable parts Γ_a^ℓ and Γ_b^ℓ, on the other hand, such that $meas\Gamma_1^\ell > 0$, $meas\Gamma_a^\ell > 0$. Let $T > 0$ and let $[0, T]$ be the time interval of interest. The Ω^ℓ body is subject to $\boldsymbol{f}_0^\ell$ forces and volume electric charges of density q_0^ℓ. The bodies are assumed to be clamped on $\Gamma_1^\ell \times [0, T]$. The surface tractions $\boldsymbol{f}_2^\ell$ act on $\Gamma_2^\ell \times [0, T]$. We also assume that the electrical potential vanishes on $\Gamma_a^\ell \times [0, T]$ and a surface electric charge of density q_2^ℓ is prescribed on $\Gamma_b^\ell \times [0,T]$. The two bodies can enter in contact along the common part $\Gamma_3^1 = \Gamma_3^2 = \Gamma_3$. We use an electro-elastic-viscoplastic constitutive law with damage given by

$$\begin{gathered}\boldsymbol{\sigma}^\ell = \mathcal{A}^\ell \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell) + \mathcal{G}^\ell \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell) + (\mathcal{E}^\ell)^* \nabla \varphi^\ell + \\ \int_0^t \mathcal{F}^\ell \Big(\boldsymbol{\sigma}^\ell(s) - \mathcal{A}^\ell \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell(s)) - (\mathcal{E}^\ell)^* \nabla \varphi^\ell,\ \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell(s)) \Big)\, ds,\end{gathered} \tag{2}$$

$$\boldsymbol{D}^\ell = \mathcal{E}^\ell \varepsilon(\boldsymbol{u}^\ell) - \mathcal{B}^\ell \nabla \varphi^\ell, \tag{3}$$

where $\boldsymbol{D}^\ell$ is the electric displacement field, $\boldsymbol{u}^\ell$ is the displacement field, $\boldsymbol{\sigma}^\ell$ and $\boldsymbol{\varepsilon}(\boldsymbol{u}^\ell)$ represent the stress and the linearized strain tensor, respectively. Here $\mathcal{A}^\ell$ is a given nonlinear function, $\mathcal{F}^\ell$ is the relaxation tensor, and $\mathcal{G}^\ell$ represents the elasticity operator. $E(\varphi^\ell) = -\nabla \varphi^\ell$ is the electric field, $\mathcal{E}^\ell = (e_{ijk})$ represents the third order piezoelectric tensor, $(\mathcal{E}^\ell)^*$ is its transposition. It follows from (2) that at each time moment, the stress tensor $\boldsymbol{\sigma}^\ell(t)$ is split into three parts: $\boldsymbol{\sigma}^\ell(t) = \boldsymbol{\sigma}_V^\ell(t) + \boldsymbol{\sigma}_E^\ell(t) + \boldsymbol{\sigma}_R^\ell(t)$, where $\boldsymbol{\sigma}_V^\ell(t) = \mathcal{A}^\ell \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell(t))$ represents the purely viscous part of the stress, $\boldsymbol{\sigma}_E^\ell(t) = (\mathcal{E}^\ell)^* \nabla \varphi^\ell(t)$ represents the electric part of the stress and $\boldsymbol{\sigma}_R^\ell(t)$ satisfies a rate-type elastic-viscoplastic relation

$$\boldsymbol{\sigma}_R^\ell(t) = \mathcal{G}^\ell \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell(t)) + \int_0^t \mathcal{F}^\ell \big(\boldsymbol{\sigma}_R^\ell(s), \varepsilon(\boldsymbol{u}^\ell(s)) \big)\, ds. \tag{4}$$

Various results, examples and mechanical interpretations in the study of elastic-viscoplastic materials of the form (4) can be found in [6, 9] and the references therein. Note also that when $\mathcal{F}^\ell = 0$, the constitutive law (2) becomes the Kelvin-Voigt electro-viscoelastic constitutive relation

$$\boldsymbol{\sigma}^\ell(t) = \mathcal{A}^\ell \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell(t)) + \mathcal{G}^\ell \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell(t)) + (\mathcal{E}^\ell)^* \nabla \varphi^\ell(t). \tag{5}$$

Dynamic contact problems with the Kelvin-Voigt materials of the form (5) can be found in [3]. The normal compliance contact condition was first considered in [12] in the study of dynamic problems with linearly elastic and viscoelastic materials and then it was used in various references, see, e.g., [11, 17]. This condition allows the interpenetration of the body's surface into the obstacle and it was justified by considering the interpenetration and deformation of surface asperities.

We need to introduce some notation and preliminary material. Here and below, $\mathbb{S}^d$ represents the space of the second-order symmetric tensors on $\mathbb{R}^d$. We recall that the inner products and the corresponding norms on $\mathbb{S}^d$ and $\mathbb{R}^d$ are given by

$$\begin{aligned} \boldsymbol{u}^\ell.\boldsymbol{v}^\ell &= u_i^\ell.v_i^\ell, \quad \left|\boldsymbol{v}^\ell\right| = (\boldsymbol{v}^\ell.\boldsymbol{v}^\ell)^{\frac{1}{2}}, \quad \forall \boldsymbol{u}^\ell, \boldsymbol{v}^\ell \in \mathbb{R}^d, \\ \boldsymbol{\sigma}^\ell.\boldsymbol{\tau}^\ell &= \sigma_{ij}^\ell.\tau_{ij}^\ell, \quad \left|\boldsymbol{\tau}^\ell\right| = (\boldsymbol{\tau}^\ell.\boldsymbol{\tau}^\ell)^{\frac{1}{2}}, \quad \forall \boldsymbol{\sigma}^\ell, \boldsymbol{\tau}^\ell \in \mathbb{S}^d. \end{aligned}$$

Here and below, the indices i and j run between 1 and d, and the summation convention over repeated indices is adopted. With these assumptions, the classical formulation of the dynamic problem for the friction contact with normal compliance and normal damped response between two elasto-viscoplastic piezoelectric bodies with damage is the following.

Problem P. For $\ell = 1, 2$, find a displacement field $\boldsymbol{u}^\ell : \Omega^\ell \times [0,T] \longrightarrow \mathbb{R}^d$, a stress field $\boldsymbol{\sigma}^\ell : \Omega^\ell \times [0,T] \longrightarrow \mathbb{S}^d$, an electric potential field $\varphi^\ell : \Omega^\ell \times [0,T] \longrightarrow \mathbb{R}$, a damage field $\boldsymbol{\beta}^\ell : \Omega^\ell \times [0,T] \longrightarrow \mathbb{R}$ and an electric displacement field $\boldsymbol{D}^\ell : \Omega^\ell \times [0,T] \longrightarrow \mathbb{R}^d$ such that

$$\begin{aligned}&\boldsymbol{\sigma}^\ell = \mathcal{A}^\ell \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell) + \mathcal{B}^\ell \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell) + (\mathcal{E}^\ell)^* \nabla \varphi^\ell + \\ &\int_0^t \mathcal{G}^\ell \Big(\boldsymbol{\sigma}^\ell(s) - \mathcal{A}^\ell \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell(s)) - (\mathcal{E}^\ell)^* \nabla \varphi^\ell, \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell(s)), \boldsymbol{\beta}^\ell(s)\Big)\, ds\end{aligned} \quad \text{in } \Omega^\ell \times (0,T), \tag{6}$$

$$\boldsymbol{D}^\ell = \mathcal{E}^\ell \varepsilon(\boldsymbol{u}^\ell) - \boldsymbol{B}^\ell \nabla \varphi^\ell \quad \text{in } \Omega^\ell \times (0,T), \tag{7}$$

$$\dot{\boldsymbol{\beta}}^\ell - \kappa^\ell \Delta \boldsymbol{\beta}^\ell + \partial_{k^\ell}(\boldsymbol{\beta}^\ell) \ni \boldsymbol{S}^\ell(\boldsymbol{\sigma}^\ell - \mathcal{A}^\ell \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell) - (\mathcal{E}^\ell)^* \nabla \varphi^\ell(s), \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell), \boldsymbol{\beta}^\ell), \tag{8}$$

$$\rho^\ell \ddot{\boldsymbol{u}}^\ell = \operatorname{Div} \boldsymbol{\sigma}^\ell + \boldsymbol{f}_0^\ell \quad \text{in } \Omega^\ell \times (0,T), \tag{9}$$

$$\operatorname{div} \boldsymbol{D}^\ell - q_0^\ell = 0 \quad \text{in } \Omega^\ell \times (0,T), \tag{10}$$

$$\boldsymbol{u}^\ell = 0 \quad \text{on } \Gamma_1^\ell \times (0,T), \tag{11}$$

$$\boldsymbol{\sigma}^\ell \boldsymbol{\nu}^\ell = \boldsymbol{f}_2^\ell \quad \text{on } \Gamma_2^\ell \times (0,T), \tag{12}$$

$$\begin{cases} \sigma_\nu^1 = \sigma_\nu^2 \equiv \sigma_\nu, \\ -\sigma_\nu = p_\nu([u_\nu] - g) + q_\nu([\dot{u}_\nu]) \end{cases} \quad \text{on } \Gamma_3 \times (0,T), \tag{13}$$

$$\begin{cases} \boldsymbol{\sigma}_\tau^1 = -\boldsymbol{\sigma}_\tau^2 \equiv \boldsymbol{\sigma}_\tau, \\ \|\boldsymbol{\sigma}_\tau\| \leqslant p_\tau([u_\tau] - g) + q_\tau([\dot{u}_\tau]) \end{cases} \quad \text{on } \Gamma_3 \times (0,T), \tag{14}$$

$$[\dot{\boldsymbol{u}}_\tau] \neq 0 \Rightarrow \boldsymbol{\sigma}_\tau = -(p_\tau([u_\tau] - g) + q_\tau([\dot{u}_\tau])).\frac{[\dot{\boldsymbol{u}}_\tau]}{[\dot{\boldsymbol{u}}_\tau]} \quad \text{on } \Gamma_3 \times (0,T), \tag{15}$$

$$\frac{\partial \boldsymbol{\beta}^\ell}{\partial v^\ell} \quad \text{on } \Gamma^\ell \times (0,T), \tag{16}$$

$$\varphi^\ell = 0 \quad \text{on } \Gamma_a^\ell \times (0,T), \tag{17}$$

$$\boldsymbol{D}^\ell.\boldsymbol{\nu}^\ell = q_2^\ell \quad \text{on } \Gamma_b^\ell \times (0,T), \tag{18}$$

$$\begin{cases} \boldsymbol{D}^1.\boldsymbol{\nu}^1 = \boldsymbol{D}^2.\boldsymbol{\nu}^2 = \boldsymbol{D}, \\ \boldsymbol{D} = \boldsymbol{\psi}([\boldsymbol{u}_v] - g)\phi_l(\varphi^1 + \varphi^2 - \varphi_0) \end{cases} \quad \text{on } \Gamma_3 \times (0,T), \tag{19}$$

$$\boldsymbol{u}^\ell(0) = \boldsymbol{u}_0^\ell, \ \ \boldsymbol{\beta}^\ell(0) = \boldsymbol{\beta}_0^\ell \quad \text{in } \Omega^\ell. \tag{20}$$

First, equations (6) and (7) represent the electro-elastic-viscoplastic constitutive law with damage, the evolution of the field is governed by the inclusion of parabolic type given by the relation (8), where $\boldsymbol{S}^\ell$ is the mechanical source of the damage growth, assumed to be a rather general function of the strains, and the damage itself, $\partial\varphi_{k^\ell}$, is the sub differential of the indicator function of the admissible damage functions set K^ℓ. Next, equations (9) and (10) are the steady equations for the stress and electric-displacement field, respectively, in which "Div" and "div" denote the divergence operator for tensor and vector-valued functions, i.e.,

$$\mathrm{Div}\,\boldsymbol{\sigma}^\ell = (\sigma^\ell_{ij,j}), \quad \mathrm{div}\,\boldsymbol{D}^\ell = (D^\ell_{i,i}).$$

We use these equations since the process is assumed to be mechanically dynamic and electrically quasi-static. Conditions (11) and (12) are the displacement and traction boundary conditions, whereas (17) and (18) represent the electric boundary conditions; the displacement field and the electrical potential vanish on Γ_1^ℓ and Γ_a^ℓ , respectively, while the forces and free electric charges are prescribed on Γ_2^ℓ and Γ_b^ℓ , respectively.
We turn to the boundary conditions (13) and (14) which describe the mechanical and electrical conditions on the potential contact surface Γ_3. The normal compliance function p_ν in(13) is described below, and g represents the gap in the reference configuration between Γ_3 and the foundation, measured along the direction of ν^ℓ. When positive, $[u_\nu]-g$ represents the interpenetration of the surface asperities into those of the foundation. This condition was first introduced in [10] and used in a large number of papers, see, for instance, [4, 7, 8, 14] and the references therein. Condition (14) is the associated friction law, where p_τ is a given function. According to (14), the tangential shear cannot exceed the maximum frictional resistance $p_\tau([u_\nu]-g)$, the so-called friction bound. Moreover, when sliding commences, the tangential shear reaches the friction bound and opposes the motion. Frictional contact conditions of the form (13), (14) have been used in various papers, see, e.g., [5, 6, 17] and the references therein.

