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Progressive Type–II Censoring Power Function Distribution Under Binomial Removals

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Abstract: Recently, progressive censoring received significant attention in many applications in engineering system reliability and survival analysis. Different lifetime models are used in the literature for progressive censoring, such as the Pareto, exponential, generalized exponential, Gompertz, Burr Type−XII, Rayleigh, generalized logistic, and exponentiated gamma distributions. A power function model is characterized by its simple mathematical structure and is easily implemented to determine failure rates and reliability values. The model is found to be useful in modeling electrical components. This work considers the estimation problem for a power function model based on progressive Type−II censoring using binomial removals. A simulation study was performed to investigate the behavior of the estimators using different sample sizes, parameter values and censored proportions. As an illustration, an application to failure time data set is presented.

Keywords: *censoring; estimation; power function distribution; reliability; simulation; survival analysis.*

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1 Introduction

The theory of reliability systems plays an important role in industry, manufacturing, safety engineering and quality. The lifetime of equipment or apparatus is a random time from the beginning of the operation until the appearance of a complete failure. Reliability is the ability of a system to perform its stated purpose adequately for a specified period of time under specified operational conditions. The system defined here could be an electronic or mechanical hardware product, a software product, a manufacturing process. For example, in the case of a mechanical system, a failure is a breakdown of some of its parts or an increase in vibration above the permitted level. The reliability characteristics are usually expressed in terms of the lifetime. Modeling and analyzing lifetime data are important issues for engineering reliability, industry, quality control, and clinical trials, etc. Different lifetime data can be modeled by different continuous probability distributions such as exponential, Lindley, Weibull, lognormal, and Frechet as well as their generalizations [1, 2].

In reliability and survival analysis, it is difficult to collect lifetime data for all components under consideration due to time and cost constraints. Various types of censoring schemes can be used for such purpose based on the model and available information using both parametric and nonparametric methods. Recently, progressive censoring sampling is of special importance in reliability and survival analysis. Progressive censoring was first introduced by Cohen [3]. Extensive studies are available in the literature related to the progressive censoring [4–6]. Different parametric survival models have been considered in progressive censoring using binomial removals, they are the Type-II generalized logistic distribution [7], the exponential distribution [8], the generalized exponential distribution [9], the exponentiated gamma distribution [10], the Pareto distribution [11–13], the Rayleigh distribution [14], the Burr Type-XII distribution [15], and the Gompertz distribution [16]. For more details about Type-I and Type-II censored samples, one can refer to Salah [17], Lin *et al.* [18], Balakrishnan [19], Balakrishnan *et al.* [20], and Salah [21, 22].

Type-II progressively censored life test is conducted as follows. For n identical units in a test, at the time of the first failure, R_1 units from the remaining $n-1$ survival items are removed. At the time of the second failure, R_2 units from the remaining $n-R_1-1$ items are removed, and so forth. Finally, at the time of $m-$th failure, the reaming survival units, would be R_m can be removed. In this case, censoring takes place progressively in m stages. Clearly, this scheme includes, as special cases, the complete sample situation (when $m = n$ and $R_1 = ... = R_m = 0$) and the conventional Type$-$II right censoring situation (when $R_1 = ... = R_{m-1} = 0$ and $R_m = n - m$). The corresponding scheme $(r_1, r_2, \ldots, r_m)$ is known as the progressive Type-II right censoring scheme.

Different versions of the power function distributions are reported in the literature [23]. These power function distributions can be easily implemented to determine the failure rates and reliability values compared to other distributions such as lognormal, Weibull, logistic and others. The particular parameterization of the power distribution function to be considered in this work has the following cumulative distribution function (CDF) form:

$$F(x) = 1 - \left(\frac{\theta - x}{\theta - \alpha}\right)^{\beta}, \ \alpha < x < \theta, \ \beta > 0, \tag{1}$$

where θ and α are the scale parameters and β is the shape parameter.

The probability density function (PDF) is given by

$$f(x) = \frac{\beta}{\theta-\alpha}\left(\frac{\theta-x}{\theta-\alpha}\right)^{\beta-1}, \ \alpha < x < \theta, \ \beta > 0. \tag{2}$$

The power function distribution is a member of the Beta family of distributions. Sarhan and Pandey [24] obtained the best linear unbiased estimates of the parameters of the above power distribution function in terms of $k-th$ upper record values. The power function distribution has applications in industrial and mechanical engineering [24]. Meniconi and Barry [25] explored the performance of the power function distribution on certain electrical components and showed that it is the most suitable distribution function as compared to the lognormal, Weibull and exponential models. Statistical properties of the power function distribution were reported by Johnson *et al.* [26].

This work considers progressive Type-II censoring for a power function distribution with binomial removals. The maximum likelihood estimators (MLEs) of the model parameters are determined. A simulation study is performed to determine the behavior of the MLEs via bias and the root mean square error (RMSE) using different sample sizes, parameter values and censored proportions. An example related to lifetime data of electronic devices will be presented to illustrate the approach developed in this work.

