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Delay-Independent Stability Conditions for a Class of Nonlinear Mechanical Systems

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Abstract: A mechanical system with linear gyroscopic forces and nonlinear homogeneous dissipative and positional forces is studied. The case is considered where there is a time-varying delay in positional forces. With the aid of the decomposition method and the Razumikhin approach, conditions are obtained ensuring that the trivial equilibrium position of the system under investigation is asymptotically stable for any nonnegative, continuous and bounded delay. Estimates for the convergence rate of motions are derived. The developed approach is used in a problem of stabilization of mechanical systems via controls with delay in a feedback law. An example is given to demonstrate the effectiveness of the obtained results.

Keywords: *mechanical system; nonlinear forces; stability; time-varying delay; decomposition; stabilization.*

Mathematics Subject Classification (2010): 34K20, 93D30.

1 Introduction

Systems of high-dimensional second-order differential equations are widely used as mathematical models of gyroscopic devices [1–3]. An effective approach to the analysis of stability and other dynamic properties of such models consists of the decomposition of the complete system into first-order precession and nutation subsystems.

The justification of the correctness of such a decomposition for linear stationary gyroscopic systems was given in [1, 2] by the Lyapunov first method via the expansion of the roots of the characteristic equations in series with respect to negative powers of a large parameter. It was proved that, for sufficiently large values of the parameter, the asymptotic stability of the isolated nutation and precession subsystems implies the same property for the complete system.

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Another approach to the justification of decomposition of gyroscopic systems into precession and nutation subsystems was proposed in [4]. This approach is based on the Lyapunov direct method. Therefore, its application turned out to be effective not only for linear time-invariant systems, but also for some classes of nonlinear and time-varying systems (see [5–9]).

In particular, in [7], it was used for the stability analysis of mechanical systems with linear gyroscopic forces and nonlinear homogeneous dissipative and positional forces. A special form of decomposition was constructed and new conditions of the asymptotic stability of equilibrium positions were found.

In the present paper, we will consider the same class of nonlinear mechanical systems as in [7] under the additional assumption that there is a time-varying delay in positional forces. Our objective is to study the impact of delay on the stability of equilibrium positions. It is well known (see, for instance, [10–12]) that an introduction of a delay might destroy stability. With the aid of the decomposition method and a special technique for the application of the Razumikhin theorem to nonlinear time-delay systems developed in [13, 14], we will obtain conditions providing the asymptotic stability of equilibrium positions for any nonnegative, continuous and bounded delay. In addition, we will derive estimates for the convergence rate of motions. Moreover, we will show that the obtained results can be effectively used for the stabilization of mechanical systems via controls with delay in a feedback law.

2 Background and Problem Formulation

In this paper, $\mathbb{R}$ denotes the field of real numbers, $\mathbb{R}^n$ is the n-dimensional Euclidean space with the associated norm $\|\cdot\|$ of a vector, the notation $\mathbb{R}^{n\times n}$ is used for the vector space of $n \times n$ matrices.

Definition 2.1 (see [15,16]) A function $f(x) : \mathbb{R}^n \mapsto \mathbb{R}$ is called homogeneous of the order $\lambda \in \mathbb{R}$ if $f(cx) = c^\lambda f(x)$ for any $c > 0$ and $x \in \mathbb{R}^n$.

Remark 2.1 In the present contribution, the homogeneity with respect to the standard dilation is considered [16, 17].

Let motions of a mechanical system be modeled by the equations

$$A\ddot{q}(t) + (B(\dot{q}(t)) + G)\dot{q}(t) + Q(q(t)) = 0. \tag{1}$$

Here $q(t), \dot{q}(t) \in \mathbb{R}^n$ are the vectors of generalized coordinates and velocities, respectively, $A, G \in \mathbb{R}^{n\times n}$ are constant matrices, the entries of the matrix $B(\dot{q}) \in \mathbb{R}^{n\times n}$ are continuous for $\dot{q} \in \mathbb{R}^n$ homogeneous functions of the order $\nu > 0$, the components of the vector $Q(q) \in \mathbb{R}^n$ are continuously differentiable for $q \in \mathbb{R}^n$ homogeneous functions of the order $\mu > 1$.