The relation (16) describes a homogeneous Neumann boundary condition, where $\partial\boldsymbol{\beta}^\ell|\partial v^\ell$ is the normal derivative of $\boldsymbol{\beta}^\ell$. (17) and (18)) represent the electric boundary conditions. Next, (19) is the electrical contact condition on Γ_3, introduced in [11]. It may be obtained as follows. First, unlike the previous papers on the piezoelectric contact, we assume that the contact surface is electrically conductive and its potential is maintained at $\boldsymbol{\varphi}_0$. When there is no contact at a point on the surface (i.e., $[u_v] < g$), the gap is assumed to be an insulator (say, it is filled with air), there are no free electrical charges on the surface and the normal component of the electric displacement field vanishes. Thus,

$$[u_v] < g \;\Rightarrow\; \boldsymbol{D}^\ell.\boldsymbol{\nu}^\ell = 0. \tag{21}$$

During the process of contact (i.e., when $[u_v] \geqslant g$) the normal component of the electric displacement field or the free charge is assumed to be proportional to the difference between the potential of the foundation and the body's surface potential, with $\boldsymbol{k}$ as the proportionality factor. Thus

$$[u_v] \geqslant g \;\Rightarrow\; \boldsymbol{D}^\ell.\boldsymbol{\nu}^\ell = \boldsymbol{k}(\varphi^1 + \varphi^2 - \varphi_0). \tag{22}$$

We combine (21), (22) to obtain

$$\boldsymbol{D}^\ell.\boldsymbol{\nu}^\ell = \boldsymbol{k}_{\chi_{[0,\infty)}}([\boldsymbol{u}_v] - g)(\varphi^1 + \varphi^2 - \varphi_0), \tag{23}$$

where $\chi_{[0,\infty)}$ is the characteristic function of the interval $[0,\infty)$; that is,

$$\chi_{[0,\infty)}(r) = \begin{cases} 0 & \text{if } r < 0, \\ 1 & \text{if } r \geqslant 0. \end{cases}$$

Condition (23) describes the perfect electrical contact and is somewhat similar to the well-known Signorini contact condition. Both conditions may be over idealizations in many applications. To make it more realistic, we regularize condition (23) and write it as (19), in which $\boldsymbol{k}_{\chi_{[0,\infty)}}([\boldsymbol{u}_v] - g)$ is replaced with $\boldsymbol{\psi}$ which is a regular function and which will be described below, and $\boldsymbol{\phi}_l$ is the truncation function

$$\boldsymbol{\phi}_l(s) = \begin{cases} -l & \text{if } s < -l, \\ s & \text{if } -l \leq s \leq l, \\ l & \text{if } s > l, \end{cases}$$

where l is a large positive constant. We note that this truncation does not pose any practical limitations on the applicability of the model, since l may be arbitrarily large, higher than any possible peak voltage in the system, and therefore in applications $\boldsymbol{\phi}_l(\varphi^1 + \varphi^2 - \varphi_0) = \varphi^1 + \varphi^2 - \varphi_0$. The reasons for the regularization (19) of (23) are mathematical. First, we need to avoid the discontinuity in the free electric charge when the contact is established and, therefore, we regularize the function $\boldsymbol{k}_{\chi_{[0,\infty)}}$ in (23) with a Lipschitz continuous function $\boldsymbol{\psi}$. A possible choice is

$$\boldsymbol{\psi}(r) \begin{cases} 0 & \text{if } r < 0, \\ k\delta r & \text{if } 0 \leq r \leq l/\delta, \\ k & \text{if } r > \delta, \end{cases} \tag{24}$$

where $\delta > 0$ is a small parameter. This choice means that during the process of contact the electrical conductivity increases as the contact among the surface asperities improves, and stabilizes when the penetration $[\boldsymbol{u}_v] - g$ reaches the value δ. Secondly, we need the term $\boldsymbol{\phi}_l(\varphi^1 + \varphi^2 - \varphi_0) = \varphi^1 + \varphi^2 - \varphi_0$ to control the boundednes of $[\varphi] - \varphi_0$. Note that $\boldsymbol{\psi} \equiv 0$ in (19), then

$$\boldsymbol{D}^\ell.\boldsymbol{\nu}^\ell = 0 \quad \text{on } \Gamma_3 \times (0,T), \tag{25}$$

which decouples the electrical and mechanical problems on the contact surface. Condition (25) models the case when the obstacle is a perfect insulator and was used in [3,14,19,20]. Condition (19), instead of (25), introduces a strong coupling between the mechanical and the electric boundary conditions and leads to a new and non-standard mathematical model. Because of the friction condition (14), which is non-smooth, we do not expect the problem to have, in general, any classical solutions. Finally, in equation (20) $\boldsymbol{u}_0^\ell$ is the initial displacement, and $\boldsymbol{\beta}_0^\ell$ is the initial damage. To obtain the variational formulation of the problem (6),we introduce for the bonding field the set

$$\mathcal{Z} = \left\{\theta \in L^\infty\big(0,T; L^2(\Gamma_3)\big);\ 0 \leq \theta(t) \leq 1 \ \ \forall t \in [0,T], \text{ a.e. on } \Gamma_3\right\}.$$

For this reason, we derive in the next section a variational formulation of the problem and investigate its solvability. Moreover, variational formulations are also starting points for the construction of finite element algorithms for this type of problems.

3 Variational Formulation and the Main Result

We use the standard notation for the L^p and the Sobolev spaces associated with Ω^ℓ and Γ^ℓ and, for a function $\zeta^\ell \in H^1(\Omega^\ell)$, we still write ζ^ℓ to denote its trace on Γ^ℓ. We recall that the summation convention applies to a repeated index. For the electric displacement field we use two Hilbert spaces

$$\mathcal{W}^\ell = L^2(\Omega^\ell)^d, \quad \mathcal{W}_1^\ell = \{\boldsymbol{D}^\ell \in \mathcal{W}^\ell : \operatorname{div} \boldsymbol{D}^\ell \in L^2(\Omega^\ell)\},$$

endowed with the inner products

$$(\boldsymbol{D}^\ell, \boldsymbol{E}^\ell)_{\mathcal{W}^\ell} = \int_{\Omega^\ell} \boldsymbol{D}_i^\ell.\boldsymbol{E}_i^\ell dx, \quad (\boldsymbol{D}^\ell, \boldsymbol{E}^\ell)_{\mathcal{W}_1^\ell} = (\boldsymbol{D}^\ell, \boldsymbol{E}^\ell)_{\mathcal{W}^\ell} + (\operatorname{div} \boldsymbol{D}^\ell, \operatorname{div} \boldsymbol{E}^\ell)_{L^2(\Omega^\ell)},$$

and the associated norms $\|.\|_{\mathcal{W}^\ell}$ and $\|.\|_{\mathcal{W}_1^\ell}$, respectively. The electric potential field is to be found in

$$W^\ell = \{\boldsymbol{\xi}^\ell \in H^1(\Omega^\ell): \quad \boldsymbol{\xi} = 0 \quad on \quad \Gamma_a^\ell\}.$$

Since $meas\Gamma_a^\ell > 0$, the Friedrichs-Poincaré inequality holds, thus,

$$\|\nabla(\boldsymbol{\xi}^\ell)\|_{\mathcal{W}^\ell} \geq c_F \|\boldsymbol{\xi}^\ell\|_{H_1^\ell(\Omega^\ell)} \quad \forall \boldsymbol{\xi}^\ell \in W^\ell, \tag{26}$$

where $c_F > 0$ is a constant which depends only on Ω^ℓ and Γ_a^ℓ. On W^ℓ , we use the inner product

$$(\boldsymbol{\varphi}^\ell, \boldsymbol{\psi}^\ell)_{W^\ell} = (\nabla \boldsymbol{\varphi}^\ell . \nabla \boldsymbol{\psi}^\ell)_{\mathcal{W}^\ell}$$

and let $\|.\|_{W^\ell}$ be the associated norm. It follows from (26) that $\|.\|_{H^1(\Omega^\ell)}$ and $\|.\|_{W^\ell}$ are equivalent norms on W^ℓ and therefore $(W^\ell, \|.\|_{W^\ell})$ is a real Hilbert space. Moreover, by the Sobolev trace theorem, there exists a constant c_0 depending only on Ω^ℓ, Γ_a^ℓ and Γ_a^ℓ such that

$$\|\boldsymbol{\xi}^\ell\|_{L^2(\Gamma_3)} \leq c_0 \|\boldsymbol{\xi}^\ell\|_{W^\ell} \quad \forall \boldsymbol{\xi}^\ell \in W^\ell. \tag{27}$$

We also introduce the spaces

$$\mathrm{E}_0^\ell = L^2(\Omega^\ell), \quad \mathrm{E}_1^\ell = H^1(\Omega^\ell).$$

We recall that when $\boldsymbol{D}^\ell \in \mathcal{W}_1^\ell$ is a sufficiently regular function, the Green type formula holds:

$$(\boldsymbol{D}^\ell, \nabla \boldsymbol{\xi}^\ell)_{\mathcal{W}^\ell} + (\operatorname{div} \boldsymbol{D}^\ell, \boldsymbol{\xi}^\ell)_{\mathcal{W}^\ell} = \int_{\Gamma^\ell} \boldsymbol{D}^\ell . \boldsymbol{v}^\ell \boldsymbol{\xi}^\ell da, \quad \forall \boldsymbol{\xi}^\ell \in H^1(\Omega^\ell). \tag{28}$$

For the stress and strain variables, we use the real Hilbert spaces

$$Q^\ell = \{\boldsymbol{\tau}^\ell = (\tau_{i,j}^\ell);\ \tau_{i,j}^\ell = \tau_{j,i}^\ell \in L^2(\Omega^\ell)\} = L^2(\Omega^\ell)_{sym}^{d\times d},$$
$$Q_1^\ell = \{\boldsymbol{\sigma}^\ell = (\sigma_{i,j}^\ell) \in Q^\ell : \operatorname{div} \boldsymbol{\sigma}^\ell = (\sigma_{ij,j}^\ell) \in \mathcal{W}^\ell\}$$

endowed with the respective inner products

$$(\boldsymbol{\sigma}^\ell, \boldsymbol{\tau}^\ell)_{Q^\ell} = \int_{\Omega^\ell} \sigma_{i,j}^\ell . \tau_{i,j}^\ell dx, \quad (\boldsymbol{\sigma}^\ell, \boldsymbol{\tau}^\ell)_{Q_1^\ell} = (\boldsymbol{\sigma}^\ell, \boldsymbol{\tau}^\ell)_{Q^\ell} + (\operatorname{div} \boldsymbol{\sigma}^\ell . \operatorname{Div} \boldsymbol{\tau}^\ell)_{\mathcal{W}^\ell}$$

and the associated norms $\|.\|_{Q^\ell}$ and $\|.\|_{Q_1^\ell}$. For the displacement variable we use the real Hilbert space

$$H_1^\ell = \left\{ \boldsymbol{u}^\ell = (u_i) \in \mathcal{W}^\ell : \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell) \in \boldsymbol{Q}^\ell \right\}$$

endowed with the inner product

$$(\boldsymbol{u}^\ell, \boldsymbol{v}^\ell)_{H_1^\ell} = (\boldsymbol{u}^\ell, \boldsymbol{v}^\ell)_{\mathcal{W}^\ell} + (\boldsymbol{\varepsilon}(\boldsymbol{u}^\ell), \boldsymbol{\varepsilon}(\boldsymbol{v}^\ell))_{\boldsymbol{Q}^\ell}$$

and the norm $\|.\|_{H_1^\ell}$. When $\boldsymbol{\sigma}^\ell$ is a regular function, the following Green's type formula holds:

$$(\boldsymbol{v}^\ell, \boldsymbol{\varepsilon}(\boldsymbol{v}^\ell)_{\boldsymbol{Q}^\ell} + (\operatorname{Div} \boldsymbol{\sigma}^\ell, \boldsymbol{v}^\ell)_{\mathcal{W}^\ell} = \int_{\Gamma^\ell} \boldsymbol{\sigma}^\ell . \boldsymbol{v}^\ell \boldsymbol{V}^\ell da, \quad \forall \boldsymbol{v}^\ell \in H_1^\ell. \tag{29}$$

Next, we define the space

$$V^\ell = \{\boldsymbol{v}^\ell \in H_1^\ell : \boldsymbol{v} = 0 \quad \textbf{on} \;\; \Gamma_1^\ell\}.$$