2 Model

Assume the lifetime random variable follows the power function distribution given in equation (1), it is a realistic assumption to assume the location parameter (lower bound) $\alpha = 0$, the cumulative distribution function (CDF) reduces to

$$F(x) = 1 - \left(\frac{\theta-x}{\theta}\right)^{\beta}, \ 0 < x < \theta, \ \beta > 0, \tag{3}$$

where θ is the scale parameter and β is the shape parameter.

The probability density function (PDF) reduces to

$$f(x) = \frac{\beta}{\theta}\left(\frac{\theta-x}{\theta}\right)^{\beta-1}, \ 0 < x < \theta, \ \beta > 0. \tag{4}$$

The reliability function is given by

$$r(x) = P(T > x) = \left(\frac{\theta-x}{\theta}\right)^{\beta}, 0 < x < \theta, \beta > 0.$$

The hazard rate function is given by

$$h(x) = \frac{f(x)}{R(x)} = \frac{\beta}{\theta-x}, 0 < x < \theta, \beta > 0.$$

Figure 1 shows a graphical representation of the probability density function (PDF) for the values of the shape parameter β of 0.2, 0.7, 1.5, 3, 4 and $\theta = 1$. The probability density function exhibits various behaviors depending on the values of the shape parameter β. Figure 2 shows the graphical representation of the hazard function using selected shape parameter values. According to Figure 2, it is seen that the power function distribution is characterized by increasing J-shaped hazard rates.

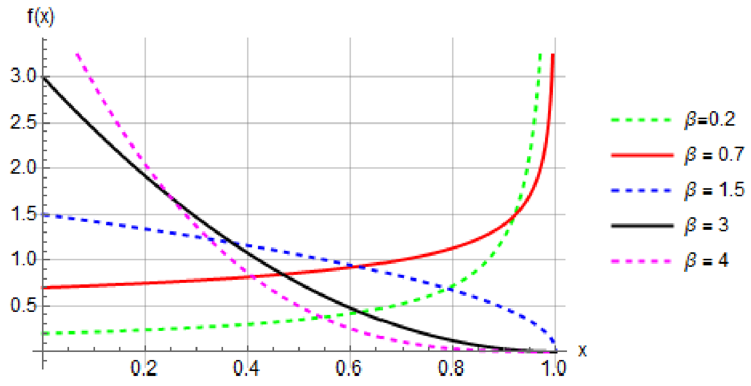

Figure 1: PDF plot of power function distribution.

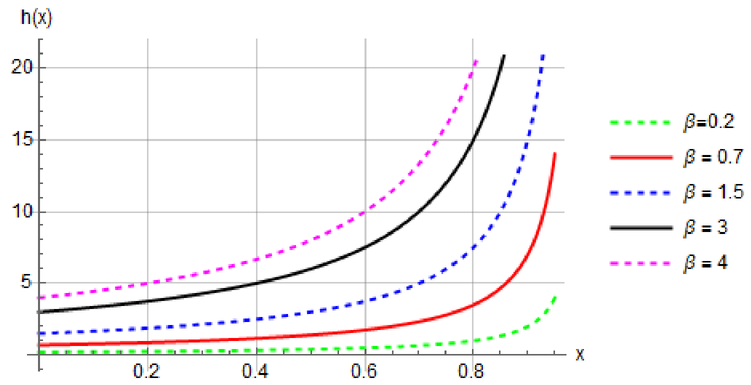

Figure 2: Hazard rate curves of power function distribution.

3 Maximum Likelihood Estimation

We first consider estimating the parameters based on the complete observed sample $x_1, ..., x_n$. Let $x_{(1)}, ..., x_{(n)}$ be the corresponding order statistics. Given the sample, the likelihood function of the density in (4) is

$$L(\beta, \theta) \equiv L(\beta, \theta | x_1, ..., x_n) = \left(\frac{\beta}{\theta}\right)^n \prod_{j=1}^{n} \left(1 - \frac{x_j}{\theta}\right)^{\beta - 1} I_{(x_{(n)}, \infty)^{(\theta)}}, \beta > 0,$$

where $I_A(x)$ is the zero−one indicator function. We notice that the support of the density depends on the scale parameter and therefore the MLE may not be calculated directly as a solution to the likelihood equations.

For fixed $\beta = \beta_0$, the limits of the likelihood function when approaching its boundaries are given by

$$\lim_{\theta \downarrow x_{(n)}} L(\beta_0, \theta) = \begin{cases} \infty, & 0 < \beta_0 < 1, \\ 0, & \beta_0 > 1, \\ \left(x_{(n)}\right)^{-n}, & \beta_0 = 1, \end{cases}$$