The system (1) has the trivial equilibrium position

$$q = \dot{q} = 0. \tag{2}$$

Stability of this equilibrium position was studied in [7] with the aid of the decomposition method. The auxiliary isolated subsystems

$$G\dot{y}(t) = -Q(y(t)), \tag{3}$$

$$A\dot{z}(t) = -(B(z(t)) + G)z(t) \tag{4}$$

were constructed, and the following constraints were imposed on the equations under consideration.

Assumption 2.1 The inequality $\mu > \nu + 1$ holds.

Assumption 2.2 The matrix A is symmetric and positive definite, while the matrix G is skew-symmetric and nonsingular.

Assumption 2.3 The function $\dot{q}^\top B(\dot{q})\dot{q}$ is positive definite.

Assumption 2.4 There exists a continuously differentiable homogeneous of the order $\nu + 1$ vector function $w(z) \in \mathbb{R}^n$ such that

$$\frac{\partial w(z)}{\partial z} A^{-1} G z = B(z)z$$

for $z \in \mathbb{R}^n$.

Assumption 2.5 The zero solution of the subsystem (3) is asymptotically stable.

Remark 2.2 Assumptions 2.2 and 2.3 imply that the system (1) is under the action of linear gyroscopic forces $-G\dot{q}$, nonlinear homogeneous dissipative forces $-B(\dot{q})\dot{q}$ and nonlinear homogeneous positional forces $-Q(q)$.

Let us note that nonlinear homogeneous forces are widely used in mathematical models of mechanical systems (see, e.g., [18–23]). Such forces can be related to both physical configurations and purely nonlinear material properties. Moreover, nonlinear homogeneous functions provide smooth approximations of non-smooth forces [24].

Remark 2.3 From the conditions imposed on the matrix G, it follows that n should be an even number.

Remark 2.4 A criterion for the fulfilment of Assumption 2.4 was obtained in [7].

In [7], it was proved that, under Assumptions 2.1–2.5, the equilibrium position (2) of the system (1) is asymptotically stable.

Remark 2.5 It is known [7] that Assumption 2.1 cannot be relaxed.

The objective of this paper is to study the impact of delay in positional forces on the stability of the equilibrium position. We consider the system

$$A\ddot{q}(t) + (B(\dot{q}(t)) + G)\dot{q}(t) + Q(q(t)) + D(q(t - \tau(t)) = 0, \tag{5}$$

where the components of the vector $D(q) \in \mathbb{R}^n$ are continuously differentiable for $q \in \mathbb{R}^n$ homogeneous functions of the order μ, $\tau(t)$ is a nonnegative, continuous and bounded for $t \geq 0$ delay, and the remaining notation is the same as for (1).

For a given delay $\tau(t)$, denote $h = \sup_{t\geq 0} \tau(t)$. Let the initial functions for the solutions of (5) belong to the space $C^1([-h, 0], \mathbb{R}^n)$ of continuously differentiable functions $\varphi(\theta) : [-h, 0] \mapsto \mathbb{R}^n$ with the uniform norm

$$\|\varphi\|_h = \max_{\theta\in[-h,0]} \left(\|\varphi(\theta)\| + \|\dot{\varphi}(\theta)\|\right).$$

We will look for conditions ensuring the delay-independent asymptotic stability of the equilibrium position (2) of the system (5).

3 Stability Analysis

Instead of (3), construct a new isolated subsystem in the form

$$G\dot{y}(t) = -Q(y(t)) - D(y(t)). \tag{6}$$

Assumption 3.1 The zero solution of the subsystem (6) is asymptotically stable.

