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Capacity and Anisotropic Sobolev Spaces with Zero Boundary Values

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Abstract: The aim of this work is to study the capacity theory in anisotropic Sobolev spaces. In particular, we will give main properties of capacity, including monotonicity, countable subadditivity and several convergence results. Moreover, we will define the anisotropic Sobolev space with zero boundary values $B_0^{1,\vec{p}}(\Omega)$, where Ω is an open bounded set of $\mathbb{R}^N (N \geq 2)$, $\vec{p} = (p_0, p_1, ..., p_N)$ and $1 < p_0, p_1, ..., p_N < \infty$. This allows us to prove that the Dirichlet energy integral has a minimizer in the anisotropic Sobolev space with zero boundary values $B_0^{1,\vec{p}}(\Omega)$.

Keywords: *capacity; anisotropic Sobolev space with zero boundary values; Dirichlet energy.*

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1 Introduction

The notion of capacity is an essential tool in the study of nonlinear potential theory, which allows us to measure sets more precisely than the usual Lebesgue measure, to see that functions are better defined almost everywhere (quasi everywhere). Capacities play a key role in the study of solutions of partial differential equations, for example, Boccardo et al. studied in [6] the existence and non existence of solutions of the following problem:

$$(\mathcal{P}) \begin{cases} -\triangle u + u \mid \nabla u \mid^2 = \mu & in \ \Omega, \\ u = 0 & on \ \partial\Omega, \end{cases}$$

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where Ω is a bounded open set in $\mathbb{R}^N, N \geq 2$, and μ is a Radon measure on Ω. More precisely, the authors proved the existence of a solution u in $H_0^1(\Omega)$ for the problem $(\mathcal{P})$ if and only if the measure μ does not charge the sets of capacity zero in Ω. Capacity is also a tool to understand the point-wise behavior of functions in Sobolev spaces.

The theory of nonlinear potential was studied by Maz'ya and Khalvin in [13] and by Meyers in [20] in the L^p space ($1 < p < +\infty$) by introducing the concept of capacity in those spaces which allowed very rich applications in functional analysis, harmonic analysis, theory of probabilities and partial differential equations. The Sobolev capacity for constant exponent spaces has found a great number of applications, see Maz'ya [19], Evans and Gariepy [8], and Heinonen et al. in [12]. Also, Kilpeläinen introduced in [14] the weighted Sobolev capacity and discussed the role of capacity in the point-wise definitions of functions in Sobolev spaces involving the weights of Muckenhoup's A_p-class. On the other hand, Harjulehto et al. [9] generalized the Sobolev capacity to the variable exponent case. Later, this notion was defined in Orlicz spaces in [4] by N. Aissaoui and A. Benkirane and in Musielak-Orlicz space by M. C. Hassib, Y. Akdim, A. Benkirane and N. Aissaoui in [2, 3].

In a recent work [5], we have defined the $C_{k,\vec{p}}$ capacity in anisotropic Sobolev spaces. Also, we proved that $C_{k,\vec{p}}$ is a Choquet capacity.

The Sobolev space with zero boundary values was classically defined as a completion of compactly supported smooth functions with respect to the Sobolev space [18]. Indeed, the Sobolev space with zero boundary is essential to specify or compare boundary values of Sobolev functions. This is particularly important in connection with boundary value problems in the calculus of variations and partial differential equations and with comparison principales in potential theory. Then, the variable exponent Sobolev space with zero boundary values has been defined in [10] following a method developed by Kilpeläinen, Kinnunen and Martio in [16] for metric measure spaces. On the other hand, this notion was generalized by M. C. Hassib and Y. Akdim [11] to weighted variable exponent Sobolev spaces on metric measure spaces. In [22], T. Ohno and T. Shimomura studied the Musielak-Orlicz-Sobolev space with zero boundary values on any metric space endowed with a Borel regular measure.

Our goal in this work is to study the anisotropic Sobolev space with zero boundary values using the concept of capacity.