and

$$\lim_{\theta \uparrow \infty} L(\beta_0, \theta) = 0, \forall \beta_0 > 0.$$

For $0 < \beta_0 \leq 1$, $L(\beta_0, \theta)$ is maximized at $\theta = x_{(n)}$. However, for $\beta_0 > 1$, $L(\beta_0, \theta)$ attains its maximum at some $\theta > x_{(n)}$ and not at $x_{(n)}$. To illustrate this, the graphs of $L(\beta_0, \theta)$ based on a sample of size 10 from the power function distribution with $\theta = 2$ and $\beta_0 = 1.2,\ 2$, and 4, respectively, are displayed in Figures 3, 4 and 5. The values of $x_{(n)}$ are approximately 1.99, 1.79 and 0.56, respectively. Notice that for $\beta_0 = 1.2$, the maximum of $L(\theta)$ is approximately 1.75, which is very close to $x_{(n)}$=1.71, for $\beta_0 = 2$, we have $x_{(n)}$=1.22 and $L(\theta)$ is approximately maximized at $\theta = 1.49$, and for $\beta_0 = 4$, $x_{(n)} = 0.75$ and $L(\theta)$ attains its maximum at $\theta = 1.83$, approximately. We observe that the maximizer of $L(\theta)$ deviates from $x_{(n)}$ with increasing β.

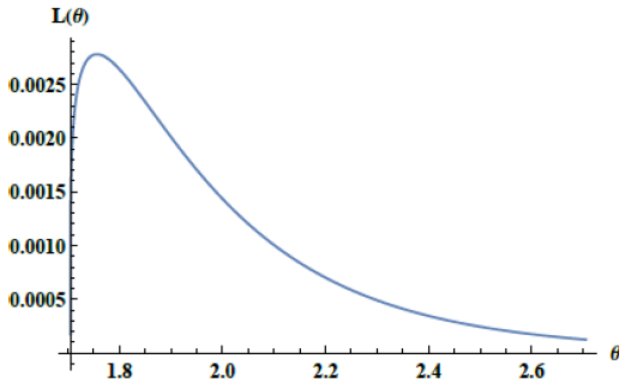

Figure 3: Likelihood function based on a sample of size 10 generated from the power function distribution with $\beta = 1.2$ and $= 2$.

Now we investigate the MLE of the parameter vector $\boldsymbol{\theta} = (\beta, \theta)$. Necessary conditions for the existence and uniqueness of the MLE of a parameter vector $\boldsymbol{\theta} = (\theta_1, ..., \theta_k)$ are in [27]:

1. $L(\boldsymbol{\theta})$ is a twice continuously differentiable likelihood function varying in a connected open subset $\boldsymbol{\Theta} \subset R^k$.

2. $L(\boldsymbol{\theta})$ satisfies the following two conditions:

   (i) $\lim_{\boldsymbol{\theta} \to \partial \boldsymbol{\Theta}} L(\boldsymbol{\theta}) = 0,$

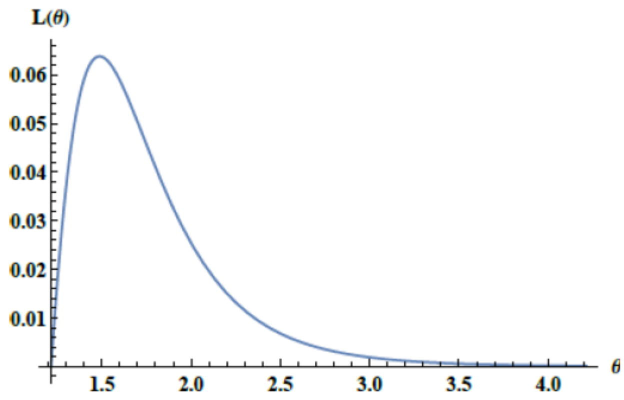

Figure 4: Likelihood function based on a sample of size 10 generated from the power function distribution with $\beta = 2$ and $= 2$.

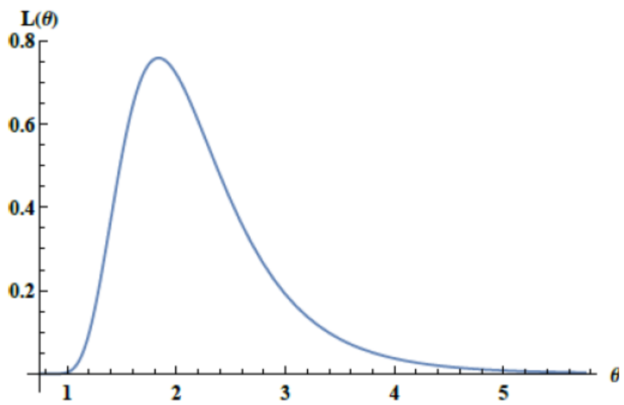

Figure 5: Likelihood function based on a sample of size 10 generated from the power function distribution with $\beta = 4$ and $= 2$.

(ii) The Hessian matrix of second partial derivatives

$$\boldsymbol{H} = \begin{pmatrix} \frac{\partial^2 L}{\partial \theta^2} & \frac{\partial^2 L}{\partial \theta \partial \beta} \\ \frac{\partial^2 L}{\partial \beta \partial \theta} & \frac{\partial^2 L}{\partial \beta^2} \end{pmatrix}$$

is negative definite at every point $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ for which the vector$\nabla L = (\partial L/\partial \theta_i) = 0$.

