Nonlinear Dynamics and Systems Theory, **21** (5) (2021) 481–493

Optimal Estimation of Unknown Data of Cauchy Problem for First Order Linear Impulsive Systems of Ordinary Differential Equations from Indirect Noisy Observations of Their Solutions

O. G. Nakonechnyi and Yu. K. Podlipenko *

Taras Shevchenko National University of Kyiv, Kyiv, Ukraine

Received: March 4, 2020; Revised: October 18, 2021

Abstract: This paper is concerned with Cauchy problems for first-order systems of impulsive linear ordinary differential equations with unknown right-hand sides, initial conditions, and jumps of solutions at impulse points entering into the statement of these problems which are assumed to be subjected to some quadratic restrictions. From indirect noisy observations of their solutions on a finite system of intervals, optimal, in a certain sense, estimates of images of unknown data under linear continuous operators are obtained. It is shown how to apply the obtained results for finding the guaranteed estimates of unknown coefficients of the nonlinear Gompers equation which is widely used in population dynamics.

Keywords: *optimal estimate; guaranteed estimate; noisy observations; impulsive ordinary differential equations.*

Mathematics Subject Classification (2010): 34A30, 34A37, 34A55.

1 Introduction

In this paper, for Cauchy problems for systems of linear impulsive ordinary differential equations, we propose a novel technique of finding optimal estimates of images of their data under linear continuous operators. We assume that the right-hand sides of equations, initial conditions, and jumps of solutions at impulse points entering into the statement of these problems are unknown and belong to certain ellipsoids in the corresponding function spaces.

* Corresponding author: mailto:youripodlipenko@gmail.com

For solving such estimation problems, we need supplementary data (observations of solutions of the above Cauchy problems). By observations of unknown solutions we mean functions that are linear transformations of the same solutions distorted by additive random noises. Such a kind of observations is motivated by the fact that unknown solutions often cannot be observed directly. Here we use indirect noisy observations of solutions on a finite system of intervals.

Under the condition that unknown correlation functions of noises in observations belong to some special sets, it is established that such estimates and estimation errors are expressed explicitly via solutions of special uniquely solvable systems of linear impulsive ordinary differential equations.

For this, we first solve the problem of guaranteed (minimax) estimation of values of linear functionals from the above-mentioned right-hand sides and obtain the boundary value problems, not depending on the specific form of linear functionals, that generate the guaranteed estimates. Further, we apply these results for obtaining the optimal estimates.

Notice that this work is a continuation of our earlier studies set forth in [3] and [4], where we elaborate the guaranteed (minimax) estimation method for the case of the problem of estimation of linear functionals from unknown solutions and right-hand sides of first order linear periodic systems of ordinary differential equations.

2 Preliminaries and Auxiliary Results

Let $\mathbb{C}$ denote the field of complex numbers, Λ^* denote the matrix complex conjugate and transpose of a matrix Λ. Let $[t_0, T]$ be a closed interval of $\mathbb{R}$, and $\{t_i\}$ be a given strictly increasing sequence of impulse points in (t_0, T) such that $t_0 < t_1 < \cdots < t_q < t_{q+1} := T$.

A Cauchy problem for a system of first order linear impulsive differential equations on $[t_0, T]$ is a problem of the form

$$\frac{dx(t)}{dt} = A(t)x(t) + B(t)f(t) \quad \text{for a.e.} \quad t \in (t_0, T], \tag{1}$$

$$\Delta x \mid_{t=t_i} = B_i x(t_i) + C_i g_i, \quad i = 1, \dots, q, \quad x(t_0) = Cx_0, \tag{2}$$

where $A(t) = [a_{ij}(t)]$ is an $n \times n$-matrix with $a_{ij}(\cdot) \in L^2(t_0, T)$, $B(t) = [b_{ij}(t)]$ is an $n \times r$-matrix with $b_{ij}(\cdot)$ being piecewise continuous on $[0, T]$, $f(t)$ is a vector-function such that $f(t) \in \mathbb{C}^r$ and $f \in (L^2(t_0, T))^r$, B_i, C_i, g_i, C and x_0 are $n \times n$, $n \times k$, $k \times 1$, $n \times m$, and $m \times 1$ constant matrices, respectively, $\Delta x(t) \mid_{t=t_i} = x(t_i^+) - x(t_i)$ denotes the jumps of $x(t)$ at the points of impulses t_i, with $x(t_i^+) = \lim_{t \to t_i+} x(t)$.

By a solution of this problem, we mean a function $x(t) \in \mathcal{A}$ that is left continuous, satisfies the equation (1) almost everywhere (a.e.) on $(t_0, T]$, and the conditions (2), where by $\mathcal{A}$ we denote a class of left continuous functions $y(t) \in \mathbb{C}^n$ defined on $[t_0, T]$ such that $y(\cdot) \mid_{(t_{i-1}, t_i)} \in (W_2^1(t_{i-1}, t_i))^n$, $i = 1, \dots, q+1$. Here $W_2^1(a, b) = \{u(t) \in L^2(a, b) \text{ such that } \frac{du(t)}{dt} \in L^2(a, b)\}$.

Further we will assume that the following conditions are valid:

$$\det(E + B_i) \neq 0, \quad i = 1, \dots, q. \tag{3}$$

Under the conditions (3), the problem (1), (2) as well as the problem

$$-\frac{dz(t; u)}{dt} = A^*(t)z(t; u) + g(t) \quad \text{for a.e.} \quad t \in [t_0, T),$$

$$\Delta z(t)\mid_{t=t_i} = -(E+B_i^*)^{-1}B_i^* z(t_i) + g_i', \quad i=1,\dots,q, \quad z(T)=z_0,$$

that is adjoint of nonhomogeneous problem (1), (2), are uniquely solvable for any vector-functions $f(t) \in \mathbb{C}^r$, $g(t) \in \mathbb{C}^n$ such that $f \in (L^2(t_0,T))^r$, $g \in (L^2(t_0,T))^n$ and for any vectors $g_i \in \mathbb{C}^k$, $g_i' \in \mathbb{C}^n$ $x_0 \in \mathbb{C}^m$, $z_0 \in \mathbb{C}^n$.