Theorem 3.1 *Let Assumptions 2.1–2.4 and 3.1 be fulfilled. Then the equilibrium position (2) of the system (5) is asymptotically stable for any nonnegative, continuous and bounded for $t \geq 0$ delay $\tau(t)$.*

Proof. Define new variables by the formulae

$$z(t) = \dot{q}(t), \quad Gy(t) + w(z(t)) = A\dot{q}(t) + Gq(t),$$

where the vector function $w(z)$ satisfies the conditions of Assumption 2.4.

We obtain the system

$$\begin{aligned} G\dot{y}(t) =& -Q(y(t)) - D(y(t)) + \Big(Q(y(t)) - Q\left(y(t) - G^{-1}Az(t) + G^{-1}w(z(t))\right)\Big) \\ &+ \Big(D(y(t)) - D\left(y(t-\tau(t)) - G^{-1}Az(t-\tau(t)) + G^{-1}w(z(t-\tau(t)))\right)\Big) \\ &+ \frac{\partial w(z(t))}{\partial z}A^{-1}\Big(B(z(t))z(t) + Q\left(y(t) - G^{-1}Az(t) + G^{-1}w(z(t))\right)\Big) \\ &+ \frac{\partial w(z(t))}{\partial z}A^{-1}D\left(y(t-\tau(t)) - G^{-1}Az(t-\tau(t)) + G^{-1}w(z(t-\tau(t)))\right), \\ A\dot{z}(t) =& -(B(z(t)) + G)z(t) - Q\left(y(t) - G^{-1}Az(t) + G^{-1}w(z(t))\right) \\ &- D\left(y(t-\tau(t)) - G^{-1}Az(t-\tau(t)) + G^{-1}w(z(t-\tau(t)))\right). \end{aligned} \tag{7}$$

For a solution $(y^\top(t), z^\top(t))^\top$ of (7), denote by $(y_t^\top, z_t^\top)^\top$ the restriction of the solution to the segment $[t-h, t]$, i.e., $(y_t^\top, z_t^\top)^\top : \theta \mapsto (y^\top(t+\theta), z^\top(t+\theta))^\top$, $\theta \in [-h, 0]$. Let

$$\|(y_t^\top, z_t^\top)^\top\|_h = \max_{\theta\in[-h,0]} \|(y^\top(t+\theta), z^\top(t+\theta))^\top\|.$$

The system (6) is homogeneous. Therefore, from Assumption 3.1 it follows (see [15,16]) that, for any number $\gamma_1 > \mu$, there exists a continuously differentiable for $y \in \mathbb{R}^n$ homogeneous of the order $\gamma_1 - \mu + 1$ Lyapunov function $V_1(y)$ such that the inequalities

$$a_1\|y\|^{\gamma_1-\mu+1} \leq V_1(y) \leq a_2\|y\|^{\gamma_1-\mu+1}, \tag{8}$$

$$\left\|\frac{\partial V_1(y)}{\partial y}\right\| \leq a_3\|y\|^{\gamma_1-\mu}, \quad \dot{V}_1\big|_{(6)} \leq -a_4\|y\|^{\gamma_1}$$

are valid for $y \in \mathbb{R}^n$. Here $a_i > 0$, $i = 1, 2, 3, 4$.

Furthermore, it should be noted that the zero solution of (4) is asymptotically stable, and a Lyapunov function for this subsystem can be chosen as follows:

$$V_2(z) = \left(z^\top Az\right)^{(\gamma_2-\nu)/2},$$

where $\gamma_2 > \nu + 1$.

Next, construct the function

$$V(y,z) = V_1(y) + \eta V_2(z), \tag{9}$$

where η is a positive parameter. The function (9) is positive definite and satisfies the estimates

$$a_1\|y\|^{\gamma_1-\mu+1} + \eta a_5\|z\|^{\gamma_2-\nu} \le V(y,z) \le a_2\|y\|^{\gamma_1-\mu+1} + \eta a_6\|z\|^{\gamma_2-\nu}$$

for $y, z \in \mathbb{R}^n$, where a_5, a_6 are positive coefficients.