The present paper is organized as follows. In the second section, we recall some preliminary results on anistropic Sobolev spaces and some properties of capacities. In Section 3, we develop a capacity theory in this space by including monotonicity, countable subadditivity and serval convergence results, we define the anisotropic Sobolev space with zero boundary values and we show some of its properties. As an application of our results, we consider, in Section 4, the Dirichlet energy and we prove that it has a minimizer in anisotropic Sobolev spaces with zero boundary values.

2 Preliminaries

2.1 Anisotropic Sobolev spaces

Let Ω be an open bounded domain in $\mathbb{R}^N (N \geq 2)$ with boundary $\partial\Omega$.
Let $1 < p_0, p_1, ..., p_N < \infty$ and denote

$$\vec{p} = (p_0, p_1, ..., p_N),\ D^0 u = u \text{ and } D^i u = \frac{\partial u}{\partial x_i} \quad \text{for } i = 1, ..., N.$$

Set

$$\underline{p} = \min\{p_0, p_1, ..., p_N\}, \text{ then } \underline{p} > 1.$$

The anisotropic Sobolev space $W^{1,\vec{p}}(\Omega)$ is defined as follows:

$$W^{1,\vec{p}}(\Omega) = \{u \in L^{p_0}(\Omega) \ \ and \ \ D^i u \in L^{p_i}(\Omega), i = 1, ..., N\}.$$

We recall that the $W^{1,\vec{p}}(\Omega)$ is a separable, reflexive Banach space (see [1]) with respect to the norm

$$\|u\|_{W^{1,\vec{p}}(\Omega)} = \sum_{i=0}^{N} \|D^i u\|_{L^{p_i}(\Omega)}.$$

We recall also the space $W_0^{1,\vec{p}}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to this norm. The theory of such anisotropic spaces was developed in [25], [21], [23], [24]. It was shown that $C_0^\infty(\Omega)$ is dense in $W_0^{1,\vec{p}}(\Omega)$ and $W_0^{1,\vec{p}}(\Omega)$ is a reflexive Banach space. For any $\vec{p} = (p_0, p_1, ..., p_N)$, with $1 < p_i < \infty$, $i = 0, 1, ..., N$, the dual space of the anisotropic Sobolev space $W_0^{1,\vec{p}}(\Omega)$ is equivalent to $W^{-1,\vec{p'}}(\Omega)$, where $\vec{p'} = (p'_0, p'_1, ..., p'_N)$ and $p'_i = \dfrac{p_i}{p_i - 1}$ for all $i = 0, 1, ..., , N$.

Proposition 2.1 *Let $p \in [1, +\infty[$ and $(f_n)_n$ be a sequence in $(L^p(\mu), \|.\|_p)$ whose series of norms $\sum_n \|f_n\|_p$ converges. Then the series of functions $\sum_n f_n$ converges for the norm $\|.\|_p$ and we have $\|\sum_n f_n\|_p \leq \sum_n \|f_n\|_p$.*

Proof. For $n \in \mathbb{N}^*$ fixed, according to the Minkowski inequality, we have

$$\left\| \sum_{k=0}^{n} |f_k| \right\|_p \leq \sum_{k=0}^{n} \|f_k\|_p \leq \sum_{k=0}^{+\infty} \|f_k\|_p.$$

It follows from the monotone convergence theorem that

$$\left(\int_\Omega \left(\sum_{k=0}^{+\infty} |f_k| \right)^p d\mu \right)^{\frac{1}{p}} \leq \sum_{k=0}^{+\infty} \|f_k\|_p.$$

Thus,

$$\left\| \sum_{k=0}^{+\infty} f_k \right\|_p \leq \sum_{k=0}^{+\infty} \|f_k\|_p.$$

Proposition 2.2 *[[7]] Let E be a Banach space. If $(f_n)_n$ converges weakly to f in E, then $\|f_n\|$ is bounded and $\|f\| \leq \liminf \|f_n\|$.*

By the application of Proposition 2.1, we have the following result.