These assertions follow from the results contained in [6], [2], [5].

3 Statement of the Problem of Guaranteed Estimation of Linear Functionals Defined on Unknown Cauchy Data

Let us give the definition of guaranteed estimates of linear functionals defined on solutions to the problem (1), (2) from observations of these solutions on a finite system of intervals.

Let Ω_j^i, $j=1,\dots,M_i$, be a given system of subintervals of (t_{i-1},t_i), $F := (f,g_1,\dots,g_q,x_0) \in \mathcal{H} := (L^2(t_0,T))^r \times \mathbb{C}^{kq} \times \mathbb{C}^m$.

The problem is to estimate the expression

$$l(F) = \int_{t_0}^{T} (f(t), l_0(t))_r dt + \sum_{i=1}^{q} (g_i, a_i)_k + (x_0, a)_m \tag{4}$$

from observations of the form

$$y_j^i(t) = H_j^i(t)x(t) + \xi_j^i(t), \quad t \in \Omega_j^i, \quad j=1,\dots,M_i, \quad i=1,\dots,q+1, \tag{5}$$

in the class of estimates

$$\widehat{l(F)} = \sum_{i=1}^{q+1}\sum_{j=1}^{M_i} \int_{\Omega_j^i} (y_j^i(t), u_j^i(t))_l dt + c, \tag{6}$$

linear with respect to observations (5); here $x(t)$ is the state of a system described by problem (1), (2), $l_0 \in (L^2(t_0,T)^r$, $a_i \in \mathbb{C}^k$, $a \in \mathbb{C}^m$, $H_j^i(t)$ are $l \times n$ matrices with the entries that are piecewise continuous complex-valued functions on $\bar{\Omega}_j^i$, $u_j^i(t)$ are vector-functions belonging to $(L^2(\Omega_j^i))^l$, $c \in \mathbb{C}$, and by $(\cdot,\cdot)_d$ we denote the inner product in $\mathbb{C}^d$.

We suppose that the vector-function f and vectors $g_1,\dots,g_q,x_0$ are unknown and the element $F=(f,g_1,\dots,g_q,x_0)$ belongs to the set G_1, where

$$G_1 = \Big\{F \in \mathcal{H} : f \in (L^2(t_0,T))^r, g_i \in \mathbb{C}^k, x_0 \in \mathbb{C}^m,$$
$$\sum_{i=1}^{q} (Q_i(g_i - g_i^0), g_i - g_i^0)_k + (Q_0(x_0 - x_0^0), x_0 - x_0^0)_m$$
$$+ \int_{t_0}^{T} (Q(t)(f(t) - f_0(t)), f(t) - f_0(t))_r \, dt \le 1\Big\},$$

$\xi := (\xi_1^1(\cdot),\dots,\xi_{M_1}^1(\cdot),\dots,\xi_1^{q+1}(\cdot),\dots,\xi_{M_{q+1}}^{q+1}(\cdot)) \in G_2$, where $\xi_j^i(\cdot)$ are observation errors in (5), that are realizations of random vector-functions $\xi_j^i(t) = \xi_j^i(\omega,t) \in \mathbb{C}^l$, and G_2 denotes the set of random elements ξ, whose components have zero means, $\mathbb{E}\xi_j^i(\cdot) =$

0, with Lebesgue square integrable second moments on Ω_j^i, and unknown correlation matrices $R_j^i(t,s)=\mathbb{E}\xi_j^i(t)(\xi_j^i)^*(s)$ satisfying the condition

$$\sum_{i=1}^{q+1}\sum_{j=1}^{M_i}\int_{\Omega_j^i}\mathrm{Tr}\,[D_j^i(t)R_j^i(t,t)]dt\leq 1, \tag{7}$$

where $f_0\in(L^2(t_0,T))^r$ is a prescribed vector-function, $g_1^0,\dots,g_q^0\in\mathbb{C}^k$ and $x_0^0\in\mathbb{C}^m$ are prescribed vectors, $D_j^i(t)$ and $Q(t)$ are known Hermitian positive definite $l\times l$ and $r\times r$-matrices with entries which are complex-valued continuous functions on $\bar\Omega_j^i$ and $[t_0,T]$, correspondingly, Q_i, $i=0,1,\dots,q$, are Hermitian positive definite matrices with constant elements for which there exist their inverse matrices $(D_j^i)^{-1}(t)$, $Q^{-1}(t)$, and Q_i^{-1}, $\mathrm{Tr}\,D:=\sum_{i=1}^l d_{ii}$ denotes the trace of the matrix $D=\{d_{ij}\}_{i,j=1}^l$.

Set $u:=(u_1^1(\cdot),\dots,u_{M_1}^1(\cdot),\dots,u_1^{q+1}(\cdot),\dots,u_{M_{q+1}}^{q+1}(\cdot))\in H$, where $H:=(L^2(\Omega_1^1))^l\times\dots\times(L^2(\Omega_{M_1}^1))^l\times\dots,\times(L^2(\Omega_1^{q+1}))^l\times\dots\times(L^2(\Omega_{M_{q+1}}^{q+1}))^l$. The norm in space H is defined by

$$\|u\|_H=\Big\{\sum_{i=1}^{q+1}\sum_{j=1}^{M_i}\|u_j^i(\cdot)\|_{(L^2(\Omega_j^i))^l}\Big\}^{1/2}.$$

Definition 3.1 The estimate

$$\widehat{\widehat{l(F)}}=\sum_{i=1}^{q+1}\sum_{j=1}^{M_i}\int_{\Omega_j^i}(y_j^i(t),\hat u_j^i(t))_l dt+\hat c,$$

in which vector-functions $\hat u_j^i(\cdot)$, and a number $\hat c$ are determined from the condition

$$\inf_{u\in H,c\in\mathbb{C}}\sigma(u,c)=\sigma(\hat u,\hat c),$$

where

$$\sigma(u,c)=\sup_{F\in G_1,\,\xi\in G_2}\mathbb{E}|l(F)-\widehat{\widehat{l(F)}}|^2,$$

will be called the guaranteed (minimax) estimate of expression (4). The quantity

$$\sigma:=\{\sigma(\hat u,\hat c)\}^{1/2}$$

will be called the error of the guaranteed estimation of $l(F)$.