Consider its derivative along the solutions of (7). We obtain that there exists a number $\delta > 0$ such that

$$\begin{gathered}\dot V\Big|_{(7)} \le -a_4\|y(t)\|^{\gamma_1} - \eta b_1\|z(t)\|^{\gamma_2}\\ +b_2\Big(\eta\|y(t)\|^{\mu}\|z(t)\|^{\gamma_2-\nu-1} + \|y(t)\|^{\gamma_1-1}\|z(t)\|\\ +\|y(t)\|^{\gamma_1-\mu}\|z(t)\|^{\mu} + \|y(t)\|^{\gamma_1-\mu}\|z(t)\|^{2\nu+1}\Big)\\ +b_3\left(\|y(t)\|^{\gamma_1-\mu} + \eta\|z(t)\|^{\gamma_2-\nu-1}\right)\Big\|D\left(y(t) - G^{-1}Az(t) + G^{-1}w(z(t))\right)\\ -D\left(y(t-\tau(t)) - G^{-1}Az(t-\tau(t)) + G^{-1}w(z(t-\tau(t)))\right)\Big\|\end{gathered}$$

for $\|(y_t^\top, z_t^\top)^\top\|_h < \delta$. Here b_1, b_2, b_3 are positive constants.

With the aid of the Young inequality, it can be verified that if δ and η are sufficiently small and

$$\max\left\{1;\ \frac{\mu}{2\nu+1}\right\} < \frac{\gamma_1}{\gamma_2} \le \frac{\mu}{\nu+1},$$

then

$$\begin{gathered}\dot V\Big|_{(7)} \le -\frac{1}{2}a_4\|y(t)\|^{\gamma_1} - \frac{1}{2}\eta b_1\|z(t)\|^{\gamma_2}\\ +b_3\left(\|y(t)\|^{\gamma_1-\mu} + \eta\|z(t)\|^{\gamma_2-\nu-1}\right)\Big\|D\left(y(t) - G^{-1}Az(t) + G^{-1}w(z(t))\right)\\ -D\left(y(t-\tau(t)) - G^{-1}Az(t-\tau(t)) + G^{-1}w(z(t-\tau(t)))\right)\Big\|\end{gathered}$$

for $\|(y_t^\top, z_t^\top)^\top\|_h < \delta$.

Assume that the following conditions are fulfilled for a solution $(y^\top(t), z^\top(t))^\top$ of (7):

(i) $\|(y_t^\top, z_t^\top)^\top\|_h < \delta$,

(ii) $V(y(\theta), z(\theta)) \le 2V(y(t), z(t))$ for $\theta \in [t-h, t]$.

Using (ii) and the estimates (8), we arrive at the inequalities

$$\|y(\theta)\| \le c_1\left(\|y(t)\| + \|z(t)\|^{\frac{\gamma_2-\nu}{\gamma_1-\mu+1}}\right), \tag{10}$$

$$\|z(\theta)\| \le c_2\left(\|y(t)\|^{\frac{\gamma_1-\mu+1}{\gamma_2-\nu}} + \|z(t)\|\right) \tag{11}$$

for $\theta \in [t-h, t]$, where c_1 and c_2 are positive constants.

Let $\gamma_1 = \gamma_2\mu/(\nu+1)$. Then $\gamma_1 - \mu + 1 > \gamma_2 - \nu$. With the aid of (10), (11) and the mean value theorem, it can be shown (see [13, 14]) that

$$\Big\| D\left(y(t) - G^{-1}Az(t) + G^{-1}w(z(t))\right)$$

$$-D\left(y(t-\tau(t)) - G^{-1}Az(t-\tau(t)) + G^{-1}w(z(t-\tau(t)))\right)\Big\|$$

$$\le \tilde{c}\left(\|y(t)\|^{\gamma_1-\mu+1} + \|z(t)\|^{\gamma_2-\nu}\right)^{\frac{\mu-1}{\gamma_1-\mu+1}+\frac{1}{\gamma_2-\nu}},$$

where $\tilde{c} = \text{const} > 0$.