These conditions also apply to the log−likelihood function with $\lim_{\boldsymbol{\theta}\to\partial\boldsymbol{\Theta}} l(\boldsymbol{\theta})=0$ being replaced by $\lim_{\boldsymbol{\theta}\to\partial\boldsymbol{\Theta}} L(\boldsymbol{\theta})=-\infty$.

The domain of the likelihood function of the power function distribution based on a sample of size n is the rectangle $(0,\infty)\times(x_{(n)},\infty)$ which is an open connected set in R^2. The boundaries of $\boldsymbol{\Theta}$ are the lines $\theta=x_{(n)},\beta\in(0,\infty)$ and $\beta=0,\theta\in(x_{(n)},\infty)$. It is clear that $\lim_{\boldsymbol{\theta}\to\partial\boldsymbol{\Theta}} L(\boldsymbol{\theta})=0$ when approaching each of these two lines. So, to prove the existence and uniqueness of the MLE, it remains to show that the Hessian matrix $\mathbf{H}$ is nonnegative definite at the zeros of the first partial derivatives of $L(\boldsymbol{\theta})$ or, equivalently, of $l(\boldsymbol{\theta})$.

Given the observed sample $x_1,...,x_n$, the log-likelihood function is

$$l(\beta,\theta)\equiv\log L(\beta,\theta)=n\,\log\,\beta-n\,\log\,\theta+(\beta-1)\sum_{j=1}^{n}\log\left(1-\frac{x_{(j)}}{\theta}\right). \tag{5}$$

For $\beta>0$, we have

$$\frac{\partial l(\beta,\theta)}{\partial\beta}=\frac{n}{\beta}+\sum_{j=1}^{n}\log\left(1-\frac{x_{(j)}}{\theta}\right)=0, \tag{6}$$

and for $\theta>x_{(n)}$, we have

$$\frac{\partial l(\beta,\theta)}{\partial\theta}=-\frac{n}{\theta}+(\beta-1)\sum_{j=1}^{n}\frac{x_{(j)}}{\theta\left(\theta-x_{(j)}\right)}=0. \tag{7}$$

Solving (6) for β, we obtain

$$\beta\equiv\beta(\theta)=-\frac{n}{\sum_{j=1}^{n}\log\left(1-\frac{x_{(j)}}{\theta}\right)} \tag{8}$$

and solving (7) for β, we have

$$\beta-1=\frac{n}{\sum_{j=1}^{n}x_j\left(\theta-x_j\right)^{-1}}. \tag{9}$$

Since

$$\frac{\partial^2 l(\beta,\theta)}{\partial\beta^2}=-\frac{n}{\beta^2}<0,$$

it follows for fixed θ, $l(\beta,\theta)$ is maximized at

$$\beta(\theta)=-\frac{n}{\sum_{j=1}^{n}\log\left(1-\frac{x_{(j)}}{\theta}\right)},$$

provided that the maximum exists. Replacing β by $\beta(\theta)$, the log-likelihood (5) can be written as

$$\begin{aligned} l(\theta)\equiv\sup_{\beta} l(\beta,\theta) &= n\,\log\beta(\theta)-n\,\log\theta+(\beta(\theta)-1)\sum_{j=1}^{n}\log\left(1-\frac{x_{(j)}}{\theta}\right)\\ &= n\,\log\beta(\theta)-n\,\log\theta+(\beta(\theta)-1)\frac{-n}{\beta(\theta)}\\ &= n\,\log\frac{\beta(\theta)}{\theta}+\frac{n}{\beta(\theta)}-n. \end{aligned}$$

The second partial derivatives of $l(\beta,\theta)$ are

$$\frac{\partial^2 l(\beta,\theta)}{\partial\beta^2} = -\frac{n}{\beta^2}, \tag{10}$$

$$\frac{\partial^2 l(\beta,\theta)}{\partial\theta^2} = \frac{n}{\theta^2} \text{-}(\beta-1)\sum_{j=1}^{n}\frac{x_{(j)}(2\theta-x_{(j)})}{\theta^2(\theta-x_{(j)})^2}, \tag{11}$$

$$\frac{\partial^2 l(\beta,\theta)}{\partial\theta\partial\beta} = \frac{\partial^2 l(\beta,\theta)}{\partial\beta\partial\theta} = \sum_{j=1}^{n}\frac{x_{(j)}}{\theta(\theta-x_{(j)})}. \tag{12}$$

From (9) and (12), we have $\frac{\partial^2 l(\beta,\theta)}{\partial\theta\partial\beta} = \frac{1}{\theta}\sum_{j=1}^{n}\frac{x_{(j)}}{(\theta-x_{(j)})} = \frac{1}{\theta}\frac{n}{\beta-1}$. Thus, $\mathbf{H}$ can be written as

$$H = \begin{pmatrix} -\frac{n}{\beta^2} & \frac{n}{\theta(\beta-1)} \\ \frac{n}{\theta(\beta-1)} & \frac{n}{\theta^2} - (\beta-1)\sum_{j=1}^{n}\frac{x_j(2\theta-x_j)}{\theta^2(\theta-x_j)^2} \end{pmatrix}.$$