Thus, a guaranteed estimate is an estimate minimizing the maximal mean-square estimation error calculated for the worst-case realization of the perturbations.

4 Representations for Guaranteed Estimates and Estimation Errors of $l(F)$

In this section we deduce equations that generate the minimax estimates.

For any fixed $u\in H$, introduce the vector-function $z(t;u)$ as a unique solution to the problem

$$-\frac{dz(t;u)}{dt}=A^*(t)z(t;u)-\sum_{i=1}^{q+1}\sum_{j=1}^{M_i}\chi_{\Omega_j^i}(t)(H_j^i)^*(t)u_j^i(t)\quad\text{for a.e.}\quad t\in[t_0,T), \tag{8}$$

$$\Delta z(t;u)\,|_{t=t_i} = -(E+B_i^*)^{-1}B_i^* z(t_i;u), \quad i=1,\dots,q, \quad z(T;u)=0, \tag{9}$$

where

$$\chi_\Omega(t) = \begin{cases} 1 & \text{if } t\in\Omega, \\ 0 & \text{if } t\notin\Omega \end{cases}$$

is a characteristic function of the set Ω.

The unique solvability of this problem follows from condition (3).

Lemma 4.1 *Finding the minimax estimate of functional $l(F)$ is equivalent to the problem of optimal control of the system (8), (9) with the cost function*

$$I(u) = \int_{t_0}^{T} (Q^{-1}(t)(B^*(t)z(t;u)+l_0(t)), B^*(t)z(t;u)+l_0(t))_r dt$$

$$+\sum_{i=1}^{q} (Q_i^{-1}(C_i^*(E+B_i^*)^{-1}z(t_i;u)+a_i), C_i^*(E+B_i^*)^{-1}z(t_i;u)+a_i)_k$$

$$+ (Q_0^{-1}(a+C^*z(t_0;u)), a+C^*z(t_0;u))_m$$
$$+\sum_{i=1}^{q+1}\sum_{j=1}^{M_i}\int_{\Omega_j^i} ((D_j^i)^{-1}(t)u_j^i(t), u_j^i(t))_l dt \to \inf_{u\in H}. \tag{10}$$

Proof. Let x be a solution to problem (1), (2). From (4)−(6), we obtain

$$\widehat{l(F)} = \sum_{i=1}^{q+1}\sum_{j=1}^{M_i}\int_{\Omega_j^i} (y_j^i(t), u_j^i(t))_l dt + c$$

$$= \sum_{i=1}^{q+1}\int_{t_{i-1}}^{t_i} (x(t), \sum_{j=1}^{M_i}\chi_{\Omega_j^i}(t)(H_j^i)^*(t)u_j^i(t))_n dt + \sum_{i=1}^{q+1}\sum_{j=1}^{M_i}\int_{\Omega_j^i} (\xi_j^i(t), u_j(t))_l dt + c.$$

Transform the first term in the right-hand side of this equality. Applying the integration by parts formula, we have

$$\sum_{i=1}^{q+1}\int_{t_{i-1}}^{t_i} (x(t), \sum_{j=1}^{M_i}\chi_{\Omega_j^i}(t)(H_j^i)^*(t)u_j^i(t))_n dt = -\sum_{i=1}^{q+1}\int_{t_{i-1}}^{t_i}\Big(x(t), -\frac{dz(t;u)}{dt} - A^*(t)z(t;u)\Big)_n dt$$

$$= -\sum_{i=1}^{q+1}\Big((x(t_{i-1}^+), z(t_{i-1}^+;u))_n - (x(t_i), z(t_i;u))_n\Big) - \sum_{i=1}^{N+1}\int_{t_{i-1}}^{t_i}\Big(\frac{dx(t)}{dt} - A(t)x(t), z(t;u)\Big)_n dt$$
$$= -(x(t_0), z(t_0;u))_n$$
$$-\sum_{i=1}^{q}(C_i g_i, (E+B_i^*)^{-1}z(t_i))_n - \sum_{i=1}^{q+1}\int_{t_{i-1}}^{t_i}\Big(B(t)f(t), z(t;u)\Big)_n dt.$$

Here we have used the fact that

$$\sum_{i=1}^{q+1}\Big((x(t_{i-1}^+), z(t_{i-1}^+;u))_n - (x(t_i), z(t_i;u))_n\Big)$$

$$= (x(t_0), z(t_0; u))_n + \sum_{i=1}^{q} \Big((x(t_i^+), z(t_i^+; u))_n - (x(t_i), z(t_i; u))_n \Big)$$

and

$$\sum_{i=1}^{q} \Big((x(t_i^+), z(t_i^+; u))_n - (x(t_i), z(t_i; u))_n \Big)$$

$$= \sum_{i=1}^{q} \Big(((E + B_i)x(t_i) + C_i g_i, (E - (E + B_i^*)^{-1} B_i^*) z(t_i; u))_n - (x(t_i), z(t_i; u))_n \Big)$$

$$= \sum_{i=1}^{q} \Big(((E + B_i)x(t_i), (E - (E + B_i^*)^{-1} B_i^*) z(t_i; u))_n - (x(t_i), z(t_i; u))_n \Big)$$

$$+ \sum_{i=1}^{q} (C_i g_i, (E - (E + B_i^*)^{-1} B_i^*) z(t_i; u))_n$$

$$= \sum_{i=1}^{q} \Big(((E + B_i)x(t_i), (E + B_i^*)^{-1} z(t_i; u))_n - (x(t_i), z(t_i; u))_n \Big)$$

$$+ \sum_{i=1}^{q} (C_i g_i, (E + B_i^*)^{-1} z(t_i; u))_n = \sum_{i=1}^{q} (g_i, C_i^* (E + B_i^*)^{-1} z(t_i; u))_k.$$