Applying the Young inequality once again, we obtain that, for an appropriate choice of δ, the estimate

$$\dot{V}\Big|_{(7)} \le -\frac{1}{3}a_4\|y(t)\|^{\gamma_1} - \frac{1}{3}\eta b_1\|z(t)\|^{\gamma_2} \tag{12}$$

holds. Hence the Lyapunov function (9) satisfies the conditions of the Razumikhin theorem (see [10]). Therefore, the zero solution of (7) is asymptotically stable for any nonnegative, continuous and bounded delay. From the relationship between the variables $y(t), z(t)$ and $q(t), \dot{q}(t)$, it follows that the equilibrium position (2) of the system (5) possesses the same property.

4 Estimates of Motions

In this section, we will show that, with the aid of the Lyapunov function (9) and the differential inequalities method (see [25,26]), estimates for the convergence rate of motions of (5) to the equilibrium position (2) can be derived.

Let Assumptions 2.1–2.4 and 3.1 be fulfilled. Consider the function (9) with $\gamma_1 = \gamma_2\mu/(\nu+1)$. According to the proof of Theorem 3.1, for an appropriate choice of η and δ, the fulfilment of (i) and (ii) implies that (12) holds.

Using inequalities (8) and (12), we obtain

$$\dot{V}\Big|_{(7)} \le -dV^{\frac{\gamma_1}{\gamma_1-\mu+1}}(y(t), z(t)),$$

where $d = \text{const} > 0$.

Applying the approach developed in [14], one can verify the existence of positive numbers $\Delta, \alpha_1, \alpha_2$ such that if the initial conditions of a solution $(y^\top(t), z^\top(t))^\top$ of (7) satisfy the inequalities $t_0 \ge 0$, $\|(y_{t_0}^\top, z_{t_0}^\top)^\top\|_h < \Delta$, then

$$\|y(t)\| \le \alpha_1(t - t_0 + 1)^{-\frac{1}{\mu-1}},$$

$$\|z(t)\| \le \alpha_2(t - t_0 + 1)^{-\omega} \tag{13}$$

for $t \ge t_0$, where

$$\omega = \frac{1}{(\mu-1)(\nu+1)}\left(\mu - \frac{\mu-\nu-1}{\gamma_2-\nu}\right). \tag{14}$$

It is worth noting that, to obtain more precise estimate (13) in the sense of minimization of the exponent, one should pass to the limit in (14) as $\gamma_2 \to \infty$.

Taking into account the relationship between the variables $y(t), z(t)$ and $q(t), \dot{q}(t)$, we arrive at the following theorem.

Theorem 4.1 *Let Assumptions 2.1–2.4 and 3.1 be fulfilled. Then, for any $\rho \in (0,1)$ and any nonnegative, continuous and bounded for $t \geq 0$ delay $\tau(t)$, there exist positive numbers $\tilde{\Delta}, \beta_1, \beta_2$ such that if for a solution $q(t)$ of (5) the inequalities $t_0 \geq 0$, $\|q_{t_0}\|_h < \tilde{\Delta}$ hold, then*

$$\|q(t)\| \leq \beta_1 (t - t_0 + 1)^{-\frac{1}{\mu-1}}, \qquad \|\dot{q}(t)\| \leq \beta_2 (t - t_0 + 1)^{-\frac{\rho\mu}{(\mu-1)(\nu+1)}}$$

for $t \geq t_0$.

5 Control Synthesis

Let the system (1) be of the form

$$A\ddot{q}(t) + (B(\dot{q}(t)) + G)\dot{q}(t) + \frac{\partial \Pi(q(t))}{\partial q} = 0. \tag{15}$$

Here $\Pi(q)$ is a twice continuously differentiable for $q \in \mathbb{R}^n$ homogeneous of the order $\mu + 1$ function. Thus, the positional forces in (15) are potential.