Lemma 2.1 *Let (f_n) be a sequence in $W^{1,\vec{p}}(\Omega)$ whose series of norms $\sum_n \|f_n\|_{W^{1,\vec{p}}(\Omega)}$ converges. Then we have*

$$\|\sum_n f_n\|_{W^{1,\vec{p}}(\Omega)} \leq \sum_n \|f_n\|_{W^{1,\vec{p}}(\Omega)}.$$

2.2 Capacity

Definition 2.1 Let E be a topological space and T be the class of Borel sets in E, and let $C : T \to [\ 0,+\infty\ ]$ be a function.
1) The function C is called a capacity if the following axioms are satisfied:
i) $C(\emptyset) = 0$.
ii) $X \subset Y \Rightarrow C(X) \leq C(Y)$ for all X and Y in T (*monotonicity*).
iii) For all sequences $(X_n) \subset T$

$$C(\bigcup_n X_n) \leq \sum_n C(X_n)\ (countable\, subadditivity).$$

2) The capacity C is called an outer capacity if, for all $X \in T$,

$$C(X) = \inf\{C(O) : O \supset X, O \text{ is open }\}.$$

3) The capacity C is called an interior capacity if, for all $X \in T$,

$$C(X) = \sup\{C(K) : K \subset X, K \text{ is compact}\}.$$

4) A property that holds true, except perhaps on a set of capacity zero, is said to be true C-quasi everywhere $(abbreviated\, C - q.e)$.

Definition 2.2 Let f be a real-valued function being finite C-q.e and (f_n) be a sequence of real-valued function being finite C-q.e.
1) We say that (f_n) converges to f in C-capacity if

$$\forall \varepsilon > 0, \lim_{n\to+\infty} C\left(\{x : |f_n(x) - f(x)| > \varepsilon\}\right) = 0.$$

2) We say that (f_n) converges to f C-quasi uniformly $(abbreviated\ \ C - q.u)$ if
$\forall \varepsilon > 0, \exists X \in T : C(X) < \varepsilon$ and (f_n) converges to f uniformly on X^c.

Using same arguments as in Remark 1.27 in [18], we obtain the following remark.

Remark 2.1 Let $\Omega \subset \mathbb{R}^N$ and $u,\, v \in W^{1,\vec{p}}(\Omega)$, then $\max(u, v) \in W^{1,\vec{p}}(\Omega)$ and $\min(u, v) \in W^{1,\vec{p}}(\Omega)$. Moreover, for $j = 1, ..., N$, we have

$$D^j \max(u, v) = \begin{cases} D^j u & almost\ everywhere\ in\ \ \{x \in \Omega,\ u(x) \geq v(x)\}, \\ D^j v & almost\ everywhere\ in\ \ \{x \in \Omega,\ v(x) \geq u(x)\}. \end{cases}$$

3 Anisotropic Sobolev $\vec{p}$ - Capacity

In the whole of this paper, we assume that Ω is an open bounded domain in $\mathbb{R}^N (N \geq 2)$ with boundary $\partial\Omega$ and μ is a measure of Lebesegue.

Definition 3.1 The anisotropic Sobolev $\vec{p}$ - capacity of the set $E \subset \Omega$ is defined by

$$C_{\vec{p}}(E) = \inf_{u\in A(E)} \left\{\|u\|_{W^{1,\vec{p}}(\Omega)}\right\},$$

where

$$A(E) = \left\{u \in W^{1,\vec{p}}(\Omega) : u \geq 1\ \ on\ \ an\ \ open\ \ set\ \ containing\ \ E\ \ and\ \ u \geq 0\right\}.$$

If $A(E) = \phi$, we set $C_{\vec{p}}(E) = \infty$. Functions belonging to $A(E)$ are called admissible functions for E.

Lemma 3.1 *The anisotropic Sobolev $\vec{p}$ - capacity is a capacity.*

Proof.

i) It is obvious that $C_{\vec{p}}(\phi) = 0$.

ii) $A(E_2) \subset A(E_1)$ *implies* $C_{\vec{p}}(E_1) \leq C_{\vec{p}}(E_2)$ for every $E_1 \subset E_2$.

iii) Let $\varepsilon > 0$, we may assume that $\sum_{i=0}^{\infty} C_{\vec{p}}(E_i) < +\infty$.