Since

$$l(F) = \int_{t_0}^{T} (f(t), l_0(t))_r dt + \sum_{i=1}^{q} (g_i, a_i)_k + (x_0, a)_m,$$

we get

$$l(F) - \widehat{l(F)} = \int_{t_0}^{T} (f(t), l_0(t) + B^*(t) z(t; u))_r dt + \sum_{i=1}^{q} (g_i, a_i + C_i^* (E + B_i^*)^{-1} z(t_i; u))_k$$

$$+ (x_0, a + C^* z(t_0; u))_m - \sum_{i=1}^{q+1} \sum_{j=1}^{M_i} \int_{\Omega_j^i} (\xi_j^i(t), u_j(t))_l dt - c.$$

The latter equality yields

$$\mathbb{E}[l(F) - \widehat{l(F)}] = \int_{t_0}^{T} (f(t), l_0(t) + B^*(t) z(t; u))_r dt$$

$$+ \sum_{i=1}^{q} (g_i, a_i + C_i^* (E + B_i^*)^{-1} z(t_i; u))_k + (x_0, a + C^* z(t_0; u))_m - c.$$

From here on, we apply the same reasoning as in the proof of Lemma in [4] to obtain

$$\inf_{c \in \mathbb{C}} \sup_{F \in G_1, \xi \in G_2} \mathbb{E} |l(F) - \widehat{l(F)}|^2 = I(u),$$

where $I(u)$ is determined by formula (10) and the infimum over c is attained at

$$c = \int_{t_0}^{T} \Big(f_0(t), l_0(t) + B^*(t) z(t;u))_r dt \Big)_r dt + \sum_{i=1}^{q} (g_i^0, a_i + C_i^*(E + B_i^*)^{-1} z(t_i;u))_k + (x_0^0, a + C^* z(t_0;u))_m. \quad (11)$$

The proof is complete.

Further in the proof of Theorem 4.1 stated below, it will be shown that solving the optimal control problem (8)−(10) is reduced to solving some system of impulsive periodic differential equations.

Theorem 4.1 *The minimax estimate* $\widehat{\widehat{l(F)}}$ *of the expression* $l(F)$ *has the form*

$$\widehat{\widehat{l(F)}} = \sum_{i=1}^{q+1} \sum_{j=1}^{M_i} \int_{\Omega_j^i} (y_j^i(t), \hat{u}_j^i(t))_l dt + \hat{c} = l(\hat{\hat{F}}),$$

where

$$\hat{u}_j^i(t) = D_j^i(t) H_j^i(t) p(t), \quad i = 1, \dots, q+1, \quad j = 1, \dots, M_i, \quad (12)$$

$$\hat{c} = \int_{t_0}^{T} \Big(f_0(t), l_0(t) + B^*(t) \hat{z}(t))_r dt \Big)_r dt + \sum_{i=1}^{q} (g_i^0, a_i + C_i^*(E + B_i^*)^{-1} \hat{z}(t_i))_k + (x_0^0, a + C^* \hat{z}(t_0))_m,$$

$\hat{\hat{F}} := (\hat{f}, \hat{g}_1, \dots, \hat{g}_q, \hat{x}_0)$ *with*

$$\hat{f}(t) = f_0(t) + Q^{-1}(t) B^*(t) \hat{p}(t), \quad \hat{g}_i = g_i^0 + Q_i^{-1} C_i^* (E + B_i^*)^{-1} \hat{p}(t_i), \quad i = 1 \dots, q,$$
$$\hat{x}_0 = x_0^0 + Q_0^{-1}(t) C^* \hat{p}(t_0), \quad (13)$$

$p(t)$, $\hat{z}(t)$, *and* $\hat{p}(t)$ *are determined from the solution of the systems of equations*

$$-\frac{d\hat{z}(t)}{dt} = A^*(t)\hat{z}(t) - \sum_{i=1}^{q+1} \sum_{j=1}^{M_i} \chi_{\Omega_j^i}(t) (H_j^i)^*(t) D_j^i(t) H_j^i(t) p(t) \quad \text{for a.e.} \quad t \in [t_0, T), \quad (14)$$

$$\Delta \hat{z}(t) \mid_{t=t_i} = -(E + B_i^*)^{-1} B_i^* \hat{z}(t_i), \quad i = 1, \dots, q, \quad \hat{z}(T) = 0, \quad (15)$$

$$\frac{dp(t)}{dt} = A(t)p(t) + B(t) Q^{-1}(t) (B^* \hat{z}(t) + l_0(t)) \quad \text{for a.e.} \quad t \in (t_0, T], \quad (16)$$

$$\Delta p(t) \mid_{t=t_i} = B_i p(t_i) + C_i Q_i^{-1} (C_i^* (E + B_i^*)^{-1} \hat{z}(t_i) + a_i),$$
$$i = 1, \dots, q, \quad p(t_0) = C Q_0^{-1} (C^* \hat{z}(t_0) + a) \quad (17)$$