Since $meas\,\Gamma_1^\ell > 0$, Korn's inequality (e.g., [5, pp. 16-17]) holds and

$$\|\varepsilon(\boldsymbol{v}^\ell)\|_{\boldsymbol{Q}^\ell} \geq c_K \|\boldsymbol{v}^\ell\|_{H_1^\ell} \quad \forall \boldsymbol{v}^\ell \in V^\ell, \tag{30}$$

where c_K is a constant which depends only on Ω^ℓ, and Γ_1^ℓ is a constant which depends only on V^ℓ, we use the inner product

$$(\boldsymbol{u}^\ell, \boldsymbol{v}^\ell)_{V^\ell} = (\varepsilon(\boldsymbol{u}^\ell), \varepsilon(\boldsymbol{v}^\ell))_{\boldsymbol{Q}^\ell}, \|\boldsymbol{v}^\ell\|_{V^\ell} = \|\boldsymbol{v}^\ell\|_{\boldsymbol{Q}^\ell}, \tag{31}$$

and let $\|.\|_{V^\ell}$ be the associated norm. It follows from (30) that the norms $\|.\|_{H_1^\ell}$ and $\|.\|_{V^\ell}$ are equivalent on V^ℓ. Then $(V^\ell, (.)_{V^\ell})$ is a real Hilbert space. Moreover, by the Sobolev trace theorem and (27), there exists a constant $\widetilde{c}_0 > 0$ depending only on Ω^ℓ, Γ_1^ℓ and Γ_3 such that

$$\|\boldsymbol{v}^\ell\|_{L^2(\Gamma_3)^d} \leq \widetilde{c}_0 \|\boldsymbol{v}^\ell\|_{V^\ell} \quad \forall \boldsymbol{v}^\ell \in V^\ell. \tag{32}$$

In order to simplify the notations, we define the product spaces

$$\mathrm{E}_0 = \mathrm{E}_0^1 \times \mathrm{E}_0^2, \quad \mathrm{E}_1 = \mathrm{E}_1^1 \times \mathrm{E}_1^2.$$

Finally, for a real Banach space $(X, \|.\|_X)$ we use the classical notation for the spaces $L^p(0,T;X)$ and $W^{k,p}(0,T;X)$, where $1 \leq p \leq \infty$, $k = 1, 2...$; denote by $C(0,T;X)$ and $C^1(0,T;X)$ the spaces of continuous and continuously differentiable functions on $[0,T]$ with values in X, with the respective norms

$$\|x\|_{C(0,T;X)} = \max_{t\in[0,T]} \|x(t)\|_X,$$
$$\|x\|_{C^1(0,T;X)} = \max_{t\in[0,T]} \|x(t)\|_X + \max_{t\in[0,T]} \|\dot{x}(t)\|_X.$$

We complete this section with the following version of the classical theorem of Cauchy-Lipschitz (see, e.g., [18, p. 48]).

Theorem 3.1 *Assume that $(X, \|.\|_X)$ is a real Banach space and $T > 0$. Let $F(t,.) : X \to X$ is an operator defined a.e. on $(0,T)$ satisfying the following conditions:*

1. *There exists a constant $L_F > 0$ such that*

$$\|F(t,x) - F(t,y)\|_X \leq L_F \|x - y\|_X \quad \forall x, y \in X, \quad \text{a.e.} \quad t \in (0,T).$$

2. *There exists $p \geq 1$ such that $t \mapsto F(t,x) \in L^p(0,T;X) \quad \forall x \in X$.*

Then for any $x_0 \in X$, there exists a unique function $x \in W^{1,p}(0,T;X)$ such that

$$\dot{x}(t) = F(t, x(t)), \quad \text{a.e.} \quad t \in (0,T),$$
$$x(0) = x_0.$$

Theorem 3.1 will be used in Section 3 to prove the unique solvability of the intermediate problem involving the bonding field. Moreover, if X_1 and X_2 are the real Hilbert spaces, then $X_1 \times X_1$ denotes the product Hilbert space endowed with the canonical inner product $(.,.)_{X_1\times X_1}$. Recall that the dot represents the time derivative.

In the study of the Problem **P**, we consider the following assumptions: we assume that the *viscosity operator* $\mathcal{A}^\ell : \Omega^\ell \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies:

$$\left\{\begin{array}{l} \text{(a) There exists } L_{\mathcal{A}^\ell} > 0 \text{ such that} \\ \qquad |\mathcal{A}^\ell(\boldsymbol{x},\boldsymbol{\xi}_1) - \mathcal{A}^\ell(\boldsymbol{x},\boldsymbol{\xi}_2)| \le L_{\mathcal{A}^\ell}|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2| \\ \qquad \forall\, \boldsymbol{\xi}_1,\boldsymbol{\xi}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega^\ell. \\ \text{(b) There exists } m_{\mathcal{A}^\ell} > 0 \text{ such that} \\ \qquad (\mathcal{A}^\ell(\boldsymbol{x},\boldsymbol{\xi}_1) - \mathcal{A}^\ell(\boldsymbol{x},\boldsymbol{\xi}_2))\cdot(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) \ge m_{\mathcal{A}^\ell}|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2|^2 \\ \qquad \forall\, \boldsymbol{\xi}_1,\boldsymbol{\xi}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega^\ell. \\ \text{(c) The mapping } \boldsymbol{x} \mapsto \mathcal{A}^\ell(\boldsymbol{x},\boldsymbol{\xi}) \text{ is Lebesgue measurable on } \Omega^\ell, \\ \qquad \text{for any } \boldsymbol{\xi} \in \mathbb{S}^d. \\ \text{(d) The mapping } \boldsymbol{x} \mapsto \mathcal{A}^\ell(\boldsymbol{x},\mathbf{0}) \text{ belongs to } Q^\ell. \end{array}\right. \tag{33}$$

The *elasticity operator* $\mathcal{B}^\ell : \Omega^\ell \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies:

$$\left\{\begin{array}{l} \text{(a) There exists } L_{\mathcal{B}^\ell} > 0 \text{ such that} \\ \qquad |\mathcal{B}^\ell(\boldsymbol{x},\boldsymbol{\xi}_1) - \mathcal{B}^\ell(\boldsymbol{x},\boldsymbol{\xi}_2)| \le L_{\mathcal{B}^\ell}|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2| \\ \qquad \forall\, \boldsymbol{\xi}_1,\boldsymbol{\xi}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega^\ell. \\ \text{(b) The mapping } \boldsymbol{x} \mapsto \mathcal{B}^\ell(\boldsymbol{x},\boldsymbol{\xi}) \text{ is Lebesgue measurable on } \Omega^\ell, \\ \qquad \text{for any } \boldsymbol{\xi} \in \mathbb{S}^d. \\ \text{(c) The mapping } \boldsymbol{x} \mapsto \mathcal{B}^\ell(\boldsymbol{x},\mathbf{0}) \text{ belongs to } Q^\ell. \end{array}\right. \tag{34}$$

The *viscoplasticity operator* $\mathcal{G}^\ell : \Omega^\ell \times \mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R} \times \ \to \mathbb{S}^d$ satisfies:

$$\left\{\begin{array}{l} \text{(a) There exists a constant } L_{\mathcal{G}^\ell} > 0 \text{ such that} \\ \qquad \|\mathcal{G}^\ell(\boldsymbol{x},\boldsymbol{\sigma}_1,\boldsymbol{\varepsilon}_1,\boldsymbol{\theta}_1,\boldsymbol{\varsigma}_1) - \mathcal{G}^\ell(\boldsymbol{x},\boldsymbol{\sigma}_2,\boldsymbol{\varepsilon}_2,\boldsymbol{\theta}_2,\boldsymbol{\varsigma}_2)\| \le \\ \qquad L_{\mathcal{G}^\ell}\big(\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\| + \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| + \|\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2\| + \|\boldsymbol{\varsigma}_1 - \boldsymbol{\varsigma}_2\|\big), \\ \qquad \forall \boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2,\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2 \in \mathbb{S}^d,\ \forall \boldsymbol{\theta}_1,\boldsymbol{\theta}_2,\boldsymbol{\zeta}_1,\boldsymbol{\zeta}_2 \in \mathbb{R} \quad \text{a.e. } \boldsymbol{x} \in \Omega^\ell. \\ \text{(b) The mapping } \boldsymbol{x} \mapsto \mathcal{G}^\ell(\boldsymbol{x},\boldsymbol{\sigma},\boldsymbol{\varepsilon},\boldsymbol{\theta},\boldsymbol{\varsigma}) \text{ is Lebesgue measurable on } \Omega^\ell, \\ \qquad \text{for } \boldsymbol{\sigma},\boldsymbol{\varepsilon} \in \mathbb{S}^d, \text{ for all } \boldsymbol{\theta},\boldsymbol{\zeta} \in \mathbb{R}. \\ \text{(c) The mapping } \boldsymbol{x} \mapsto \mathcal{G}^\ell(\boldsymbol{x},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0}) \text{ belongs to } \mathrm{Q}^\ell. \end{array}\right. \tag{35}$$

The *electric permittivity operator* $\boldsymbol{B}^\ell = (b^\ell_{ij}) : \Omega^\ell \times \mathbb{R}^d \to \mathbb{R}^d$ verifies:

$$\left\{\begin{array}{l} \text{(a) } \boldsymbol{B}^\ell(\boldsymbol{x},\mathbf{E}) = (b^\ell_{ij}(\boldsymbol{x})E_j) \quad \forall \mathbf{E} = (E_i) \in \mathbb{R}^d, \text{ a.e. } \boldsymbol{x} \in \Omega^\ell. \\ \text{(b) } b^\ell_{ij} = b^\ell_{ji},\ b^\ell_{ij} \in L^\infty(\Omega^\ell), \quad 1 \le i,j \le d. \\ \text{(c) There exists } m_{\boldsymbol{B}^\ell} > 0 \text{ such that } \boldsymbol{B}^\ell\mathbf{E}.\mathbf{E} \ge m_{\boldsymbol{B}^\ell}|\mathbf{E}|^2 \\ \qquad \forall \mathbf{E} = (E_i) \in \mathbb{R}^d, \text{ a.e. } \boldsymbol{x} \in \Omega^\ell. \end{array}\right. \tag{36}$$

The *piezoelectric tensor* $\mathcal{E}^\ell : \Omega^\ell \times \mathbb{S}^d \to \mathbb{R}^d$ satisfies:

$$\left\{\begin{array}{l} \text{(a) } \mathcal{E}^\ell(\boldsymbol{x},\tau) = (e^\ell_{ijk}(\boldsymbol{x})\tau_{jk}), \quad \forall \tau = (\tau_{ij}) \in \mathbb{S}^d \text{ a.e. } \boldsymbol{x} \in \Omega^\ell. \\ \text{(b) } e^\ell_{ijk} = e^\ell_{ikj} \in L^\infty(\Omega^\ell),\ 1 \le i,j,k \le d. \end{array}\right. \tag{37}$$

The damage source function $\boldsymbol{S}^\ell : \Omega^\ell \times \mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R} \times \; \to \mathbb{S}^d$ satisfies:

$$\left\{\begin{array}{l}\text{(a) There exists a constant } M_{\boldsymbol{S}^\ell} > 0 \text{ such that}\\ \quad \|\boldsymbol{S}^\ell(\boldsymbol{x},\boldsymbol{\sigma}_1,\boldsymbol{\varepsilon}_1,\boldsymbol{\beta}_1) - \boldsymbol{S}^\ell(\boldsymbol{x},\boldsymbol{\sigma}_2,\boldsymbol{\varepsilon}_2,\boldsymbol{\beta}_2)\| \le M_{\boldsymbol{S}^\ell}\big(\|\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2\| + \|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2\|\\ \quad + \|\boldsymbol{\beta}_1-\boldsymbol{\beta}_2\|\big), \;\; \forall \boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2,\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \forall \boldsymbol{\beta}_1,\boldsymbol{\beta}_2 \in \mathbb{R} \text{ a.e. } \boldsymbol{x}\in\Omega^\ell.\\ \text{(b) The mapping } \boldsymbol{x} \mapsto \boldsymbol{S}^\ell(\boldsymbol{x},\boldsymbol{\sigma},\boldsymbol{\varepsilon},\boldsymbol{\beta}) \text{ is Lebesgue measurable on } \Omega^\ell,\\ \quad \text{for any } \boldsymbol{\sigma},\boldsymbol{\varepsilon} \in \mathbb{S}^d, \text{ and}, \boldsymbol{\beta} \in \mathbb{R}.\\ \text{(c) The mapping } \boldsymbol{x} \mapsto \boldsymbol{S}^\ell(\boldsymbol{x},\mathbf{0},\mathbf{0},\mathbf{0}) \text{ belongs to } \mathrm{L}^2(\Omega^\ell).\end{array}\right. \tag{38}$$