We will suppose that the potential energy $\Pi(q)$ is a negative definite function. Then, under Assumptions 2.2, 2.3, the equilibrium position (2) of (15) is unstable (see [2, 26]).

Next, consider the corresponding control system

$$A\ddot{q}(t) + (B(\dot{q}(t)) + G)\dot{q}(t) + \frac{\partial \Pi(q(t))}{\partial q} = U, \tag{16}$$

where U is a control vector. Our objective is to design a feedback control law stabilizing the equilibrium position under the constraint that there exists a delay in the control scheme.

Let

$$U = -\varepsilon \|q(t - \tau(t))\|^{\mu-1} G q(t - \tau(t)). \tag{17}$$

Here ε is a positive parameter.

Theorem 5.1 *If Assumptions 2.1–2.4 are fulfilled, then the equilibrium position (2) of the system (16) closed by the control (17) is asymptotically stable for any $\varepsilon > 0$ and any continuous, nonnegative and bounded for $t \geq 0$ delay.*

Proof. To prove the theorem, it is sufficient to show the fulfilment of Assumption 3.1.

The corresponding subsystem (6) takes the form

$$\dot{y}(t) = -G^{-1} \frac{\partial \Pi(y(t))}{\partial y} - \varepsilon \|y(t)\|^{\mu-1} y(t). \tag{18}$$

Consider the Lyapunov function

$$V_1(y) = -\Pi(y). \tag{19}$$

This function is positive definite. Differentiating (19) along the solutions of (18), we obtain

$$\dot{V}_1\big|_{(18)} = \varepsilon(\mu + 1)\|y(t)\|^{\mu-1}\Pi(y(t)) \leq -\tilde{a}\varepsilon \|y(t)\|^{2\mu},$$

where $\tilde{a}$ is a positive constant. Thus, the zero solution of (18) is asymptotically stable.

The application of Theorem 3.1 completes the proof.

Remark 5.1 Theorem 5.1 guarantees the asymptotic stability of the equilibrium position for any value of parameter ε. Hence, the control forces (17) may be arbitrary small compared with the destabilizing potential forces.

6 Example

Consider the control system

$$\ddot{q}(t) + b\|\dot{q}(t)\|^{\nu}\dot{q}(t) + G\dot{q}(t) - \|q(t)\|^2 q(t) = U, \tag{20}$$

where $n = 2$, $q(t) = (q_1(t), q_2(t))^\top$,

$$G = \begin{pmatrix} 0 & g \\ -g & 0 \end{pmatrix},$$

b, g and ν are positive parameters, $U = (u_1, u_2)^\top$ is a control vector.

It should be noted that from the results of [7] it follows that Assumption 2.4 is fulfilled for the system (20), and the vector function $w(z)$ can be defined by the formula

$$w(z) = b\|z\|^{\nu} G^{-1} z.$$

Applying Theorem 5.1, we obtain that, under the condition $\nu < 2$, the control law

$$U = -\varepsilon\|q(t - \tau(t))\|^2 Gq(t - \tau(t))$$

stabilizes the equilibrium position $q = \dot{q} = 0$ of (20) for any positive values of b, g, ε and any continuous, nonnegative and bounded for $t \geq 0$ delay $\tau(t)$.

7 Conclusion

In the present paper, an approach to the decomposition of stability problem for a class of nonlinear mechanical systems is developed. Instead of the stability analysis for the original time-delay second order system (5), it is proposed to study stability for simpler delay-free first order isolated subsystems (4) and (6). It is worth noting that, unlike the classical decomposition conditions for linear gyroscopic systems [1, 2], to justify the decomposition of (5), Theorem 3.1 does not require the presence of a large parameter in the equations under study. It is shown that with the aid of the Lyapunov function constructed in the proof of the theorem, estimates of convergence rate of motions can be derived. An application of the developed approach to the control design for a mechanical system is presented. An interesting direction for further research is an extension of the obtained results to nonlinear mechanical systems with delay and switched force fields.

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