and

$$-\frac{d\hat{p}(t)}{dt}=A^*(t)\hat{p}(t)-\sum_{i=1}^{q+1}\sum_{j=1}^{M_i}\chi_{\Omega_j^i}(t)(H_j^i)^*(t)D_j^i(t)[H_j^i(t)\hat{x}(t)-y_j^i(t)] \quad \textit{for a.e. } t\in[t_0,T), \tag{18}$$

$$\Delta\hat{p}(t)\mid_{t=t_i}=-(E+B_i^*)^{-1}B_i^*\hat{p}(t_i), \quad i=1,\dots,q, \quad \hat{p}(T)=0, \tag{19}$$

$$\frac{d\hat{x}(t)}{dt}=A(t)\hat{x}(t)+B(t)(Q^{-1}(t)B^*(t)\hat{p}(t)+f_0(t)) \quad \textit{for a.e.} \quad t\in(t_0,T], \tag{20}$$

$$\Delta\hat{x}(t)\mid_{t=t_i}=B_i\hat{x}(t_i)+C_iQ_i^{-1}(C_i^*(E+B_i^*)^{-1}\hat{p}(t_i)+g_i),$$
$$i=1,\dots,q, \quad \hat{x}(t_0)=CQ_0^{-1}(C^*\hat{p}(t_0)+x_0^0), \tag{21}$$

respectively. Problems (14) – (17) *and* (18) – (21) *are uniquely solvable. Equations* (18) – (21) *are fulfilled with probability* 1.

The minimax estimation error σ *is determined by the formula*

$$\sigma=[l(\hat{P})]^{1/2}, \tag{22}$$

where

$$\hat{P}=\Big(Q^{-1}(\cdot)(l_0(\cdot)+B^*(\cdot)\hat{z}(\cdot)),Q_1^{-1}(C_1^*(E+B_1^*)^{-1}\hat{z}(t_1)+a_1),$$
$$\dots,Q_q^{-1}(C_q^*(E+B_q^*)^{-1}\hat{z}(t_q)+a_q),Q_0^{-1}(C^*\hat{z}(t_0)+x_0^0)\Big).$$

Proof. It is not difficult to verify, using the representation (1.21) from [2], that $I(u)$ is a weakly lower semicontinuous strictly convex functional on H. Therefore, since

$$I(u)\geq\sum_{i=1}^{q+1}\sum_{j=1}^{M_i}\int_{\Omega_j^i}((D_j^i)^{-1}(t)u_j^i(t),u_j^i(t))_l dt\geq c\|u\|_H^2 \quad \forall u\in H, \ \text{c=const},$$

by Theorems 13.2 and 13.4 (see [1]), there exists one and only one element $\hat{u}\in H$ such that $I(\hat{u})=\inf_{u\in H}I(u)$. Hence, for any fixed $v\in H$ and $\tau\in\mathbb{R}$, the functions $s_1(\tau):=I(\hat{u}+\tau v)$ and $s_2(\tau):=I(\hat{u}+i\tau v)$ reach their minimums at a unique point $\tau=0$ so that

$$\frac{1}{2}\frac{d}{d\tau}I(\hat{u}+\tau v)\Big|_{\tau=0}=0 \quad \text{and} \quad \frac{1}{2}\frac{d}{d\tau}I(\hat{u}+i\tau v)\Big|_{\tau=0}=0, \tag{23}$$

where $i=\sqrt{-1}$. Since $z(t;\hat{u}+\tau v)=z(t;\hat{u})+\tau z(t;v)$ and $z(t;\hat{u}+i\tau v)=z(t;\hat{u})+i\tau z(t;v)$, from (10) and (23), we obtain

$$0=\int_{t_0}^{T}\Big(Q^{-1}(t)(B^*(t)z(t;\hat{u})+l_0(t)),B^*(t)z(t;v)\Big)_r dt+(Q_0^{-1}(C^*z(t_0;\hat{u})+a),C^*z(t_0;v))_m$$
$$+\sum_{i=1}^{q}(Q_i^{-1}(C_i^*(E+B_i^*)^{-1}z(t_i;\hat{u})+a_i),C_i^*(E+B_i^*)^{-1}z(t_i;v))_k$$
$$+\sum_{i=1}^{q+1}\sum_{j=1}^{M_i}\int_{\Omega_j^i}((D_j^i)^{-1}(t)\hat{u}_j^i(t),v_j^i(t))_l dt. \tag{24}$$

Let $p(t)$ be a solution of the problem

$$\frac{dp(t)}{dt} = A(t)p(t) + B(t)Q^{-1}(t)(B^*z(t;\hat{u}) + l_0(t)) \quad \text{for a.e.} \quad t \in (t_0, T],$$

$$\Delta p(t)\mid_{t=t_i} = B_i p(t_i) + C_i Q_i^{-1}(C_i^*(E + B_i^*)^{-1}\hat{z}(t_i;\hat{u}) + a_i),$$
$$i = 1, \ldots, q, \quad p(t_0) = CQ_0^{-1}(C^*\hat{z}(t_0) + a).$$

Taking this into account, transform the first summand in the right-hand side of (24). We have

$$\int_{t_0}^{T}\Big(Q^{-1}(t)(B^*(t)z(t;\hat{u})+l_0(t)), B^*(t)z(t;v)\Big)_r dt = \sum_{i=1}^{q+1}\int_{t_{i-1}}^{t_i}\Big(\frac{dp(t)}{dt} - A(t)p(t), z(t;v)\Big)_n dt$$

$$= \sum_{i=1}^{q+1}\Big((p(t_i), z(t_i;v))_n - (p(t_{i-1}^+), z(t_{i-1}^+;v))_n\Big) - \sum_{i=1}^{q+1}\int_{t_{i-1}}^{t_i}\Big(p(t), \frac{dz(t;v)}{dt} + A^*(t)z(t;v)\Big)_n dt$$

$$= -\sum_{i=1}^{q}(Q_i^{-1}(C_i^*(E + B_i^*)^{-1}z(t_i;\hat{u}) + a_i), C_i^*(E + B_i^*)^{-1}z(t_i;v))_k,$$

$$-(Q_0^{-1}(C^*z(t_0;\hat{u}) + a), C^*z(t_0;v))_m - \int_{t_0}^{T}\Big(p(t), \sum_{i=1}^{q+1}\sum_{j=1}^{M_i}\chi_{\Omega_j^i}(t)(H_j^i)^*(t)v_j^i(t)\Big)_n dt. \quad (25)$$