Let (E_i) be a subset of Ω $\Big($ if $\sum_{i=0}^{\infty} C_{\vec{p}}(E_i) = +\infty$, there is nothing to show$\Big)$, then

$$\forall i \in \mathbb{N}, \;\; C_{\vec{p}}(E_i) < +\infty,$$

therefore, we choose $u_i \in A(E_i)$ so that

$$\|u_i\|_{W^{1,\vec{p}}(\Omega)} \leq C_{\vec{p}}(E_i) + \varepsilon \times 2^{-i-1}, \quad i = 0, 1, 2, ...$$

Let $v = \sup u_i$, we show that v is an admissible function for $\bigcup_{i=0}^{+\infty} E_i$.

Indeed, for all $i \in \mathbb{N}$, we have by Lemma 2.1 that

$$\|\sup u_i\|_{W^{1,\vec{p}}(\Omega)} \leq \|\sum_{i=0}^{+\infty} u_i\|_{W^{1,\vec{p}}(\Omega)} \leq \sum_{i=0}^{+\infty} \|u_i\|_{W^{1,\vec{p}}(\Omega)},$$

thus,

$$\|v\|_{W^{1,\vec{p}}(\Omega)} \leq \sum_{i=0}^{+\infty} \|u_i\|_{W^{1,\vec{p}}(\Omega)} \leq \sum_{i=0}^{+\infty} C_{\vec{p}}(E_i) + \varepsilon,$$

which implies that $v \in W^{1,\vec{p}}(\Omega)$. Since $u_i \in A(E_i)$, there exists an open set $O_i \supset E_i$ such that $u_i \geq 1$ on O_i for every i= 0,1,2,..., it follows that $v = \sup u_i \geq 1$ on $\bigcup_{i=1}^{+\infty} O_i$ which is an open set containing $\bigcup_{i=0}^{+\infty} E_i$.

Hence we conclude that $C_{\vec{p}}$ is a capacity.

Lemma 3.2 *Let $E \subset \Omega$. The anistropic Sobolev $\vec{p}$ - capacity of E is given by*

$$C_{\vec{p}}(E) = \inf_{u \in B(E)} \left\{ \|u\|_{W^{1,\vec{p}}(\Omega)} \right\},$$

where

$$B(E) = \Big\{ u \in A(E) : 0 \leq u \leq 1 \Big\}.$$

Proof. Clearly, we have

$$B(E) \subset A(E),$$

thus,

$$C_{\vec{p}}(E) \leq \inf_{u \in B(E)} \Big\{ \|u\|_{W^{1,\vec{p}}(\Omega)} \Big\}.$$

For the reverse inequality, let $\varepsilon > 0$ and let $u \in A(E)$ such that

$$\|u\|_{W^{1,\vec{p}}(\Omega)} \leq C_{\vec{p}}(E) + \varepsilon.$$

Then we have $v = \max(0, \min(u, 1)) \in B(E)$. Thus, $v \leq u$ and by Remark 2.1, we have

$$\left| \frac{\partial v}{\partial x_j} \right| \leq \left| \frac{\partial u}{\partial x_j} \right| \ \ for\ j = 1, ..., N\ almost\ everywhere.$$

Thus,

$$\inf_{u \in B(E)} \left\{ \|u\|_{W^{1,\vec{p}}(\Omega)} \right\} \leq \|v\|_{W^{1,\vec{p}}(\Omega)} \leq \|u\|_{W^{1,\vec{p}}(\Omega)} \leq C_{\vec{p}}(E) + \varepsilon.$$

Letting $\varepsilon \to 0$, we obtain

$$\inf_{u \in B(E)} \left\{ \|u\|_{W^{1,\vec{p}}(\Omega)} \right\} \leq C_{\vec{p}}(E).$$