The determinant of $\mathbf{H}$ is

$$D = -\frac{n}{\beta^2}\left[\frac{n}{\theta^2} - (\beta-1)\sum_{j=1}^{n}\frac{x_j(2\theta-x_j)}{\theta^2(\theta-x_j)^2}\right] - \left(\frac{n}{\theta(\beta-1)}\right)^2. \tag{13}$$

Completing the square of the numerator of the term inside the sum on the right-hand side of (13), we get

$$\begin{aligned} D &= -\frac{n}{\beta^2}\left[\frac{n}{\theta^2} + (\beta-1)\sum_{j=1}^{n}\frac{\left\{(\theta-x_j)^2-\theta^2\right\}}{\theta^2(\theta-x_j)^2}\right] - \left(\frac{n}{\theta(\beta-1)}\right)^2 \\ &= -\frac{n}{\beta^2}\left[\frac{n\beta}{\theta^2} - (\beta-1)\sum_{j=1}^{n}\frac{1}{(\theta-x_j)^2}\right] - \left(\frac{n}{\theta(\beta-1)}\right)^2 \\ &= -\frac{n}{\beta^2}\left[\frac{n\beta}{\theta^2} + \frac{n\beta^2}{\theta^2(\beta-1)^2} - (\beta-1)\sum_{j=1}^{n}\frac{1}{(\theta-x_j)^2}\right]. \end{aligned}$$

For D to be negative definite, we need to show that the term between the square brackets is positive. That is,

$$\frac{n\beta}{\theta^2}\left(1+\frac{\beta}{(\beta-1)^2}\right) - (\beta-1)\sum_{j=1}^{n}\frac{1}{(\theta-x_j)^2} > 0, \text{ or}$$

$$\frac{n\beta}{\theta^2}\left(1+\frac{\beta}{(\beta-1)^2}\right) > (\beta-1)\sum_{j=1}^{n}\frac{1}{(\theta-x_j)^2}.$$

Then for each $j = 1,\dots,n$, we have $\theta - x_j < \theta$, which implies that

$$\sum_{j=1}^{n}\frac{1}{(\theta-x_j)^2} > \sum_{j=1}^{n}\frac{1}{\theta^2} = \frac{n}{\theta^2}.$$

So, the above inequality reduces to

$$\frac{n\beta}{\theta^2(\beta-1)}\left(1+\frac{\beta}{(\beta-1)^2}\right)>\frac{n}{\theta^2}.$$

Multiplying both sides of the above inequality by θ^2/n, after some algebra, we get

$$\frac{\beta^3-\beta^2+1-(\beta-1)^3}{(\beta-1)^3}>0.$$

We have noticed earlier that for $0<\beta\leq 1$, the MLE is $X_{(n)}$, so we only examine here the case $\beta>1$. After expanding the numerator and noticing that the denominator is positive, the last inequality reduces to $2\beta^2-3\beta+2=2(\beta-1)^2+\beta>0$, which is true for all β and hence for $\beta>1$. Thus,

$$\frac{n\beta}{\theta^2}+\frac{n\beta^2}{\theta^2(\beta-1)^2}-(\beta-1)\sum_{j=1}^{n}\frac{1}{(\theta-x_j)^2}>0,\forall\beta>1,\text{and}$$

$$-\frac{n}{\beta^2}\left[\frac{n\beta}{\theta^2}+\frac{n\beta^2}{\theta^2(\beta-1)^2}-(\beta-1)\sum_{j=1}^{n}\frac{1}{(\theta-x_j)^2}\right]<0,\forall\beta>1.$$

We have shown that the Hessian matrix $\mathbf{H}$ is negative definite at the zeros of the first partial derivatives of the log-likelihood function. Thus, all necessary conditions for the existence and uniqueness of the MLE are met.

Let $(X_1,R_1),(X_2,R_2),\ldots,(X_m,R_m)$ be a progressively censored sample, where $X_1<X_2<\ldots<X_m$. With a predetermined number of removals, such as $R_1=r_1,R_2=r_2,\ldots,R_m=r_m$, the conditional likelihood function can be written as [3]

$$L(\theta,\beta;x|R=r)=A\prod_{i=1}^{m}f(x_i)\left(1-F(x_i)\right)^{r_i},\tag{14}$$

where $A=n(n-r-1)...(n-\sum_{i=1}^{m-1}r_i+1)$.

After substituting (3) and (4) into equation (14), the likelihood function becomes

$$L(\theta,\beta;x|R=r)=A\prod_{i=1}^{m}\frac{\beta}{\theta}\left(\frac{\theta-x_i}{\theta}\right)^{\beta-1}\left(\left(\frac{\theta-x_i}{\theta}\right)^{\beta}\right)^{r_i}.\tag{15}$$

Suppose that an individual unit being removed from the test at the i^{th} failure,$i=1,2,\ldots,m-i$, is independent of the others but with the same probability p. Therefore, $R_i, i=1,2,..,m-1$, follows a binomial distribution with parameters $n-m-\sum_{k=1}^{m-1}r_k$ and p. Thus,

$$P(R_1=r_1)=\binom{n-m}{r_1}p^{r_1}(1-p)^{n-m-r_1},\tag{16}$$

$$P\left(R_i=r_i|R_{i-1}=r_{i-1},\ldots,R_1-r_1\right)$$
$$=\binom{n-m-\sum_{k=1}^{m-1}r_k}{r_i}p^{r_i}(1-p)^{n-m-\sum_{k=1}^{i}r_k},\ i=1,2,...,m-1,\tag{17}$$

where $0 \leq r_i \leq n - m - \sum_{j=1}^{i-1} r_j$.