From (24) and (25), we find

$$\sum_{i=1}^{q+1}\sum_{j=1}^{M_i}\int_{\Omega_j^i}((D_j^i)^{-1}(t)\hat{u}_j^i(t), v_j^i(t))_l dt = \sum_{i=1}^{q+1}\sum_{j=1}^{M_i}\int_{\Omega_j^i}(p(t), (H_j^i)^*(t)v_j^i(t))_n dt$$

for any $v := (v_1^1(\cdot), \ldots, v_{M_1}^1(\cdot), \ldots, v_1^{q+1}(\cdot), \ldots, v_{M_{q+1}}^{q+1}(\cdot)) \in H$, whence $\hat{u}_j^i(t)$, $i = 1, \ldots, q+1$, $j = 1, \ldots, M_i$, are defined by (12). Setting $u = \hat{u}$ in (11), (8) and (9) and denoting $\hat{z}(t) = z(t;\hat{u})$, we see that $\hat{z}(t)$ and $p(t)$ satisfy system (14) – (17); the unique solvability of this system follows from the fact that the functional $I(u)$ has one minimum point $\hat{u}$.

Now let us establish that $\sigma = [l(\hat{P})]^{1/2}$. Substituting expression (12) into (10), we obtain

$$\sigma^2 = \int_{t_0}^{T}(Q^{-1}(t)(B^*(t)\hat{z}(t)+l_0(t)), B^*(t)\hat{z}(t)+l_0(t))_r dt + (Q_0^{-1}(a+C^*\hat{z}(t_0)), a+C^*\hat{z}(t_0))_m$$

$$+\sum_{i=1}^{q}(Q_i^{-1}(C_i^*(E + B_i^*)^{-1}\hat{z}(t_i) + a_i), C_i^*(E + B_i^*)^{-1}\hat{z}(t_i) + a_i)_k$$

$$+\sum_{i=1}^{q+1}\sum_{j=1}^{M_i}\int_{\Omega_j^i}(H_j^i(t)p(t), D_j^i(t)H_j^i(t)p(t))_l dt. \quad (26)$$

However,

$$\int_{t_0}^{T}(Q^{-1}(t)(B^*(t)\hat{z}(t) + l_0(t)), B^*(t)\hat{z}(t))_r dt = \sum_{i=1}^{q+1}\int_{t_{i-1}}^{t_i}\Big(\frac{dp(t)}{dt} - A(t)p(t), \hat{z}(t)\Big)_n dt$$

$$= \sum_{i=1}^{q+1}\Big((p(t_i),\hat z(t_i))_n - (p(t_{i-1}^{+}),\hat z(t_{i-1}^{+}))_n\Big) - \sum_{i=1}^{q+1}\int_{t_{i-1}}^{t_i}\Big(p(t),\frac{d\hat z(t)}{dt} + A^*(t)\hat z(t)\Big)_n dt$$

$$= -\sum_{i=1}^{q} Q_i^{-1}(C_i^*(E+B_i^*)^{-1}\hat z(t_i) + a_i), C_i^*(E+B_i^*)^{-1}\hat z(t_i))_k$$

$$-(Q_0^{-1}(C^* z(t_0;\hat u) + a), C^* z(t_0;v))_m - \sum_{i=1}^{q+1}\sum_{j=1}^{M_i}\int_{\Omega_j^i}(H_j^i(t)p(t), D_j^i(t)H_j^i(t)p(t))_l dt.$$

From here and from (26) it follows (22).

The representation $\widehat{\widehat{l(F)}} = l(\hat{\hat F})$ can be proved in much the same way as the representation

$$\widehat{\widehat{l(F)}} = \sum_{i=1}^{q+1}\sum_{j=1}^{M_i}\int_{\Omega_j^i}(y_j^i(t),\hat u_j^i(t))_l dt + \hat c.$$

This completes the proof.

Remark 4.1 In the representation $\widehat{\widehat{l(F)}} = l(\hat{\hat F})$ of the guaranteed mean square estimate of $l(F)$, where $F := (f, g_1, \dots, g_q, x_0)$, $\hat{\hat F} := (\hat f, \hat g_1, \dots, \hat g_q, \hat x_0)$ with $\hat f(t) = f_0(t) + Q^{-1}(t)B^*(t)\hat p(t)$, $\hat g_i = g_i^0 + Q_i^{-1}C_i^*(E+B_i^*)^{-1}\hat p(t_i)$, $i = 1\dots,q$, $\hat x_0 = x_0^0 + Q_0^{-1}(t)C^*\hat p(t_0)$, the vector-function $\hat f(t)$ and vectors $\hat g_i$, and $\hat x_0$ do not depend on a specific form of the functional l.

5 Optimal Estimation Problem of Unknown Cauchy Data

Now consider the problem of finding the optimal estimate of the vector $g = LF$ among the estimates of the form

$$\hat g = \sum_{i=1}^{q+1}\sum_{j=1}^{M_i} U_j^i y_j^i(\cdot) + C; \tag{27}$$

here $y_j^i(\cdot)$ are observations (5), L is a linear continuous operator acting from the space $\mathcal{H}$ into a separable complex Hilbert space V with the inner product $(\cdot,\cdot)$ and the norm $\|\cdot\|$, U_j^i are linear continuous operators acting from $(L^2(\Omega_j^i))^l$ to V, $C \in V$.