Theorem 3.1 *The anisotropic Sobolev $\vec{p}$ - capacity is an outer capacity.*

Proof. Indeed, by monotonicity, we have

$$C_{\vec{p}}(E) \leq \inf\{C_{\vec{p}}(O) : E \subset O\ \ is\ \ open\}.$$

To prove the other inequality, let $\varepsilon > 0$ and take $u \in A(E)$ such that

$$\|u\|_{W^{1,\vec{p}}(\Omega)} \leq C_{\vec{p}}(E) + \varepsilon.$$

Since $u \in A(E)$, there is an open set O containing E such that $u \geq 1$ on O.
This implies that

$$C_{\vec{p}}(O) \leq \|u\|_{W^{1,\vec{p}}(\Omega)} \leq C_{\vec{p}}(E) + \varepsilon.$$

The claim follows by letting $\varepsilon \to 0$.

Proposition 3.1 *Let μ be a Lebesegue measure on Ω and $E \subset \Omega$, then*

$$\mu(E) \leq \mu(\Omega) C_{\vec{p}}(E).$$

Proof. If $C_{\vec{p}}(E) = \infty$, there is nothing to prove. Thus we may assume that $C_{\vec{p}}(E) < \infty$. Let $\varepsilon > 0$ and take $u \in A(E)$ such that

$$\|u\|_{W^{1,\vec{p}}(\Omega)} \leq C_{\vec{p}}(E) + \varepsilon.$$

There is an open $O \supset E$ such that $u \geq 1$ in O and $u \geq 0$, thus

$$\mu(E) \leq \mu(O) \leq \int_O |u| d\mu \leq \int_\Omega |u| d\mu.$$

On the other hand, by Hölder's inequality, we have

$$\int_\Omega |u| d\mu \leq (\mu(\Omega))^{1-\frac{1}{p_0}} \|u\|_{W^{1,\vec{p}}(\Omega)} \leq \mu(\Omega) \left(C_{\vec{p}}(E) + \varepsilon \right).$$

The claim follows by letting $\varepsilon \to 0$.

Theorem 3.2 *Let (k_n) be a decreasing sequence of compacts and $k = \bigcap_{n\in\mathbb{N}} k_n$, then*

$$\lim_{n\to\infty} C_{\vec{p}}(k_n) = C_{\vec{p}}(k).$$

Proof. First, we observe that $C_{\vec{p}}(k) \leq \lim_{n\to\infty} C_{\vec{p}}(k_n)$. On the other hand, let O be an open set such that $k \subset O$, thus

$$k \cap O^c = \phi.$$

The sequence $(k_n^{'})$ defined for all n by $k_n^{'} = k_n \cap O^c$ is a decreasing sequence of compacts that satisfies $\bigcap_{n\in\mathbb{N}} k_n^{'} = \emptyset$. Then, there exists n_0 such that $k_{n_0}^{'} = \emptyset$. Hence, for all $n \geq n_0$, $k_n^{'} = \emptyset$ and then $k_n \subset O$, for all $n \geq n_0$. Therefore,

$$\lim_{n\to\infty} C_{\vec{p}}(k_n) \leq C_{\vec{p}}(O).$$

And since $C_{\vec{p}}$ is an outer capacity, we obtain the claim by taking infimum over all open sets O containing k.

Proposition 3.2 *If there exists $u \in W^{1,\vec{p}}(\Omega)$ such that $u = +\infty$ on an open set containing E, then $C_{\vec{p}}(E) = 0$.*

Proof. Let $u \in W^{1,\vec{p}}(\Omega)$ be such that $u = +\infty$ on an open set O containing E, then $u \geq \alpha, \quad$ for all $\alpha > 0$. Therefore,

$$\forall \alpha > 0,\ C_{\vec{p}}(E) \leq \frac{1}{\alpha}\|u\|_{W^{1,\vec{p}}(\Omega)}.$$

Letting $\alpha \to +\infty$, we obtain $C_{\vec{p}}(E) = 0$.