The full likelihood function takes the following form:

$$L(\theta, \beta, p; x, r) = L(\theta, \beta; x|R) = r(P(R = r|p)), \tag{18}$$

where $P(R = r|p))$ is the joint conditional distribution and is given by

$$P(R = r|p) =$$

$$P(R_1 = r_1)P(R_2 = r_2|R_1 = r_1)...P(R_{m-1} = r_{m-1}|R_{m-2} = r_{m-2}, ..., R_1 = r_1) \tag{19}$$
$$= \frac{\text{(n-m)!p}^{\sum_{i=1}^{m-1} r_i}(1-p)^{(m-1)(n-m)\sum_{i=1}^{m-1}(m-i)r_i}}{\left(n - m - \sum_{i=1}^{m-1} r_i\right)\ !\ \prod_{i=1}^{m-1} r_i!}.$$

Using equations (15), (18) and (19), we can write the full likelihood function as

$$L(\theta, \beta, p; x, r) = AL_1(\theta, \beta) L_2(p),$$

where

$$L_1(\theta, \beta) = \prod_{i=1}^{m} \frac{\beta}{\theta}\left(\frac{\theta - x_i}{\theta}\right)^{\beta - 1}\left(\left(\frac{\theta - x_i}{\theta}\right)^{\beta}\right)^{r_i}$$
$$= \left(\frac{\beta}{\theta}\right)^{m} \prod_{i=1}^{m}\left(\frac{\theta - x_i}{\theta}\right)^{\beta(1+r_i)-1},$$
$$L_2(p) = p^{\sum_{i=1}^{m-1} r_i}(1-p)^{(m-1)(n-m)\sum_{i=1}^{m-1}(m-i)r_i}$$

and

$$A = \frac{c(n-m)!}{\left(n - m - \sum_{i=1}^{m-1} r_i\right)\ !\ \prod_{i=1}^{m-1} r_i!}.$$

It is clear that A is parameter free and $L_2(p)$ is independent of θ and β.

The MLE of β can be obtained by maximizing

$$L_1(\beta, \theta) = \prod_{i=1}^{m} \frac{\beta}{\theta}\left(\frac{\theta - x_i}{\theta}\right)^{\beta(r_i+1)-1}$$

or, equivalently, the log-likelihood function

$$l_1(\beta, \theta) = m\log(\beta) - m\log(\theta) + \sum_{i=1}^{m}[\beta(r_i + 1) - 1]\log\left(\frac{\theta - x_i}{\theta}\right). \tag{20}$$

The corresponding likelihood equations are

$$\frac{\partial l_1}{\partial \beta} = \frac{m}{\beta} + \sum_{i=1}^{m}(r_i + 1)\log\left(\frac{\theta - x_i}{\theta}\right) = 0, \tag{21}$$

$$\frac{\partial l_1}{\partial \theta} = -\frac{m}{\theta} + \sum_{i=1}^{m}[\beta(r_i + 1) - 1]\log\frac{x_i}{\theta(\theta - x_i)} = 0. \tag{22}$$

Solving (21) for β, we obtain

$$\beta \equiv \beta(\theta) = -\frac{m}{\sum_{i=1}^{m}(r_i+1)\log\left(\frac{\theta - x_i}{\theta}\right)}. \tag{23}$$

Substitute (23) into (22) to obtain

$$\sum_{i=1}^{m}[\beta(\theta)(r_i+1)-1]\log\frac{x_i}{\theta\,(\theta - x_i)} = \frac{m}{\theta}. \tag{24}$$

As clarified before, the MLE of θ is $X_{(m)}$ if $0 < \beta(r_i+1) \le 1$, in this case the MLE of β is

$$\hat{\beta} = -\frac{m}{\sum_{i=1}^{m}(r_i+1)\log\left(\frac{x_{(m)} - x_i}{x_{(m)}}\right)}.$$

For $\beta(r_i+1) > 1$, we use numerical methods to solve (24) for θ and then apply (23) to solve for β. The MLE of p is easily derived by maximizing $\log L_2(p)$:

$$\hat{p} = \frac{\sum_{i=1}^{m-1} r_i}{\sum_{i=1}^{m-1} r_i + (m-1)(n-m) - \sum_{i=1}^{m-1}(m-i)r_i}.$$

4 Numerical Results

4.1 Simulation study

A simulation study was performed to deduce the behavior of the maximum likelihood estimators. Different sample sizes, namely, n = 25, 50 and 100 were used. Different combinations of the parameter values of θ and β were considered. The values of the parameter p used in the simulation study are 0.25, 0.5 and 0.75. The simulation results were based on 1000 replicates. The means and root mean square errors (RMSE) of the maximum likelihood estimators for the three parameters p, θ and β are displayed in Tables 1, 2 and 3.