The *normal damped response function* $q_r : \Gamma_3 \times \mathbb{R} \to \mathbb{R}, (r = v, \tau)$ satisfies:

$$\left\{\begin{array}{l}\text{(a) There exists a constant } C_1^r, C_2^r \text{ such that}\\ \quad |q_\tau(x,\boldsymbol{d})| \le C_1^r|\boldsymbol{d}| + C_2^r, \quad \forall\, d \in \mathbb{R}^d \text{ a.e. } \; x\in\Gamma_3.\\ \text{(b) } (q_r(x,\boldsymbol{d}_1) - q_r(x,\boldsymbol{d}_2))(\boldsymbol{d}_1-\boldsymbol{d}_2) \ge 0, \;\; \forall\, \boldsymbol{d}_1,\boldsymbol{d}_2 \in \mathbb{R}^d \text{ a.e. } \; x\in\Gamma_3.\\ \text{(c) The mapping } \boldsymbol{x} \mapsto q_r(x,d) \text{ is measurable on } \Gamma_3 \;\text{ for any } \; d\in\mathbb{R}^d.\\ \text{(d) The mapping } \boldsymbol{x} \mapsto q_r(x,d) \text{ is continuous on } \mathbb{R}^{\mathrm{d}} \;\text{ a.e. } \; x\in\Gamma_3.\end{array}\right. \tag{39}$$

The *normal compliance functions* $p_r : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+, (r = v, \tau)$ satisfies:

$$\left\{\begin{array}{l}\text{(a) There exists a constant } C_1^r, C_2^r \text{ such that}\\ \quad |p_r(x,\boldsymbol{d})| \le C_1^r|\boldsymbol{d}| + C_2^r, \quad \forall\, d \in \mathbb{R}^d \text{ a.e. } \; x\in\Gamma_3.\\ \text{(b) } (p_r(x,\boldsymbol{d}_1) - p_r(x,\boldsymbol{d}_2))(\boldsymbol{d}_1-\boldsymbol{d}_2) \ge 0, \quad \forall\, \boldsymbol{d}_1,\boldsymbol{d}_2 \in \mathbb{R}^d \;\text{ a.e. } \; x\in\Gamma_3.\\ \text{(c) The mapping } \boldsymbol{x} \mapsto p_r(x,d) \text{ is measurable on } \Gamma_3 \;\text{ for any } \; d\in\mathbb{R}^d.\\ \text{(d) The mapping } \boldsymbol{x} \mapsto p_r(x,d) \text{ is continuous on } \mathbb{R}^{\mathrm{d}} \;\text{ a.e. } \; x\in\Gamma_3.\end{array}\right. \tag{40}$$

An example of a normal compliance function p_ν, which satisfies conditions (40), is $p_\nu(u) = c_\nu u_+$, where $c_\nu \in L^\infty(\Gamma_3)$ is a positive surface stiffness coefficient, and $u_+ = max\{0, u\}$. The choices $p_\tau = \mu p_\nu$ and $p_\tau = \mu p_\nu(1-\delta p_\nu)_+$ in (14), where $\mu \in L^\infty(\Gamma_3)$ and $\delta \in L^\infty(\Gamma_3)$ are positive functions, lead to the usual or modified Coulomb's law of dry friction, respectively, see [5,6,21] for details. Here, μ represents the coefficient of friction and δ is a small positive material constant related to the wear and hardness of the surface. We note that if p_ν satisfies condition (40), then p_τ satisfies it too, in both examples. Therefore, we conclude that the results below are valid for the corresponding piezoelectric frictional contact models. The surface electrical conductivity function $\psi : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$ satisfies

$$\left\{\begin{array}{l}\text{(a) } \exists\, L_\psi > 0 \,\text{such that } \|\psi(\boldsymbol{x},u_1) - \psi(\boldsymbol{x},r_2)\| \le L_\psi\|u_1-u_2\|\\ \quad \forall\, u_1,u_2 \in \mathbb{R}, \;\; \text{a.e. } \boldsymbol{x}\in\Gamma_3.\\ \text{(b) } \exists\, M_\psi > 0 \,\text{such that } \|\psi(\boldsymbol{x},u)\| \le L_\psi\|u_1-u_2\|, \;\; \forall\, u \in \mathbb{R}, \; \text{a.e. } \boldsymbol{x}\in\Gamma_3.\\ \text{(c) The mapping } \boldsymbol{x}\mapsto\psi(\boldsymbol{x},u) \;\text{ is measurable on}\,\Gamma_3, \forall u \in \mathbb{R}.\\ \text{(d) } \psi(\boldsymbol{x},u) = 0, \;\text{ for all } \; u \le 0, \;\text{a.e. } \boldsymbol{x}\in\Gamma_3.\end{array}\right. \tag{41}$$

An example of a conductivity function, which satisfies condition (41), is given by (24), in which case $M_\psi = k$. Another example is provided by $\psi \equiv 0$, which models the contact with an insulated foundation, as noted in Section 2. We conclude that our results below are valid for the corresponding piezoelectric contact models.
The microcrack diffusion coefficient verifies

$$K^\ell > 0, \tag{42}$$

and the initial damage field satisfies

$$\beta_0^\ell \in K^\ell. \tag{43}$$

Finally, we assume that the gap function, the given potential and the initial displacement satisfy

$$g \in L^2(\Gamma_3) \quad g \geq 0. \text{ a.e } \text{ on } \Gamma_3, \tag{44}$$

$$\boldsymbol{\varphi}_0 \in L^2(\Gamma_3), \tag{45}$$

$$\boldsymbol{u}_0 \in \boldsymbol{V}. \tag{46}$$

The forces, tractions, volume and surface free charge densities satisfy

$$\mathbf{f}_0^\ell \in W^{1,p}(0,T;W^\ell), \quad \mathbf{f}_2^\ell \in W^{1,p}(0,T;L^2(\Gamma_2^\ell)^d), \tag{47}$$

$$q_0^\ell \in W^{1,p}(0,T;L^2(\Omega^\ell)), \quad q_2^\ell \in W^{1,p}(0,T;L^2(\Gamma_b^\ell)). \tag{48}$$

Here, $1 \leqslant p \leqslant \infty$. We define the bilinear form $a : H^1(\Omega^\ell) \times H^1(\Omega^\ell)) \longrightarrow \mathbb{R}$,

$$a(\xi^\ell, \varphi^\ell) = \sum_{\ell=1}^{2} k^\ell \int_{(\Omega)^\ell} \nabla \xi^\ell . \nabla \varphi^\ell dx. \tag{49}$$

Next, we define the four mappings $j_1 : \boldsymbol{V} \times \boldsymbol{V} \longrightarrow \mathbb{R}$, $j_2 : \boldsymbol{V} \times \boldsymbol{V} \longrightarrow \mathbb{R}$, $h : \boldsymbol{V} \times W \longrightarrow W$, $\mathbf{f} : [0,T] \to \boldsymbol{V}$ and $q : [0,T] \to W$, respectively, by

$$j_1(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_3} p_\nu([u_\nu] - g)[v_\nu] da + \int_{\Gamma_3} p_\tau([\boldsymbol{u}_\tau] - g)\|\boldsymbol{v}_\tau\| da, \tag{50}$$

$$j_2(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_3} q_\nu([u_\nu])[v_\nu] da + \int_{\Gamma_3} q_\tau([\boldsymbol{u}_\tau])\|\boldsymbol{v}_\tau\| da, \tag{51}$$

$$(h(\boldsymbol{u}, \varphi), \xi)_W = \int_{\Gamma_3} \psi([\boldsymbol{u}_v] - g)\phi_l([\varphi] - \varphi_0)\xi da, \tag{52}$$

$$(\mathbf{f}(t), \boldsymbol{v})_{V' \times V} = \sum_{\ell=1}^{2} \int_{\Omega^\ell} \mathbf{f}_0^\ell(t) \cdot \boldsymbol{v}^\ell \, dx + \sum_{\ell=1}^{2} \int_{\Gamma_2^\ell} \mathbf{f}_2^\ell(t) \cdot \boldsymbol{v}^\ell \, da, \tag{53}$$

$$(q(t), \zeta)_W = \sum_{\ell=1}^{2} \int_{\Omega^\ell} q_0^\ell(t) \zeta^\ell \, dx - \sum_{\ell=1}^{2} \int_{\Gamma_b^\ell} q_2^\ell(t) \zeta^\ell \, da \tag{54}$$

for all $\boldsymbol{u}^\ell, \boldsymbol{v}^\ell \in V^\ell$, $\varphi^\ell, \xi^\ell \in W^\ell$ and $t \in [0;T]$. We note that the definitions of h, f and q are based on the Riesz representation theorem, moreover, it follows from assumptions (38)-(46) that the integrals in (50)-(51) and (54) are well-defined. Using Green's formulas (28) and (29), it is easy to see that if $\{\boldsymbol{u}^\ell, \boldsymbol{\sigma}^\ell, \boldsymbol{D}^\ell\}$ are sufficiently regular functions which satisfy (9)-(15) and (17)-(19), then

$$(\ddot{u}, v)_{\boldsymbol{V}' \times \boldsymbol{V}} + \sum_{\ell=1}^{2} (\boldsymbol{\sigma}^\ell(t), \boldsymbol{\varepsilon}(\boldsymbol{v}^\ell))_{Q^\ell} + j_1(\boldsymbol{u}(t), \boldsymbol{v}) + j_2(\dot{\boldsymbol{u}}(t), \boldsymbol{v}) = (\mathbf{f}, \boldsymbol{v})_V, \tag{55}$$

$$\sum_{\ell=1}^{2} (\boldsymbol{D}^\ell(t), \nabla \xi^\ell)_{\mathcal{W}^\ell} - (q(t), \xi)_W = (h(\boldsymbol{u}, \varphi), \xi)_W \tag{56}$$

for all $\boldsymbol{u}^\ell, \boldsymbol{v}^\ell \in V^\ell, \varphi^\ell, \xi^\ell \in W^\ell$ and $t \in [0;T]$. We substitute (6) in (55), (7) in (56), we use the initial condition (20) and derive a variational formulation of Problem $\mathcal{P}$.

Problem $\mathcal{P}_V$. Find a displacement field $\boldsymbol{u} = (\boldsymbol{u}^1, \boldsymbol{u}^2) : [0,T] \to \boldsymbol{V}$, a stress field $\boldsymbol{\sigma} = (\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2) : [0,T] \to Q$, an electric potential field $\varphi = (\varphi^1, \varphi^2) : [0,T] \to W$, a damage field $\beta = (\beta^1, \beta^2) : [0,T] \to \mathrm{E}_1$, and an electric displacement field $\boldsymbol{D} = (\boldsymbol{D}^1, \boldsymbol{D}^2) : [0,T] \to \mathcal{W}$ such that

$$\begin{aligned}\boldsymbol{\sigma}^\ell(t) = \mathcal{A}^\ell \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell(t)) + \mathcal{B}^\ell \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell) + (\mathcal{E}^\ell)^* \nabla \varphi^\ell + \\ \int_0^t \mathcal{G}^\ell\Big(\boldsymbol{\sigma}^\ell(s) - \mathcal{A}^\ell \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell(s)) - (\mathcal{E}^\ell)^* \nabla \varphi^\ell, \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell(s)), \beta^\ell(s)\Big)\, ds\end{aligned} \quad \text{in } \Omega^\ell \times (0,T), \tag{57}$$

$$\boldsymbol{D}^\ell = \mathcal{E}^\ell \varepsilon(\boldsymbol{u}^\ell) - \boldsymbol{B}^\ell \nabla \varphi^\ell \quad \text{in } \Omega^\ell \times (0,T), \tag{58}$$

$$(\ddot{u}, v)_{\boldsymbol{V}' \times \boldsymbol{V}} + \sum_{\ell=1}^{2} (\boldsymbol{\sigma}^\ell, \boldsymbol{\varepsilon}(\boldsymbol{v}^\ell))_{Q^\ell} + j_1(\boldsymbol{u}, \boldsymbol{v}) + j_2(\dot{\boldsymbol{u}}, \boldsymbol{v}) = (\mathbf{f}, \boldsymbol{v})_{\boldsymbol{V}' \times \boldsymbol{V}}, \quad \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{59}$$

$$\sum_{\ell=1}^{2} (\mathcal{B}^\ell \nabla \varphi^\ell, \nabla \xi^\ell)_{\mathcal{W}^\ell} - \sum_{\ell=1}^{2} (\mathcal{E}^\ell \varepsilon(\boldsymbol{u}^\ell), \nabla \xi^\ell)_{\mathcal{W}^\ell} + (h(\boldsymbol{u}, \varphi), \xi)_{W^\ell} = (q, \xi)_W, \quad \xi \in W, \tag{60}$$

$$\begin{aligned}\beta(t) \in K, \quad \sum_{\ell=1}^{2} (\dot{\beta}^\ell(t), \xi^\ell - \beta^\ell(t))_{L^2(\Omega^\ell)} + a(\beta(t), \xi - \beta(t)) \geq \\ \sum_{\ell=1}^{2} (\mathcal{S}^\ell(\boldsymbol{\sigma}^\ell(t) - \mathcal{A}^\ell \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell(t)) - (\mathcal{E}^\ell)^* \nabla \varphi^\ell(t), \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell(t)), \xi^\ell - \beta^\ell(t))_{L^2(\Omega^\ell)}, \quad \xi \in K,\end{aligned} \tag{61}$$

$$\boldsymbol{u}^\ell(0) = \boldsymbol{u}_0^\ell, \quad \dot{\boldsymbol{u}}^\ell(0) = \boldsymbol{v}_0^\ell, \quad \boldsymbol{\beta}^\ell(0) = \boldsymbol{\beta}_0^\ell. \tag{62}$$

To study Problem $\mathcal{P}_V$, we use the smallness assumption

$$M_{\psi^\ell} < \frac{m_{\boldsymbol{B}^\ell}}{c_0^2}, \tag{63}$$

where M_{ψ^ℓ}, c_0 and $m_{\boldsymbol{B}^\ell}$ are given in (41) , (27) and (36), respectively. We note that only the trace constant, the coercivity constant of $\boldsymbol{B}^\ell$ and the bound of ψ^ℓ are involved in (63); therefore, this smallness assumption involves only the geometry and the electrical part, and does not depend on the mechanical data of the problem. Moreover, it is satisfied when the obstacle is insulated since then $\psi^\ell \equiv 0$ and so $M_{\psi^\ell} = 0$. Removing this assumption remains a task for future research since it is made for mathematical reasons, and does not seem to relate to any inherent physical constraints of the problem.