Theorem 3.3 *Let u and $(u_n)_n$ be in $W^{1,\vec{p}}(\Omega)$ and consider the following propositions:*
i) $u_n \to u$ strongly in $W^{1,\vec{p}}(\Omega)$.
ii) $u_n \to u$ in $C_{\vec{p}}$-capacity .
iii) There is a subsequence (u_{n_j}) such that $u_{n_j} \to u$ in $C_{\vec{p}}$ - q.u.
iv) $(u_{n_j}) \to u$ in $C_{\vec{p}}$.- q.e.
Then we have

$$i) \Rightarrow ii) \Rightarrow iii) \Rightarrow iv).$$

Proof.

- We show that $\quad i) \Rightarrow ii)$.

By Proposition 3.2, we have u and u_n are finite $C_{\vec{p}}$ -q.e, for all n.
Let $\varepsilon > 0$, then

$$C_{\vec{p}}\Big(\Big\{x : |u_n - u|(x) > \varepsilon\Big\}\Big) \leq \frac{\|u_n - u\|_{W^{1,\vec{p}}}(\Omega)}{\varepsilon}.$$

- We show that $\quad ii) \Rightarrow iii)$.

Let $\varepsilon > 0$, then there exists u_{n_j} such that

$$C_{\vec{p}}\Big(\Big\{x : |u_{n_j} - u|(x) > 2^{-j}\Big\}\Big) \leq \varepsilon \cdot 2^{-j}.$$

We put

$$E_j = \Big\{x : |u_{n_j} - u|(x) > 2^{-j}\Big\} \ \ and \ \ G_m = \bigcup_{j \geq m} E_j.$$

Then we have

$$C_{\vec{p}}(G_m) \leq \sum_{j \geq m} \varepsilon \cdot 2^{-j} < \varepsilon.$$

On the other hand,

$$\forall x \in (G_m)^c, \forall j \geq m \ \ |u_{n_j} - u|(x) \leq 2^{-j},$$

thus

$$u_{n_j} \to u \ \ in \ C_{\vec{p}} - q.u.$$

- We show that $iii) \Rightarrow iv)$.

We have

$$\forall j \in \mathbb{N}, \exists X_j : C_{\vec{p}}(X_j) \leq \frac{1}{j},$$

thus,

$$u_{n_j} \ \ converges \ uniformly \ to \ u \ on \ (X_j)^C.$$

We put $X = \bigcap_j X_j$, then $C_{\vec{p}}(X) = 0$ and $u_{n_j} \to u$ on X^C.

As an immediate consequence of Theorem 3.3 and Proposition 3.1, we have the following result.

Corollary 3.1 *If $(u_n)_n$ is a sequence which converges to u in $W^{1,\vec{p}}(\Omega)$, then there exists a subsequence of $(u_n)_n$ which converges to u, μ a.e.*

Definition 3.2 A function $u : \Omega \to [-\infty, +\infty]$ is called a $C_{\vec{p}}$ - quasicontinuous function in Ω if for every $\varepsilon > 0$, there is a set X such that $C_{\vec{p}}(X) < \varepsilon$ and $u_{|\Omega \setminus X}$ is continuous.

Theorem 3.4 *The anistropic Sobolev $\vec{p}$ - capacity $C_{\vec{p}}$ satisfies the following properties:*

1) If O is an open set of Ω and $E \subset \Omega$ is such that $\mu(E) = 0$, then

$$C_{\vec{p}}(O) = C_{\vec{p}}(O - E).$$

2) Let u and v be $C_{\vec{p}}-$ quasicontinuous functions in Ω, we have
i) if $u = v$ almost everywhere in an open $O \subset \Omega$, then

$$u = v \ \ C_{\vec{p}} - \ \ quasi \ everywhere \ in \ \ O,$$

ii) If $u \leq v$ almost everywhere in an open $O \subset \Omega$, then

$$u \leq v \ \ C_{\vec{p}} - \ \ quasi \ everywhere \ in \ \ O.$$

Proof.