Let $\{e_1, e_2, \dots\}$ be an orthonormal basis of V. Denote by $\sigma_1(U,C)$ and $\sigma_2(U,C)$ the quantities defined by

$$\sigma_1(U,C) = \sup_{G_1,G_2}\mathbb{E}\|g-\hat g\|^2$$

and

$$\sigma_2(U,C) = \sum_{k=1}^{\infty}\sup_{G_1,G_2}\mathbb{E}|(g-\hat g, e_k)|^2,$$

respectively, where $U := (U_1^1, \dots, U_{M_1}^1, \dots, U_1^{q+1}, \dots, U_{M_{q+1}}^{q+1})$, G_1 and G_2 are defined on page 483.

Definition 5.1 The estimates $\hat{\hat g}_1$ and $\hat{\hat g}_2$, which are determined from the condition

$$\hat{\hat g}_i \in \mathrm{Argmin}_{\hat g\in\mathcal{L}}\sigma_i(U,C),$$

are called the guaranteed and optimal estimate of g, respectively, where by $\mathcal{L}$ we denote the set of all estimates of the form (27).

Parseval's formula implies that the following inequality holds:

$$\sigma_1(U,C) \le \sigma_2(U,C).$$

Lemma 5.1 *Suppose that, for an arbitrary vector $e \in V$, there holds the equality*

$$\inf_{\widehat{(g,e)}} \sup_{G_1,G_2} \mathbb{E}|(g,e) - \widehat{(g,e)}|^2 = \sup_{G_1,G_2} \mathbb{E}|(g,e) - \widehat{\widehat{(g,e)}}|^2,$$

where $\widehat{\widehat{(g,e)}} = (\hat{\hat{g}}, e)$, $\hat{\hat{g}}$ does not depend on the vector e, and $\widehat{(g,e)}$ is a linear estimate of the inner product (g,e). Then the vector $\hat{\hat{g}}$ is the optimal estimate of the vector g.

Proof. Notice that

$$\inf_{\hat{g}\in\mathcal{L}} \sigma_2(U,C) = \inf_{\hat{g}\in\mathcal{L}} \sum_{k=1}^{\infty} \sup_{G_1,G_2} \mathbb{E}|(g-\hat{g},e_k)|^2 \ge \sum_{k=1}^{\infty} \inf_{\widehat{(g,e_k)}} \sup_{G_1,G_2} \mathbb{E}|(g,e_k) - \widehat{(g,e_k)}|^2$$

$$= \sup_{G_1,G_2} \mathbb{E}|(g,e_k) - \widehat{\widehat{(g,e_k)}}|^2 = \sum_{k=1}^{\infty} \sup_{G_1,G_2} \mathbb{E}|(g-\hat{\hat{g}},e_k)|^2$$

and the lower bound is attained at $\hat{g} = \hat{\hat{g}}$. This completes the proof.

Next we obtain the optimal estimate of the element $g = LF$ using this lemma. Note first that for any $e \in V$, we have

$$(g,e) - (\hat{g},e) = (LF,e) - (\sum_{i=1}^{q+1}\sum_{j=1}^{M_i} U_j^i y_j^i(\cdot) + C, e)$$

$$= (F, L^*e)_{\mathcal{H}} - \sum_{i=1}^{q+1}\sum_{j=1}^{M_i} \int_{\Omega_j^i} (y_j^i(t), (U_j^i)^* e(t))_n dt - (C,e)$$

$$= l(F) - \widehat{l(F)},$$

where L^* and $(U_j^i)^*$ denote the adjoint operators of L and U_j^i, respectively,

$$l(F) := (F, L^*e)_{\mathcal{H}} = \int_{t_0}^{T} (f(t), l_0(t))_r dt + \sum_{i=1}^{q} (g_i, a_i)_k + (x_0, a)_m,$$

with some $l_0 \in (L^2(t_0,T)^r$, $a_i \in \mathbb{C}^k$, and $a \in \mathbb{C}^m$,

$$\widehat{l(F)} := (\widehat{F, L^*e})_{\mathcal{H}} = \sum_{i=1}^{q+1}\sum_{j=1}^{M_i} \int_{\Omega_j^i} (y_j^i(t), u_j^i(t))_n dt + c,$$

where $u_j^i(t) = (U_j^i)^* e(t)$ are vector-functions belonging to $(L^2(\Omega_j^i))^l$, $c = (C,e) \in \mathbb{C}$.

By Theorem 4.1,

$$\inf_{(\widehat{F,L^*e})_{\mathcal{H}}} \sup_{G_1,G_2} \mathbb{E}|(F,L^*e)_{\mathcal{H}} - (\widehat{F,L^*e})_{\mathcal{H}}|^2 = \sup_{G_1,G_2} \mathbb{E}|(F,L^*e)_{\mathcal{H}} - (\widehat{\widehat{F,L^*e}})_{\mathcal{H}}|^2,$$

where $(\widehat{\widehat{F,L^*e}})_{\mathcal{H}} = (\hat{\hat{F}}, L^*e)_{\mathcal{H}}$ with $\hat{\hat{F}} := (\hat{f}, \hat{g}_1, \dots, \hat{g}_q, \hat{x}_0)$ and $\hat{f}(\cdot), \hat{g}_1, \dots, \hat{g}_q, \hat{x}_0$ being determined by (13). From the latter relationship and from the fact that $\hat{\hat{F}}$ does not depend on L^*e (see Remark 1) it follows that the vector $\hat{g} = L\hat{F}$ satisfies the assumptions of Lemma 5.1. This proves the validity of the following assertion.