4 Existence and Uniqueness Result

Now, we propose our existence and uniqueness result.

Theorem 4.1 *Assume that* (32)-(48) *hold. Then there exists a unique solution*

$\{\boldsymbol{u}, \boldsymbol{\varphi}, \boldsymbol{\sigma}, \boldsymbol{D}, \beta\}$ *to Problem* $\mathcal{P}_V$. *Moreover, the solution satisfies*

$$\boldsymbol{u} \in W^{2.p}(0,T;\boldsymbol{V}) \cap C^1(0,T;\boldsymbol{V}),\ \ddot{\boldsymbol{u}} \in W^{2.p}(0,T;\boldsymbol{V}'), \tag{64}$$

$$\boldsymbol{\varphi} \in W^{1.p}(0,T;W), \tag{65}$$

$$\boldsymbol{\sigma} \in W^{1.p}(0,T;Q), \quad (\mathrm{Div}\,\sigma^1, \mathrm{Div}\,\sigma^2) \in W^{1.p}(0,T;\mathcal{W}), \tag{66}$$

$$\boldsymbol{D} \in W^{1.p}(0,T;\mathcal{W}), \tag{67}$$

$$\beta \in W^{1.2}(0,T;\mathrm{E}_0)) \cap L^2(0,T;\mathrm{E}_1). \tag{68}$$

The functions $\boldsymbol{u}, \varphi, \boldsymbol{\sigma}$, $\boldsymbol{D}$ and β, which satisfy (57)-(62), are called a weak solution to the contact Problem $\mathcal{P}$. We conclude that, under the assumptions (33)-(48) and (63), the mechanical problem (6) has a unique weak solution satisfying (64).
The regularity of the weak solution is given by(64), and in terms of electric displacement,

$$\boldsymbol{D} \in W^{1.p}(0,T;\mathcal{W}). \tag{69}$$

It follows from (80) and (47) that $\mathrm{div}\,\boldsymbol{D}^\ell(t) - q_0^\ell(t) = 0$ for all $t \in [0,T]$, and therefore the regularity (65) of φ, combined with (36), (37) and (48), implies (69). In this section we suppose that the assumptions of Theorem4.1 hold, and we consider that C is a generic positive constant which depends on Ω^ℓ, Γ_1^ℓ, Γ_3, p_ν, p_τ, q_ν, q_τ and may change from place to place.

Let a $\eta \in L^2(0,T;\boldsymbol{V}')$ be given. In the first step, we consider the following variational problem.

Problem $\mathcal{PV}_\eta^1$. Find a displacement field $\boldsymbol{u}_\eta = (\boldsymbol{u}_\eta^1, \boldsymbol{u}_\eta^2) : [0,T] \to \boldsymbol{V}$ such that

$$\begin{aligned}(\ddot{u}_\eta(t), v)_{\boldsymbol{V}'\times\boldsymbol{V}} + \sum_{\ell=1}^{2}(\mathcal{A}^\ell\varepsilon(\dot{\boldsymbol{u}}^\ell(t)),\ \varepsilon(\boldsymbol{v}^\ell))_{\mathcal{H}^\ell} + j_2(\dot{\boldsymbol{u}}(t), \boldsymbol{v})\\ +(\eta(t), v)_{\boldsymbol{V}'\times\boldsymbol{V}} = (\mathbf{f}(t), \boldsymbol{v})_{\boldsymbol{V}'\times\boldsymbol{V}} \quad \forall \boldsymbol{v} \in \boldsymbol{V}, a.e.\ t \in (0,T),\end{aligned} \tag{70}$$

$$\boldsymbol{u}^\ell(0) = \boldsymbol{u}_0^\ell, \quad \dot{\boldsymbol{u}}^\ell(0) = \boldsymbol{v}_0^\ell \quad \text{in } \Omega^\ell. \tag{71}$$

To solve Problem $\mathcal{PV}_\eta^1$, we apply an abstract existence and uniqueness result which we recall now, for the convenience of the reader. Let $\boldsymbol{V}$ and H denote real Hilbert spaces such that $\boldsymbol{V}$ is dense in H and the inclusion map is continuous, H is identified with its dual and with a subspace of the dual $\boldsymbol{V}'$ of $\boldsymbol{V}$, i.e., $\boldsymbol{V} \subset H \subset \boldsymbol{V}'$, and we say that the inclusions above define a Gelfand triple. The notations $\|.\|_{\boldsymbol{V}}$, $\|.\|_{\boldsymbol{V}'}$ and $(.,.)_{\boldsymbol{V}'\times\boldsymbol{V}}$ represent the norms on $\boldsymbol{V}$ and on $\boldsymbol{V}'$ and the duality pairing between them, respectively. The following abstract result may be found in [22, p.48].

Theorem 4.2 *Let* $\boldsymbol{V}$, H *be as above, and let* $A : \boldsymbol{V} \to \boldsymbol{V}'$ *be a hemicontinuous and monotone operator which satisfies*

$$(A\boldsymbol{v}, \boldsymbol{v})_{\boldsymbol{V}'\times\boldsymbol{V}} \geq w\|\boldsymbol{v}\|_{\boldsymbol{V}}^2 + \lambda \quad \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{72}$$

$$\|A\boldsymbol{v}\|_{\boldsymbol{V}'} \leq C(\|\boldsymbol{v}\|_{\boldsymbol{V}} + 1) \quad \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{73}$$

for some constants $w > 0$, $C > 0$ *and* $\lambda \in \mathbb{R}$. *Then, given* $\boldsymbol{u}_0 \in H$ *and* $f \in L^2(0,T;\boldsymbol{V}')$, *there exists a unique function* $\boldsymbol{u}$ *which satisfies*

$$\boldsymbol{u} \in L^2(0,T;\boldsymbol{V}) \cap C(0,T;H), \quad \dot{\boldsymbol{u}} \in L^2(0,T;\boldsymbol{V}'),$$
$$\dot{\boldsymbol{u}}(t) + A\boldsymbol{u}(t) = \mathbf{f}(t) \text{ a.e. } t \in (0,T),$$
$$\boldsymbol{u}(0) = \boldsymbol{u}_0.$$

We have the following result for the problem.

Lemma 4.1 *There exists a unique solution to Problem* $\mathcal{PV}_\eta^1$ *and it has its regularity expressed in* (64).

Proof. We define the operator $A : \boldsymbol{V} \to \boldsymbol{V}'$ by

$$(A\boldsymbol{u}, \boldsymbol{v})_{\boldsymbol{V}'\times\boldsymbol{V}} = \sum_{\ell=1}^{2} (\mathcal{A}^\ell \varepsilon(\boldsymbol{u}^\ell),\ \varepsilon(\boldsymbol{v}^\ell))_{\mathcal{H}^\ell} + j_2(\boldsymbol{u}, \boldsymbol{v}) \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}. \tag{74}$$

Let $\boldsymbol{u}_1, \boldsymbol{u}_2 \in \boldsymbol{V}$, using (74) and (51), we find

$$(A\boldsymbol{u}_1 - A\boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2)_{\boldsymbol{V}'\times\boldsymbol{V}} = \sum_{\ell=1}^{2} (\mathcal{A}^\ell \varepsilon(\boldsymbol{u}_1^\ell) - \mathcal{A}^\ell \varepsilon(\boldsymbol{u}_2^\ell),\ \varepsilon(\boldsymbol{u}_1^\ell - \boldsymbol{u}_2^\ell))_{\mathcal{H}^\ell} +$$

$$\int_{\Gamma_3} (q_\nu([u_{1\nu}]) - q_\nu([u_{2\nu}]))([u_{1\nu} - u_{2\nu}])da + \int_{\Gamma_3} (q_\tau([\boldsymbol{u}_{1\tau}]) - q_\tau([\boldsymbol{u}_{2\tau}]))\|[\boldsymbol{u}_{1\tau} - \boldsymbol{u}_{2\tau}]\|da,$$

and keeping in mind (33), (39) , we obtain

$$(A\boldsymbol{u}_1 - A\boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_1)_{\boldsymbol{V}'\times\boldsymbol{V}} \geq m\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{\boldsymbol{V}}^2 \quad \forall \boldsymbol{u}_1, \boldsymbol{u}_1 \in \boldsymbol{V}. \tag{75}$$

Use again (74) and (51), it follows that

$$(A\boldsymbol{u}_1 - A\boldsymbol{u}_2, \boldsymbol{v})_{\boldsymbol{V}'\times\boldsymbol{V}} = \sum_{\ell=1}^{2} (\mathcal{A}^\ell \varepsilon(\boldsymbol{u}_1^\ell) - \mathcal{A}^\ell \varepsilon(\boldsymbol{u}_2^\ell),\ \varepsilon(\boldsymbol{v}^\ell))_{\mathcal{H}^\ell} +$$

$$\int_{\Gamma_3} (q_\nu([u_{1\nu}]) - q_\nu([u_{2\nu}]))([v_\nu])da + \int_{\Gamma_3} (q_\tau([\boldsymbol{u}_{1\tau}]) - q_\tau([\boldsymbol{u}_{2\tau}]))\|[\boldsymbol{v}_\tau]\|da, \quad \forall \boldsymbol{v} \in \boldsymbol{V},$$

and by (32) and (33), we deduce that

$$|A\boldsymbol{u}_1 - A\boldsymbol{u}_2|_{\boldsymbol{V}'} \leq L_{\mathcal{A}^\ell}|\boldsymbol{u}_1 - \boldsymbol{u}_2|_{\boldsymbol{V}} + c_0|q_\nu([u_{1\nu}]) - q_\nu([u_{2\nu}])|_{L^2(\Gamma_3)}$$
$$+c_0|q_\tau([\boldsymbol{u}_{1\tau}]) - q_\tau([\boldsymbol{u}_{2\tau}])|_{L^2(\Gamma_3)^d}, \quad \forall \boldsymbol{u}_1, \boldsymbol{u}_2 \in \boldsymbol{V},$$

and keeping in mind the Krasnoselski theorem (see [10, p.60]), we deduce that $A : \boldsymbol{V} \to \boldsymbol{V}'$ is a continuous operator. Now, by (74), (31) and (33), we find where the positive constant $m = \min\{m_{\mathcal{A}^1}, m_{\mathcal{A}^2}\}$. Choosing $\boldsymbol{u}_2 = 0_{\boldsymbol{V}}$ in (75) we obtain

$$(A\boldsymbol{u}_1, \boldsymbol{u}_1)_{\boldsymbol{V}'\times\boldsymbol{V}} \geq m\|\boldsymbol{u}_1\|_{\boldsymbol{V}}^2 - \|Ao_{\boldsymbol{V}}\|_{\boldsymbol{V}'}^2\|\boldsymbol{u}_1\|_{\boldsymbol{V}}$$
$$\geq \frac{1}{2}m\|\boldsymbol{u}_1\|_{\boldsymbol{V}}^2 - \frac{1}{2m}\|Ao_{\boldsymbol{V}}\|_{\boldsymbol{V}'}^2 \quad \forall \boldsymbol{u}_1 \in \boldsymbol{V},$$

which implies that A satisfies condition (72) with $\omega = \frac{m}{2}$ and $\lambda = -\frac{1}{2m}\|Ao\|_{\boldsymbol{V}'}^2$. Moreover, by (74) and (33) we find

$$\|A\boldsymbol{u}_1\|_{\boldsymbol{V}'} \leq C^1\|u_1\|_{\boldsymbol{V}} + C^2 \quad \forall \boldsymbol{u}_1 \in \boldsymbol{V},$$

where $C^1 = \max\{C_{\mathcal{A}^1}^1, C_{\mathcal{A}^2}^1\}$ and $C^2 = \max\{C_{\mathcal{A}^1}^2, C_{\mathcal{A}^2}^2\}$. This inequality and (31) imply that A satisfies condition (75). Finally, we recall that by (47) and (53) we have $\mathbf{f} - \eta \in L^2(0, T; \boldsymbol{V}')$ and $\boldsymbol{v}_0 \in H$.