The following concluding remarks can be drawn based on the results shown in Tables 1, 2, and 3:

1. For a fixed value of m, as n increases, the bias and RMSE show a decreasing trend.
2. For a fixed value of n, as m increases, the bias and RMSE decrease for the maximum likelihood estimators for θ and β; on the other hand, the RMSE increases for the maximum likelihood estimator for p.
3. As the shape parameter β increases, the bias and RMSE increase.
4. As the value of the probability parameter p increases, the bias and RMSE for the estimators of p and β increase.

4.2 Study on real data

A real-life data set is considered which represents the failure times (in minutes) for a sample of 15 electronic components in an accelerated life test [5]. The data set is 1.4, 5.1, 6.3, 10.8, 12.1, 18.5, 19.7, 22.2, 23.0, 30.6, 37.3, 46.3, 53.9, 59.8, 66.2.

Table 1: Mean and RMSE of the MLEs and for p = 0.25 and different choices n, m, θ and β.

n	m	$\hat{\theta}$	$\hat{\beta}$	$\hat{p}$		$\hat{\theta}$		$\hat{\beta}$	
				Mean	RMSE	Mean	RMSE	Mean	RMSE
25	15	1	0.5	0.26926	0.07899	0.99366	0.01187	0.51518	0.14276
		1.5	1	0.26926	0.07910	1.41241	0.08126	0.91826	0.26551
		2	1.5	0.26926	0.07900	1.72610	0.17629	1.20655	0.37488
	20	1	0.5	0.29167	0.12260	0.99592	0.00899	0.57428	0.13269
		1.5	1	0.29166	0.12261	1.43108	0.06667	1.03523	0.24506
		2	1.5	0.29166	0.12261	1.76733	0.15360	1.37200	0.35504
50	30	1	0.5	0.25922	0.05237	0.99796	0.00422	0.52328	0.10064
		1.5	1	0.25922	0.05237	1.45143	0.04723	0.96558	0.18577
		2	1.5	0.25922	0.05237	1.81607	0.12262	1.30226	0.26939
	40	1	0.5	0.27079	0.07868	0.99892	0.00234	0.54803	0.08874
		1.5	1	0.27079	0.07868	1.46462	0.03431	1.02557	0.16921
		2	1.5	0.27079	0.07868	1.85091	0.09830	1.39968	0.25315
100	80	1	0.5	0.25909	0.05182	0.99969	0.00068	0.52754	0.05999
		1.5	1	0.25909	0.05182	1.48124	0.01857	1.01082	0.11601
		2	1.5	0.25909	0.05182	1.90260	0.06567	1.41151	0.17877
	90	1	0.5	0.27079	0.07868	0.99977	0.00045	0.52939	0.05726
		1.5	1	0.27079	0.07868	1.48350	0.01589	1.01832	0.11180
		2	1.5	0.27079	0.07868	1.91029	0.05957	1.42767	0.17258

The failure times were analyzed in order to validate the proposed progressive Type-II censoring scheme using the power function distribution model. The validity of the power distribution function was checked based on the maximum likelihood estimated parameters θ and β of 66.2 and 1.1573, respectively. The Kolmogorov-Smirnov (K−S) test was used for this data set. It is noted that the K−S distance between the fitted and the empirical distribution functions equals to 0.21, and the corresponding critical value at α =0.05 equals to 0.33. Thus, the power function distribution fits the above data set reasonably well.

Five progressively censored samples were generated from the above data for the values of m = 14, 13, 12, 11, 10. Uniform random removal of subjects was used to generate (r_1,r_2,...,r_m).

Progressive censoring with m =14 (1 observation removed): (1.4, 0), (5.1,1), (6.3,0), 10.8,0), (12.1,0), (18.5,0), (19.7,0), (22.2,0), (23.0,0), (30.6,0), (37.3,0), (53.9,0), (59.8,0), (66.2).

Progressive censoring with m =13 (2 observations removed): (1.4,1), (5.1,0), (6.3,0), (10.8,0), (12.1,0), (18.5,0), (22.2,0), (23.0,0), (30.6,0), (37.3,1), (53.9,0), (59.8,0), (66.2,0).

Progressive censoring with m =12 (3 observations removed): (1.4,1), (6.3,0), (10.8,0), (12.1,0), (18.5,1), (19.7,0), (22.2,0), (23.0,0), (30.6,0), (37.3,1), (59.8,0), (66.2,0).

Progressive censoring with m =11 (4 observations removed): (1.4,1), (5.1,0), (10.8,0), (12.1,1), (18.5,0), (19.7,1), (22.2,0), (23.0,0), (30.6,0), (37.3,1), (53.9,0).

Progressive censoring with m =10 (5 observations removed): (1.4,1), (5.1,0), (6.3,1), (10.8,1), (12.1,0), (18.5,0), (19.7,1), (22.2,0), (37.3,0), (46.3,1).