Theorem 5.1 *The optimal estimates* $\hat{\hat{F}}$ *and* $\hat{\hat{g}}$ *of* F *and* $g = LF$ *are determined by* $\hat{\hat{F}} = (\hat{f}(\cdot), \hat{g}_1, \ldots, \hat{g}_q, , \hat{x}_0)$ *and* $L\hat{\hat{F}}$, *respectively, where* $\hat{f}(\cdot), \hat{g}_1, \ldots, \hat{g}_q, \hat{x}_0$ *are defined by* (13).

Remark 5.1 All the results of the paper remain valid if we assume that the components $\xi_j^i(\cdot)$ of the random elements $\xi := (\xi_1^1(\cdot), \ldots, \xi_{M_1}^1(\cdot), \ldots, \xi_1^{q+1}(\cdot), \ldots, \xi_{M_{q+1}}^{q+1}(\cdot))$ entering into the set G_2 are pairwise uncorrelated and satisfy the condition

$$\int_{\Omega_j^i} \mathrm{Tr}\,[D_j^i(t) R_j^i(t,t)]dt \le 1, \quad i = 1, \ldots q+1, \quad j = 1, \ldots, M_i.$$

Let us present an example of applying the obtained results to the guaranteed estimation problem for the impulsive nonlinear differential equation.

In the population dynamics, for modeling of the processes of rapid change of the number of individuals of a population, the Gompers equation of the form

$$\frac{dx(t)}{dt} = \big(a(t) + b(t) \ln x(t)\big) x(t) \tag{28}$$

is applied. For the use of such models, it is required to know the parameters $a(t)$ and $b(t)$.

Let us show how to apply the above results, for example, for obtaining the guaranteed estimates for the function $a(t)$ by assuming that the function $b(t)$ is known and that $a(t)$ satisfies the following condition

$$\int_0^T \Big(\frac{da(t)}{dt}\Big)^2 dt \le \gamma_T^2 \quad (\gamma_T = \mathrm{const}), \quad a(0) = 0.$$

Let the function

$$v(t) = \xi(t) x(t) \tag{29}$$

be observed on the set $(0,T)\backslash(\cup_{i=1}^q \{t_i\})$, where $\xi(t)$ is a realization of a stochastic process $\xi(t,\omega) > 0$, $x(t)$ satisfies equation (28) and the conditions

$$x(0) = 1, \quad \frac{x(t_k + 0)}{x(t_k - 0)} = c_k, \tag{30}$$

where t_k, $k = 1, \ldots, q$, are given impulse points such that $0 < t_1 < \cdots < t_q < T$, c_k are prescribed numbers.

We will find the guaranteed estimate of the functional

$$L(a) = \int_0^T l(t) a(t) dt$$

in the class of estimates

$$\widehat{L(a)} = \int_0^T u(t) \ln v(t) dt,$$

where $l \in L^2(0,T)$ is a given function, $u \in L^2(0,T)$.

If we introduce the notation $\varphi_1(t) = \ln x(t)$, $\varphi_2(t) = a(t)$, $y(t) = \ln v(t)$, $\eta(t) = \ln \xi(t)$, then the guaranteed estimation problem of the functional $L(a)$ is reduced to the guaranteed estimation problem of the functional $L(\varphi_2)$ from the observations of the form

$$y(t) = \varphi_1(t) + \eta(t),$$

where $\varphi_1(t)$ and $\varphi_2(t)$ are found from solving the following system of linear impulsive differential equations:

$$\frac{d\varphi_1(t)}{dt} = \varphi_2(t) + b(t)\varphi_1(t) \quad \text{for a.e.} \quad t \in (0,T], \quad \varphi_1(0) = 0,$$

$$\frac{d\varphi_2(t)}{dt} = f(t) \quad \text{for a.e.} \quad t \in (0,T], \quad \varphi_2(0) = 0,$$

$$\varphi_1(t_k + 0) = \varphi_1(t_k - 0) + \ln c_k, \quad k = 1, \dots, q,$$

$$\varphi_2(t_k + 0) = \varphi_2(t_k - 0), \quad k = 1, \dots, q,$$

where $f(t) = \frac{da(t)}{dt}$.

Under certain restrictions on the correlation function of the process $\eta(t)$, we can apply the results of the present paper for obtaining the guaranteed estimates of the parameter $a(t)$.

6 Conclusion

The method proposed in the present paper enables one to obtain the optimal estimates of unknown data of Cauchy problems for first-order linear impulsive systems of ordinary differential equations from noisy observations of their solutions.

We deduce the boundary value problems for linear impulsive ordinary differential equations of the special kind that generate the optimal estimates.

The results presented above are aimed at elaborating mathematically justified estimation techniques for various forward and inverse problems with uncertainties describing evolution processes characterized by the combination of a continuous and abrupt change of their state.

References

[1] I. B. Badriev and M. M. Karchevsky. *Duality Methods in Applied Problems.* Publ. of Kazan State University, Kazan, 1987.

[2] D. D. Bainov and P. S. Simeonov. *Impulsive Differential Equations. Asymptotic Properties of the Solutions.* World Scientifc, 1995.

[3] O. G. Nakonechny and Yu. K. Podlipenko. The minimax approach to the estimation of solutions to first order linear systems of ordinary differential periodic equations with inexact data. arXiv:1810.07228V1, 2018, P. 1-19.

[4] O. Nakonechnyi and Y. Podlipenko. Guaranteed Estimation of Solutions of First Order Linear Systems of Ordinary Differential Periodic Equations with Inexact Data from Their Indirect Noisy Observations. In: *Proceedings of 2018 IEEE First International Conference on System Analysis and Intelligent Computing (SAIC).*

[5] I. Rachunkova and J. Tomecek. *State-Dependent Impulses Boundary Value Problems on Compact Interval.* Atlantis Press, Amsterdam-Paris-Beijing, 2015.

[6] A. M. Samoilenko and N. A. Perestyuk. *Impulsive Differential Equations.* World Scientific, 1995.