It follows now from Theorem 4.2 that there exists a unique function $\boldsymbol{v}_\eta$ which satisfies

$$\boldsymbol{v}_\eta \in L^2(0,T;\boldsymbol{V}) \cap C(0,T;H), \quad \dot{\boldsymbol{v}}_\eta \in L^2(0,T;\boldsymbol{V}'), \tag{76}$$

$$\dot{\boldsymbol{v}}_\eta(t) + A\boldsymbol{v}_\eta(t) + \eta(t) = \mathbf{f}(t), \quad a.e.\ t \in [0,T], \tag{77}$$

$$\boldsymbol{v}_\eta(0) = \boldsymbol{v}_0. \tag{78}$$

Let $\boldsymbol{u}_\eta : [0,T] \to \boldsymbol{V}$ be the function defined by

$$\boldsymbol{u}_\eta(t) = \int_0^t \boldsymbol{v}_\eta(s)ds + \boldsymbol{u}_0 \quad \forall t \in [0,T]. \tag{79}$$

It follows from (74) and (76)–(79) that $\boldsymbol{u}_\eta$ is a unique solution of the variational Problem $\mathcal{PV}^1_\eta$, and it satisfies the regularity expressed in (64).

In the second step, let $\eta \in L^2(0,T;\boldsymbol{V}')$, we use the displacement field $\boldsymbol{u}_\eta = (\boldsymbol{u}^1_\eta, \boldsymbol{u}^2_\eta)$ obtained in Lemma 4.1 and we consider the following variational problem.

Problem $\mathcal{PV}^2_\eta$. Find the electrical potential field $\varphi_\eta = (\varphi^1_\eta, \varphi^2_\eta) : [0;T] \longrightarrow W$ such that

$$\sum_{\ell=1}^{2} (\boldsymbol{B}^\ell \nabla\varphi^\ell_\eta(t) - \mathcal{E}^\ell \varepsilon(\boldsymbol{u}^\ell_\eta(t)), \nabla\xi^\ell)_{\mathcal{W}^\ell} + (h(\boldsymbol{u}_\eta(t), \varphi_\eta(t)), \xi)_W = (q(t), \xi)_W \tag{80}$$

for all $\xi \in W$, $t \in [0,T]$.

The well-posedness of Problem $\mathcal{PV}^2_\eta$ follows.

Lemma 4.2 *There exists a unique solution $\varphi_\eta = (\varphi^1_\eta, \varphi^2_\eta) \in W^{1,p}(0,T;W)$ which satisfies* (80). *Moreover, if φ_{η_1} and φ_{η_2} are the solutions of* (80) *corresponding to $\varphi_{\eta_1}, \varphi_{\eta_2} \in C(0,t;Q)$, then there exists $c > 0$ such that*

$$\|\varphi_{\eta_1}(t) - \varphi_{\eta_2}(t)\|_W \leq C\|\boldsymbol{u}_{\eta_1}(t) - \boldsymbol{u}_{\eta_2}(t)\|_{\boldsymbol{V}} \quad \forall t \in [0,T]. \tag{81}$$

Proof. We define a bilinear form $b(.,.) : W \times W \to \mathbb{R}$ such that

$$b(\varphi, \xi) = \sum_{\ell=1}^{2} (\mathcal{B}^\ell \nabla\varphi^\ell, \nabla\xi^\ell)_{H^\ell} \quad \forall \varphi,\ \xi \in W. \tag{82}$$

We use (30), (36) and (71) to show that the bilinear form $b(.,.)$ is continuous, symmetric and coercive on W. Moreover, using the Riesz representation theorem, we may define an element $q_\eta : [0,T] \to W$ such that

$$(q_\eta(t), \xi)_W = \sum_{\ell=1}^{2} (\mathcal{E}^\ell \varepsilon(\boldsymbol{u}^\ell_\eta(t)), \nabla\xi^\ell)_{H^\ell} - (h(\boldsymbol{u}_\eta(t), \varphi_\eta(t)) + q(t), \xi)_W, \quad \forall \xi \in W, t \in (0,T).$$

We apply the Lax-Milgram theorem to deduce that there exists a unique element $\varphi_\eta(t) \in W$ such that

$$b(\varphi_\eta(t), \xi) = (q_\eta(t), \xi)_W \quad \forall \xi \in W. \tag{83}$$

We conclude that $\varphi_\eta(t)$ is a solution to Problem $\mathcal{PV}^2_\eta$. Let $t_1, t_2 \in [0,T]$, it follows from (80) that

$$\|\varphi_\eta(t_1) - \varphi_\eta(t_2)\|_W \leq C\big(\|\boldsymbol{u}_\eta(t_1) - \boldsymbol{u}_\eta(t_2)\|_{\boldsymbol{V}} + \|q(t_1) - q(t_2)\|_W\big),$$

and the previous inequality, the regularity of $\boldsymbol{u}_\eta$ and q imply that $\varphi_\eta \in C(0,T;W)$. In the third step, we use the displacement field $\boldsymbol{u}_\eta$ obtained in Lemma4.1 and we consider the following initial-value problem.

In the third step, we let $\theta = (\theta^1, \theta^2) \in L^2(0.T; E_0)$ be given and consider the following variational problem for the damage field.

Problem $\mathcal{PV}_\theta$. Find the damage field $\beta_\theta = (\beta^1_\theta, \beta^2_\theta) : [0;T] \longrightarrow \mathrm{E}_1$ such that

$$\begin{aligned} &\beta_\theta(t) \in K, \qquad \sum_{\ell=1}^{2} (\dot{\beta}^\ell_\theta(t), \xi^\ell - \beta^\ell_\theta)_{L^2(\Omega^\ell)} + a(\beta_\theta(t), \xi - \beta_\theta(t)) \geq \\ &\sum_{\ell=1}^{2} (\theta^\ell(t), \xi^\ell - \beta^\ell_\theta(t))_{L^2(\Omega^\ell)} \;\; \forall \xi \in K, \; a.e \;\; t \in (0,T), \end{aligned} \tag{84}$$

$$\beta^\ell_\theta(0) = \beta^\ell_0. \tag{85}$$

The following abstract result for parabolic variational inequalities (see, e.g., [18, p.48])is valid.

Theorem 4.3 *Let $X \subset Y = Y' \subset X'$ be a Gelfand triple. Let F be a nonempty, closed, and convex set of* X. *Assume that $a(.,.) : X \times X \longrightarrow \mathbb{R}$ is a continuous and symmetric bilinear form such that for some constants $\alpha > 0$ and c_0,*

$$a(v,v) + c_0\|v\|^2_{\mathrm{Y}} \geq \alpha\|v\|^2_X.$$

Then, for every $u_0 \in F$ and $f \in L^2(0,T;\mathrm{Y})$, there exists a unique function $u \in H^1(0,T;Y) \cap L^2(0,T;X)$ such that $u(0) = u_0$, $u(t) \in F \;\; \forall t \in (0,T)$, and

$$(\dot{u}(t), v - u(t))_{X' \times X} + a(u(t), v - u(t)) \geq (f(t), v - u(t))_Y \;\; \forall v \in F \;\; a.e. \; t \in (0,T).$$

We prove next the unique solvability of Problem $\mathcal{PV}_\theta$.

Lemma 4.3 *There exists a unique solution β_θ of Problem $\mathcal{PV}_\theta$ and it satisfies*

$$\beta_\theta \in H^1(0,T;\mathrm{E}_0) \cap L^2(0,T;\mathrm{E}_1).$$

Proof. The inclusion mapping of $(\mathrm{E}_1, \|.\|_{\mathrm{E}_1})$ into $(\mathrm{E}_0, \|.\|_{\mathrm{E}_0})$ is continuous and its range is dense. We denote by E'_1 the dual space of E_1 and, identifying the dual of E_0 with itself, we can write the Gelfand triple

$$\mathrm{E}_1 \subset \mathrm{E}_0 = \mathrm{E}'_0 \subset \mathrm{E}'_1.$$

We use the notation $(.,.)_{\mathrm{E}'_1 \times \mathrm{E}_1}$ to represent the duality pairing between E_0 and E_1. We have

$$(\beta, \xi)_{\mathrm{E}'_1 \times \mathrm{E}_1} = (\beta, \xi)_{\mathrm{E}_0} \;\; \forall \beta \in \mathrm{E}_0, \; \xi \in \mathrm{E}_1$$

and we note that K is a closed convex set in E_1. Then, using (42), (49) and the fact that $\beta_\theta \in K$ in (43), it is easy to see that Lemma 4.3 is a straight consequence of Theorem 4.3. Now we use the displacement field $\boldsymbol{u}_\eta$ obtained in Lemma 4.1, φ_η obtained in Lemma 4.2 and β_θ obtained in Lemma 4.3 to construct the following Cauchy problem for the stress field.

Problem $\mathcal{PV}_{\eta,\theta}$. Find the stress field $\boldsymbol{\sigma}_{\eta,\theta} = (\boldsymbol{\sigma}^1_{\eta,\theta}, \boldsymbol{\sigma}^2_{\eta,\theta}) : [0,T] \to Q$ which is a solution of the problem

$$\boldsymbol{\sigma}^\ell_{\eta,\theta}(t) = \mathcal{B}^\ell \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell_\eta(t)) + \int_0^t \mathcal{G}^\ell(\boldsymbol{\sigma}^\ell_{\eta,\theta}(s), \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell_\eta(s)), \beta^\ell_\theta(s))\, ds, \quad \ell = 1,2, \quad \text{a,e } t \in (0.T). \tag{86}$$

Lemma 4.4 *There exists a unique solution of Problem $\mathcal{PV}_{\eta,\theta}$ and it satisfies* (66). *Moreover, if $\boldsymbol{u}^\ell_{\eta_i}$, $\beta^\ell_{\eta_i}$ and $\boldsymbol{\sigma}^\ell_{\eta_i,\theta_i}$ represent the solutions of problems $\mathcal{PV}^1_{\eta_i}$, $\mathcal{PV}_{\theta_i}$ and $\mathcal{PV}_{\eta_i,\theta_i}$, respectively, for $(\boldsymbol{\eta}_i, \boldsymbol{\theta}_i) \in W^{1.p}(0,T; Q \times \mathrm{E}_0)$, $i = 1,2$, then there exists $C > 0$ such that*

$$\begin{aligned}\|\boldsymbol{\sigma}_{\eta_1,\theta_1}(t) - \boldsymbol{\sigma}_{\eta_2,\theta_2}(t)\|^2_Q \le C\big(\|\boldsymbol{u}_{\eta_1}(t) - \boldsymbol{u}_{\eta_2}(t)\|^2_V + \\ \int_0^t \|\boldsymbol{u}_{\eta_1}(s) - \boldsymbol{u}_{\eta_2}(s)\|^2_V\, ds + \int_0^t \|\beta_{\theta_1}(s) - \beta_{\theta_2}(s)\|^2_{\mathrm{E}_0} ds\big).\end{aligned} \tag{87}$$

Proof. Let $\Lambda^\ell_{\eta,\theta} : W^{1.p}(0,T; Q^\ell) \to W^{1.p}(0,T; Q^\ell)$ be the operator given by

$$\Lambda^\ell_{\eta,\theta} \boldsymbol{\sigma}^\ell(t) = \mathcal{B}^\ell \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell_\eta(t)) + \int_0^t \mathcal{G}^\ell(\boldsymbol{\sigma}^\ell(s), \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell_\eta(s)), \beta^\ell_\theta(s))\, ds, \quad \ell = 1,2, \tag{88}$$

for all $\boldsymbol{\sigma}^\ell \in W^{1.p}(0,T; Q^\ell)$ and $t \in (0,T)$. For $\boldsymbol{\sigma}^\ell_1$, $\boldsymbol{\sigma}^\ell_2 \in W^{1.p}(0,T; Q^\ell)$ we use (35) and (88) to obtain for all $t \in (0,T)$

$$\|\Lambda^\ell_{\eta,\theta} \boldsymbol{\sigma}^\ell_1(t) - \Lambda^\ell_{\eta,\theta} \boldsymbol{\sigma}^\ell_2(t)\|_Q \le L_{\mathcal{G}^l} \int_0^t \|\boldsymbol{\sigma}^\ell_1(s) - \boldsymbol{\sigma}^\ell_2(s)\|_Q, \qquad \ell = 1,2. \tag{89}$$