The maximum likelihood estimates for the model parameters β and θ using the five

Table 2: Mean and RMSE of the MLEs $\hat{p}, \hat{\theta}$, and $\hat{\beta}$ for $p = 0.5$ and different choices of n, m, θ and β.

n	m	$\hat{\theta}$	$\hat{\beta}$	$\hat{p}$		$\hat{\theta}$		$\hat{\beta}$	
				Mean	RMSE	Mean	RMSE	Mean	RMSE
25	15	1	0.5	0.52050	0.11961	0.99366	0.01187	0.57897	0.15603
		1.5	1	0.52050	0.11961	1.41241	0.08126	1.03978	0.28820
		2	1.5	0.52050	0.11961	1.72610	0.17649	1.35124	0.40669
	20	1	0.5	0.55036	0.17118	0.99592	0.00899	0.59603	0.13753
		1.5	1	0.55036	0.17118	1.43108	0.06666	1.07318	0.25358
		2	1.5	0.55036	0.17118	1.76733	0.15360	1.42118	0.36723
50	30	1	0.5	0.51021	0.08361	0.99796	0.00422	0.55312	0.10559
		1.5	1	0.51021	0.08361	1.45143	0.04723	1.01860	0.19525
		2	1.5	0.51021	0.08361	1.81607	0.12262	1.37170	0.28424
	40	1	0.5	0.52049	0.11961	0.92545	0.04940	1.41970	0.25714
		1.5	1	0.52049	0.11961	1.46462	0.03431	1.04072	0.17182
		2	1.5	0.52049	0.11961	1.85091	0.09880	1.41970	0.25719
100	80	1	0.5	0.51021	0.08361	0.99969	0.00068	0.52127	0.06048
		1.5	1	0.51021	0.08361	1.48124	0.01857	1.01775	0.11693
		2	1.5	0.51021	0.08361	1.90260	0.06567	1.42089	0.18018
	90	1	0.5	0.52050	0.11961	0.99877	0.00045	0.53083	0.05737
		1.5	1	0.52050	0.11961	1.48347	0.01589	1.02103	0.11204
		2	1.5	0.52050	0.11961	1.91029	0.05957	1.43137	0.17302

progressive censoring schemes with m=14, 13, 12, 11 and 10 are (1.1880, 66.2), (1.1348, 66.2), (1.1882, 66.2), (1.0772, 53.9) and (0.7553, 46.3), respectively.

5 Conclusion

We develop some results on the power function distribution when progressive Type-II censoring is used with binomial removals. The maximum likelihood estimators for the model parameters were derived. The simulation results showed that as the sample size increases, the performance of the estimators improves in terms of the bias and the RMSE. The biases and the RMSEs for p and β decrease as m increases. The bias and RMSE increase with the increase in the shape parameter β. The bias and RMSE for the estimators of p and β increase with the increase in parameter p. An application of real lifetime data was conducted, it illustrates the proposed censoring scheme.

Table 3: Mean and RMSE of the MLEs and for $p = 0.75$ and different choices of n, m, θ and β.

n	m	$\hat{\theta}$	$\hat{\beta}$	$\hat{p}$		$\hat{\theta}$		$\hat{\beta}$	
				Mean	RMSE	Mean	RMSE	Mean	RMSE
25	15	1	0.5	0.76234	0.12057	0.97366	0.01187	0.60158	0.16142
		1.5	1	0.76234	0.12057	1.41241	0.08126	1.06821	0.29896
		2	1.5	0.76234	0.12057	1.72261	0.17649	1.40003	0.42009
	20	1	0.5	0.77827	0.16000	0.99592	0.00899	0.60243	0.13902
		1.5	1	0.77827	0.16000	1.43108	0.06666	1.00416	0.25630
		2	1.5	0.77827	0.16000	1.76733	0.15360	1.43519	0.37113
50	30	1	0.5	0.75463	0.08281	0.99796	0.00422	0.56249	0.10805
		1.5	1	0.75463	0.08281	1.45143	0.04723	1.03510	0.19815
		2	1.5	0.75463	0.08281	1.81607	0.12262	1.39312	0.28874
	40	1	0.5	0.76234	0.12057	0.99892	0.00234	0.55915	0.09050
		1.5	1	0.76234	0.12057	1.46462	0.03431	1.04560	0.17262
		2	1.5	0.76234	0.12057	1.85091	0.09830	1.42612	0.25041
100	80	1	0.5	0.75463	0.08281	0.99969	0.00068	0.53249	0.06060
		1.5	1	0.75463	0.08281	1.48124	0.01857	1.01997	0.11713
		2	1.5	0.75463	0.08281	1.90260	0.06567	1.42390	0.19050
	90	1	0.5	0.76234	0.12057	0.99977	0.00045	0.53132	0.05744
		1.5	1	0.76234	0.12057	1.48347	0.01589	1.02195	0.11218
		2	1.5	0.76234	0.12057	1.91029	0.05957	1.43261	0.17326

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