It follows from this inequality that for n large enough, a power $(\Lambda^\ell_{\eta,\theta})^n$ is a contraction on the Banach space $W^{1.p}(0,T; Q^\ell)$ and, therefore, there exists a unique element $\boldsymbol{\sigma}^\ell \in W^{1.p}(0,T; Q^\ell)$ such that $\Lambda^\ell_{\eta,\theta} \boldsymbol{\sigma}^\ell = \boldsymbol{\sigma}^\ell$. Moreover, $\boldsymbol{\sigma}$ is the unique solution to Problem $\mathcal{PV}_{\eta,\theta}$ and, when using (86), the regularity of $\boldsymbol{u}_\eta$, β_θ and the properties of the operators $\mathcal{B}^\ell$ and $\mathcal{G}^\ell$, it follows that $\boldsymbol{\sigma}_i \in W^{1,p}(0,T; Q)$. Consider now $(\boldsymbol{\eta}_1, \theta_1)$,$(\boldsymbol{\eta}_2, \theta_2) \in W^{1.p}(0,T; Q \times \mathrm{E}_0)$ and, for $i = 1,2$, denote $\boldsymbol{u}_{\eta_i} = \boldsymbol{u}_i$, $\boldsymbol{\sigma}_{\eta_i} = \boldsymbol{\sigma}_i$, $\beta_{\theta_i} = \beta_i$, $\varphi_{\theta_i} = \varphi_i$. We have

$$\boldsymbol{\sigma}^\ell_i(t) = \mathcal{B}^\ell \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell_i(t)) + \int_0^t \mathcal{G}^\ell(\boldsymbol{\sigma}^\ell_i(s), \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell_i(s)), \beta^\ell_i(s))\, ds, \quad \ell = 1,2, \quad \forall t \in (0,T), \tag{90}$$

and, using the properties (35) and (36) of $\mathcal{G}^\ell$ and $\mathcal{B}^\ell$, we find

$$\begin{aligned}\|\boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t)\|^2_Q \le C\,\big(\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|^2_V + \int_0^t \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|^2_V\, ds \\ + \int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|^2_V\, ds + \int_0^t \|\beta_1(s) - \beta_2(s)\|^2_{\mathrm{E}_0} ds\big), \quad \forall t \in (0,T).\end{aligned} \tag{91}$$

We use a Gronwall argument in the obtained inequality we deduced in (87), which concludes the proof of Lemma 4.4 Finally, as a consequence of these results and using the properties of the operator $\mathcal{G}^\ell$, the operator $\mathcal{E}^\ell$, the functional $\boldsymbol{S}^\ell$ and the function for $t \in [0,T]$, we consider the element

$$\Lambda(\eta,\theta)(t) = \big(\Lambda^1(\eta,\theta)(t), \Lambda^2(\eta,\theta)(t)\big) \in Q \times \mathrm{E}_0 \tag{92}$$

defined by the equations

$$(\Lambda^1(\eta,\theta)(t),\boldsymbol{v})_Q = \sum_{\ell=1}^{2} \big((\mathcal{E}^\ell)^*\nabla\varphi_\eta^\ell(t)\ \varepsilon(\boldsymbol{v}^\ell)\big)_{\mathcal{H}^\ell} + \sum_{\ell=1}^{2}\bigg(\int_0^t \mathcal{G}^\ell\big(\boldsymbol{\sigma}_{\eta,\theta}^\ell(t),\varepsilon(\boldsymbol{u}_\eta^\ell(s)),\beta_\theta^\ell(s)\big)\,ds,\ \varepsilon(\boldsymbol{v}^\ell)\bigg)_Q + j_1(\boldsymbol{u}_\eta,\boldsymbol{v}) \quad \forall v\in V, \tag{93}$$

$$\Lambda^2(\eta,\theta)(t) = \big(\boldsymbol{S}^1(\boldsymbol{\sigma}_{\eta,\theta}^1(t),\varepsilon(\boldsymbol{u}_\eta^1(t)),\beta_\theta^1(t),\boldsymbol{S}^2(\boldsymbol{\sigma}_{\eta,\theta}^2(t),\varepsilon(\boldsymbol{u}_\eta^2(t)),\beta_\theta^2(t)\big)\,. \tag{94}$$

Here, for every $(\eta,\theta)\in W^{1,p}(0,T;Q\times \mathrm{E}_0)$ the element $\Lambda(\eta,\theta)\in W^{1,p}(0,T;Q\times\mathrm{E}_0)$. $\boldsymbol{u}_\eta$, φ_θ, β_η and $\sigma_{\eta,\theta}$, represent the displacement field, the potential field, the damage field and the stress field obtained in Lemmas 4.1, 4.2, 4.3, and 4.4.

Lemma 4.5 *The mapping* Λ *has a unique fixed point* $(\eta^*,\theta^*)\in W^{1,p}(0,T;Q\times L^2(\Omega))$.

Proof. Let $t\in(0,T)$ and $(\eta_1,\theta_1),(\eta_2,\theta_2)\in W^{1,p}(0,T;Q\times\mathrm{E}_0)$. We use the notation $\boldsymbol{u}_{\eta_i}=\boldsymbol{u}_i$, $\dot{\boldsymbol{u}}_{\eta_i}=\dot{\boldsymbol{u}}_i$, $\varphi_{\eta_i}=\varphi_i$ and $\boldsymbol{\sigma}_{\eta_i,\theta_i}=\boldsymbol{\sigma}_i$ for $i=1,2$. Let us start by using (32), hypotheses, (35), (37) and (40) we have

$$\begin{aligned}\big\|\Lambda^1(\eta_1,\theta_1)(t)-\Lambda^1(\eta_2,\theta_2)(t)\big\|_Q^2 &\le \sum_{\ell=1}^{2}\big\|(\mathcal{E}^\ell)^*\nabla\varphi_1^\ell(t)-(\mathcal{E}^\ell)^*\nabla\varphi_2^\ell(t)\big\|_{Q^\ell}^2\\
&+\sum_{\ell=1}^{2}\int_0^t\big\|\mathcal{G}^\ell\big(\boldsymbol{\sigma}_1^\ell(s),\varepsilon(\boldsymbol{u}_1^\ell(s)),\beta_1^l(s)\big)-\mathcal{G}^\ell\big(\boldsymbol{\sigma}_2^\ell(s),\varepsilon(\boldsymbol{u}_2^\ell(s)),\beta_2^l(s)\big)\big\|_{\mathcal{H}^\ell}^2\,ds\\
&+C\big\|p_\nu([\boldsymbol{u}_{1\nu}(t))([\boldsymbol{u}_{1\nu}(t)])-p_\nu([\boldsymbol{u}_{1\nu}(t))([\boldsymbol{u}_{1\nu}(t)])\big\|_{L^2(\Gamma_3)}^2\\
&+C\big\|p_\tau([\boldsymbol{u}_{1\tau}(t))([\boldsymbol{u}_{1\tau}(t)])-p_\tau([\boldsymbol{u}_{1\tau}(t))([\boldsymbol{u}_{1\tau}(t)])\big\|_{L^2(\Gamma_3)}^2\\
&\le C\Big(\|\varphi_1(t)-\varphi_2(t)\|_Q^2+\int_0^t\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s))\|_{\boldsymbol{V}}^2\,ds\\
&+\int_0^t\|\boldsymbol{\sigma}_1(s)-\boldsymbol{\sigma}_2(s))\|_Q^2\,ds+\int_0^t\|\beta_1(s)-\beta_2(s)\|_{\mathrm{E}_0}^2\,ds+\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t))\|_{\boldsymbol{V}}^2\Big).\end{aligned} \tag{95}$$

We use estimates (81), (87) to obtain

$$\|\Lambda^1(\eta_1,\theta_1)(t)-\Lambda^1(\eta_2,\theta_2)(t)\|_Q^2\le C\Big(\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|_{\boldsymbol{V}}^2 +\int_0^t\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s))\|_{\boldsymbol{V}}^2\,ds+\int_0^t\|\beta_1(s)-\beta_2(s)\|_{\mathrm{E}_0}^2\,ds\Big). \tag{96}$$

By similar arguments, from (94), (87) and (38) we obtain

$$\|\Lambda^2(\eta_1,\theta_1)(t)-\Lambda^2(\eta_2,\theta_2)(t)\|_Q^2\le C\bigg(\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|_{\boldsymbol{V}}^2 +\int_0^t\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s))\|_{\boldsymbol{V}}^2\,ds+\|\beta_1(t)-\beta_2(t)\|_{\mathrm{E}_0}^2+\int_0^t\|\beta_1(s)-\beta_2(s)\|_{\mathrm{E}_0}^2\,ds\bigg). \tag{97}$$

Also, since

$$\boldsymbol{u}_i^\ell(t)=\int_0^t\boldsymbol{v}_i^\ell(s)ds+\boldsymbol{u}_0^\ell,\quad t\in[0,T],\quad \ell=1,2,$$

we have

$$\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|_{\boldsymbol{V}} \leq \int_0^t \|\boldsymbol{v}_1(s)-\boldsymbol{v}_2(s))\|_{\boldsymbol{V}}\, ds,$$

which implies

$$\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|^2_{\boldsymbol{V}} + \int_0^t \|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s))\|^2_{\boldsymbol{V}}\, ds \leq c\int_0^t \|\boldsymbol{v}_1(s)-\boldsymbol{v}_2(s))\|^2_{\boldsymbol{V}}\, ds. \tag{98}$$

Therefore,

$$\begin{aligned}\|\Lambda(\eta_1,\theta_1)(t)-\Lambda(\eta_2,\theta_2)(t)\|^2_{Q\times \mathrm{E}_0} \leq C\Big(\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|^2_{\boldsymbol{V}}+ \\ \int_0^t \|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s))\|^2_{\boldsymbol{V}}\, ds + \|\beta_1(t)-\beta_2(t)\|^2_{\mathrm{E}_0} + \int_0^t \|\beta_1(s)-\beta_2(s)\|^2_{\mathrm{E}_0}\, ds\Big).\end{aligned} \tag{99}$$

Moreover, from (70) we obtain

$$\begin{aligned}(\dot{\boldsymbol{v}}_1-\dot{\boldsymbol{v}}_2,\boldsymbol{v}_1-\boldsymbol{v}_2)_{\boldsymbol{V}'\times\boldsymbol{V}} + \sum_{\ell=1}^{2}(\mathcal{A}^\ell\varepsilon(\boldsymbol{v}_1^\ell)-\mathcal{A}^\ell\varepsilon(\boldsymbol{v}_2^\ell),\varepsilon(\boldsymbol{v}_1^\ell-\boldsymbol{v}_2^\ell))_{\mathcal{H}^\ell} \\ +j_2(\boldsymbol{v}_1,\boldsymbol{v}_1-\boldsymbol{v}_2)-j_2(\boldsymbol{v}_2,\boldsymbol{v}_1-\boldsymbol{v}_2)+(\eta_1-\eta_2,\boldsymbol{v}_1-\boldsymbol{v}_2)_{\boldsymbol{V}'\times\boldsymbol{V}}=0.\end{aligned} \tag{100}$$

We use (39) (51) to deduce that

$$j_2(\boldsymbol{v}_1,\boldsymbol{v}_1-\boldsymbol{v}_2)-j_2(\boldsymbol{v}_2,\boldsymbol{v}_1-\boldsymbol{v}_2)\geq 0. \tag{101}$$

It follows from (100) and (101) that

$$\begin{aligned}(\dot{\boldsymbol{v}}_1-\dot{\boldsymbol{v}}_2,\boldsymbol{v}_1-\boldsymbol{v}_2)_{\boldsymbol{V}'\times\boldsymbol{V}} + \sum_{\ell=1}^{2}(\mathcal{A}^\ell\varepsilon(\boldsymbol{v}_1^\ell)-\mathcal{A}^\ell\varepsilon(\boldsymbol{v}_2^\ell),\varepsilon(\boldsymbol{v}_1^\ell-\boldsymbol{v}_2^\ell))_{\mathcal{H}^\ell} \\ \leq -(\eta_1-\eta_2,\boldsymbol{v}_1-\boldsymbol{v}_2)_{\boldsymbol{V}'\times\boldsymbol{V}}.\end{aligned} \tag{102}$$

We integrate this equality with respect to time, use the initial conditions $\boldsymbol{v}_1(0)=\boldsymbol{v}_2(0)=\boldsymbol{v}_0$ and condition (33) to find

$$\begin{aligned}\min(m_{\mathcal{A}^1},m_{\mathcal{A}^2})\int_0^t \|\boldsymbol{v}_1(s)-\boldsymbol{v}_2(s))\|^2_{\boldsymbol{V}}\, ds \leq \\ -\int_0^t (\eta_1(s)-\eta_2(s),\boldsymbol{v}_1(s)-\boldsymbol{v}_2(s))_{\boldsymbol{V}'\times\boldsymbol{V}}\, ds\end{aligned}$$

for all $t\in[0,T]$. Then, using the inequality $2ab\leq \frac{a^2}{m}+mb^2$, we obtain

$$\int_0^t \|\boldsymbol{v}_1(s)-\boldsymbol{v}_2(s))\|^2_{\boldsymbol{V}}\, ds \leq C\int_0^t \|\eta_1(s)-\eta_2(s)\|^2_{\boldsymbol{V}'}\, ds \quad \forall t\in[0,T]. \tag{103}$$

Form (84), we deduce that

$$(\dot\beta_1-\dot\beta_2,\beta_1-\beta_2)_{\mathrm{E}_0}+a(\beta_1-\beta_2,\beta_1-\beta_2)\leq(\theta_1-\theta_1,\beta_1-\beta_2)_{\mathrm{E}_0} \ \ \forall t\in[0,T].$$

Integrating the previous inequality with respect to time, using the initial conditions $\beta_1(0) = \beta_2(0) = \beta_0$ and inequality $a(\beta_1 - \beta_2, \beta_1 - \beta_2) \geq 0$, we find

$$\frac{1}{2}\|\beta_1(s) - \beta_2(s))\|^2_{\mathrm{E}_0}\, ds \leq \int_0^t (\theta_1(s) - \theta_2(s), \beta_1(s) - \beta_2(s))_{\mathrm{E}_0}\, ds, \tag{104}$$

which implies

$$\|\beta_1(s) - \beta_2(s))\|^2_{\mathrm{E}_0}\, ds \leq \int_0^t \|\theta_1(s) - \theta_2(s)\|^2_{\mathrm{E}_0}\, ds + \int_0^t \|\beta_1(s) - \beta_2(s))\|^2_{L^2(\Omega)}\, ds.$$

This inequality, combined with Gronwall's inequality, leads to

$$\|\beta_1(s) - \beta_2(s))\|^2_{\mathrm{E}_0}\, ds \leq C\int_0^t \|\theta_1(s) - \theta_2(s)\|^2_{\mathrm{E}_0}\, ds \quad \forall [0,T]. \tag{105}$$

Form the previous inequality and estimates (103), (105) and (99) it follows now that

$$\|\Lambda(\eta_1, \theta_1)(t) - \Lambda(\eta_2, \theta_2)(t)\|^2_{Q\times \mathrm{E}_0} \leq C\int_0^t \|(\eta_1, \theta_1)(s) - (\eta_2, \theta_2)(s)\|^2_{Q\times \mathrm{E}_0}\, ds.$$

Reiterating this inequality n times we obtain

$$\|\Lambda^n(\eta_1, \theta_1) - \Lambda^n(\eta_2, \theta_2)\|^2_{W^{1,p}(0,T;Q\times \mathrm{E}_0)} \leq \frac{C^n T^n}{n!}\|(\eta_1, \theta_1) - (\eta_2, \theta_2)\|^2_{W^{1,p}(0,T;Q\times \mathrm{E}_0)}.$$

Thus, for n sufficiently large, Λ^n is a contraction on $W^{1,p}(0,T;Q\times \mathrm{E}_0)$, and so Λ has a unique fixed point in this Banach space.

Now, we have all the ingredients to prove Theorem 4.1.

Existence. Let $(\eta^*, \theta^*) \in W^{1,p}(0,T;Q\times\mathrm{E}_0)$ be the fixed point of Λ defined by (92)-(94), and denote

$$\boldsymbol{u}_* = \boldsymbol{u}_{\eta^*}, \quad \varphi_* = \varphi_{\eta^*}, \quad \beta_* = \beta_{\theta^*}, \tag{106}$$

$$\boldsymbol{\sigma}^\ell_* = \mathcal{A}^\ell \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell_*) + (\mathcal{E}^\ell)^* \nabla\varphi^\ell_* + \boldsymbol{\sigma}^\ell_{\eta^*\theta^*}, \quad \ell = 1, 2, \tag{107}$$

$$\boldsymbol{D}^\ell_* = \mathcal{E}^\ell \varepsilon(\boldsymbol{u}^\ell_*) - \mathcal{B}^\ell \nabla\varphi^\ell_*, \quad \ell = 1, 2. \tag{108}$$

We prove that $\{\boldsymbol{u}_*, \boldsymbol{\sigma}_*, \varphi_*, \beta_*, \boldsymbol{D}_*\}$ satisfies (57)–(62) and the regularities(64)–(67). Indeed, we write (70) for $\eta^* = \eta$ and use (106) to find

$$\begin{aligned}&(\ddot{u}_*(t), v)_{\boldsymbol{V}'\times \boldsymbol{V}} + \sum_{\ell=1}^{2} (\mathcal{A}^\ell \varepsilon(\dot{\boldsymbol{u}}^\ell_*(t)),\ \varepsilon(\boldsymbol{v}^\ell))_{\mathcal{H}^\ell} + j_2(\dot{\boldsymbol{u}}_*(t), \boldsymbol{v})\\ &+(\eta^*(t), v)_{\boldsymbol{V}'\times \boldsymbol{V}} = (\mathbf{f}(t), \boldsymbol{v})_{\boldsymbol{V}'\times \boldsymbol{V}} \quad \forall \boldsymbol{v} \in \boldsymbol{V}, a.e.\ t \in [0,T].\end{aligned} \tag{109}$$

For the equalities $\Lambda^1(\eta^*, \theta^*) = \eta^*$ and $\Lambda^2(\eta^*, \theta^*) = \theta^*$ it follows that

$$\begin{aligned}(\eta^*(t), v)_{Q\times \boldsymbol{V}} &= \sum_{\ell=1}^{2} \left(\mathcal{E}^\ell \varepsilon(\boldsymbol{u}^\ell(t)),\ \varepsilon(\boldsymbol{v}^\ell)\right)_Q + \sum_{\ell=1}^{2} \Big(\int_0^t \mathcal{G}^\ell \Big(\boldsymbol{\sigma}^\ell(s) - \mathcal{A}^\ell \varepsilon(\dot{\boldsymbol{u}}^\ell(s))\\ &-\mathcal{E}^\ell \nabla\varphi^\ell(s), \varepsilon(\boldsymbol{u}^\ell(s)), \beta(s)\Big)\, ds,\ \varepsilon(\boldsymbol{v}^\ell)\Big)_Q + j_1(\boldsymbol{u}_*(t), \boldsymbol{v})\end{aligned} \tag{110}$$

$$\theta^*(t) = \sum_{\ell=1}^{2}(\mathcal{S}^\ell(\boldsymbol{\sigma}^\ell(t) - \mathcal{A}^\ell\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\ell(t)) - (\mathcal{E}^\ell)^*\nabla\varphi^\ell(t), \boldsymbol{\varepsilon}(\boldsymbol{u}^\ell(t)), \beta^\ell(t)). \tag{111}$$

We now substitute (110) in (109) to obtain

$$\begin{aligned}(\ddot{\boldsymbol{u}}_*(t), v)_{\boldsymbol{V}'\times\boldsymbol{V}} + \sum_{\ell=1}^{2}(\mathcal{A}^\ell\varepsilon(\dot{\boldsymbol{u}}_*^\ell(t)),\ \varepsilon(\boldsymbol{v}^\ell))_{\mathcal{H}^\ell} + j_2(\dot{\boldsymbol{u}}_*(t), \boldsymbol{v})\\ + \sum_{\ell=1}^{2}\big(\mathcal{B}^\ell\varepsilon(\boldsymbol{u}_*^\ell(t)),\ \varepsilon(\boldsymbol{v}^\ell)\big)_{\mathcal{H}^\ell} + \sum_{\ell=1}^{2}\big((\mathcal{E}^\ell)^*\nabla\varphi_*^\ell,\ \varepsilon(\boldsymbol{v}^\ell)\big)_{\mathcal{H}^\ell}\\ + \sum_{\ell=1}^{2}\Big(\int_0^t \mathcal{G}^\ell\big(\boldsymbol{\sigma}_*^\ell(s) - \mathcal{A}^\ell\varepsilon(\dot{\boldsymbol{u}}_*^\ell(s)) - (\mathcal{E}^\ell)^*\nabla\varphi_*^\ell(s), \varepsilon(\boldsymbol{u}_*^\ell(s)), \beta_*^\ell(s)\big)\,ds,\ \varepsilon(\boldsymbol{v}^\ell)\Big)_{\mathcal{H}^\ell}\\ + j_1(\boldsymbol{u}_*(t), \boldsymbol{v}) = (\mathbf{f}(t), v)_{\boldsymbol{V}'\times\boldsymbol{V}},\quad \forall \boldsymbol{v} \in \boldsymbol{V},\end{aligned} \tag{112}$$

and we substitute (111) in (84) to have

$$\begin{aligned}\beta_*(t) \in K, \qquad \sum_{\ell=1}^{2}(\dot{\beta}_*^\ell(t), \xi^\ell - \beta_*^\ell(t))_{L^2(\Omega^\ell)} + a(\beta(t), \xi - \beta(t)) \geq\\ \sum_{\ell=1}^{2}\Big(\phi^\ell\big(\boldsymbol{\sigma}_*^\ell(t) - \mathcal{A}^\ell\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_*^\ell(t)) - (\mathcal{E}^\ell)^*\nabla\varphi^\ell(t), \boldsymbol{\varepsilon}(\boldsymbol{u}_*^\ell(t)), \beta_*^\ell(t)\big), \xi^\ell - \beta_*^\ell(t)\Big)_{L^2(\Omega^\ell)},\\ \forall \xi \in K,\ \text{a.e. } t \in [0, T].\end{aligned} \tag{113}$$

We write now (80) for $\eta = \eta^*$ and use (106) to see that

$$\begin{aligned}\sum_{\ell=1}^{2}(\mathcal{B}^\ell\nabla\varphi_*^\ell(t), \nabla\xi^\ell)_{H^\ell} - \sum_{\ell=1}^{2}(\mathcal{E}^\ell\varepsilon(\boldsymbol{u}_*^\ell(t)), \nabla\xi^\ell)_{H^\ell} = (q(t), \xi)_W,\\ \forall \xi \in W,\quad \text{a.e. } t \in [0, T].\end{aligned} \tag{114}$$

The relations (106)- (108), (112), (113), and (114) allow us to conclude now that $\{\boldsymbol{u}_*, \boldsymbol{\sigma}_*, \varphi_*, \beta_*, \boldsymbol{D}_*\}$ satisfies (57). Next, (62) and the regularities (64), (65), (68) follow from Lemmas 4.1, 4.2 and 4.3. Since $\boldsymbol{u}_*$ and φ_* satisfy (64) and (68), it follows from Lemma 4.4 and (107) that

$$\sigma_* \in L^2(0, T; \mathcal{H}). \tag{115}$$

We choose $v = (v^1, v^2)$ with $v^\ell = \omega^\ell \in D(\Omega^\ell)^d$ and $v^{3-\ell} = 0$ in (112) and by (106) and (53)

$$\rho^\ell\ddot{\boldsymbol{u}}_*^\ell = \operatorname{Div}\boldsymbol{\sigma}_*^\ell + \boldsymbol{f}_0^\ell,\ \text{a.e. } t \in [0, T],\quad \ell = 1, 2.$$

Also, by (47), (64) and (115) we have

$$(\operatorname{Div}\boldsymbol{\sigma}_*^1, \operatorname{Div}\boldsymbol{\sigma}_*^2) \in L^2(0, T; \boldsymbol{V}').$$

Let $t_1, t_2 \in [0, T]$, by (26), (36), (37) and (108), we deduce that

$$\|\boldsymbol{D}_*(t_1) - \boldsymbol{D}_*(t_2)\|_H \leq C\left(\|\varphi_*(t_1) - \varphi_*(t_2)\|_W + \|\boldsymbol{u}_*(t_1) - \boldsymbol{u}_*(t_2)\|_{\boldsymbol{V}}\right).$$

The regularity of $\boldsymbol{u}_*$ and φ_* given by (64) and (65) implies

$$\boldsymbol{D}_* \in C(0,T;H). \tag{116}$$

We choose $\phi = (\phi^1, \phi^2)$ with $\phi^\ell \in D(\Omega^\ell)^d$ and $\phi^{3-\ell} = 0$ in (114), and using (54), (108) we find

$$div \boldsymbol{D}_*^\ell(t) = q_0^\ell(t) \quad \forall t \in [0,T], \quad \ell = 1, 2,$$

and, by (48), (116), we obtain

$$\boldsymbol{D}_* \in C(0,T;\mathcal{W}).$$

Finally, we conclude that the weak solution $\{\boldsymbol{u}_*, \boldsymbol{\sigma}_*, \varphi_*, \beta_*, \boldsymbol{D}_*\}$ of the piezoelectric contact Problem **PV** has the regularities (64)–(67), which concludes the existence part of Theorem 4.1.

Uniqueness. The uniqueness of the solution is a consequence of the uniqueness of the fixed point of the operator Λ defined by (93)-(94) and the unique solvability of the Problems $\mathcal{PV}_\eta^1$, $\mathcal{PV}_\eta^2$, $\mathcal{PV}_\theta$ and $\mathcal{PV}_{\eta,\theta}$.

Conclusion

We presented a model for the dynamic process of frictional contact between two elasto-viscoplastic piezoelectric bodies with damage response. The contact was modeled with a normal compliance and a normal damped. The existence of the unique weak solution for the problem was established by using arguments from the parabolic inequalities, differential equations and fixed-point arguments.

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