1) By monotonicity of $C_{\vec{p}}$, we get $C_{\vec{p}}(O) \geq C_{\vec{p}}(O-E)$.
Let $\varepsilon > 0$ and let $u \in A(O-E)$ be such that

$$\|u\|_{W^{1,\vec{p}}(\Omega)} \leq C_{\vec{p}}(O-E) + \varepsilon.$$

Then there exists an open $G \subset \Omega$ with $(O-E) \subset G$ and $u \geq 1$ almost everywhere in G. Since $G \cup O$ is open, $O \subset G \cup O$ and $u \geq 1$ almost everywhere in $G \cup (O-E)$, and almost everywhere in $G \cup O$ since $\mu(E) = 0$, we have $u \in A(O)$

$$C_{\vec{p}}(O) \leq \|u\|_{W^{1,\vec{p}}(\Omega)} \leq C_{\vec{p}}(O-E) + \varepsilon$$

by letting $\varepsilon \to 0$, we deduce that $C_{\vec{p}}(O) \leq C_{\vec{p}}(O-E)$.

2) Since $C_{\vec{p}}$ is an outer capacity, we get the results by [15].

Lemma 3.3 *For any bounded open $O \subset \Omega$, we have*

$$\mu(O) = 0 \Longleftrightarrow C_{\vec{p}}(O) = 0.$$

Proof. If $\mu(O) = 0$, then, by applying Theorem 3.4, we get $C_{\vec{p}}(O) = C_{\vec{p}}(O \backslash O) = C_{\vec{p}}(\phi) = 0$. On the other hand, if $C_{\vec{p}}(O) = 0$, then, by Proposition 3.1, $\mu(O) \leq C_{\vec{p}}(O) = 0$.

Proposition 3.3 *Let $(u_n)_n$, $u \in W^{1,\vec{p}}(\Omega)$ be such that $u_n \rightharpoonup u$ weakly in $W^{1,\vec{p}}(\Omega)$, then* $\liminf(u_n) \leq u \leq \limsup(u_n)$ $\quad C_{\vec{p}}$*-q.e.*

Proof. Since $W^{1,\vec{p}}(\Omega)$ is a reflexive space, $u_n \rightharpoonup u$ weakly in $W^{1,\vec{p}}(\Omega)$. Then, by the Banach-Saks theorem, there is a subsequence denoted again by (u_n) such that the sequence (g_n) defined by $g_n = \frac{1}{n}\sum\limits_{i=1}^{n} u_i$ converges to u strongly in $W^{1,\vec{p}}(\Omega)$.
By Theorem 3.3, there is a subsequence of (g_n) denoted again by (g_n) such that

$$\lim_{n\to+\infty} g_n = u \qquad C_{\vec{p}} - q.e.$$

On the other hand,

$$\liminf u_n \leq \lim_{n\to+\infty} g_n.$$

Therefore,

$$\liminf(u_n) \leq u \qquad C_{\vec{p}} - q.e.$$

For the second inequality, it suffices to replace u_n by $(-u_n)$ in the first inequality.

3.1 Anisotropic Sobolev spaces with zero boundary values

Definition 3.3 We say that a function u belongs to the anisotropic Sobolev space with zero boundary values, and we denote $u \in B_0^{1,\vec{p}}(\Omega)$ if there is a $C_{\vec{p}}$-quasicontinuous function $\tilde{u} \in W^{1,\vec{p}}(\mathbb{R}^N)$ such that $\tilde{u} = u$ almost everywhere in Ω and $\tilde{u} = 0$ $C_{\vec{p}}$ -quasi everywhere in $\mathbb{R}^N \backslash \Omega$. The set $B_0^{1,\vec{p}}(\Omega)$ is endowed with the norm

$$||u||_{B_0^{1,\vec{p}}(\Omega)} = ||\tilde{u}||_{W^{1,\vec{p}}(\mathbb{R}^N)}.$$

Theorem 3.5 *$B_0^{1,\vec{p}}(\Omega)$ is a Banach space.*

Proof. Let $(u_n)_n$ be a Cauchy sequence in $B_0^{1,\vec{p}}(\Omega)$, for every n, there is a $C_{\vec{p}}$-quasicontinuous function $\tilde{u_n} \in W^{1,\vec{p}}(\mathbb{R}^N)$ such that $\tilde{u}_n = u_n$ almost everywhere in Ω and $\tilde{u_n} = 0 C_{\vec{p}}$-quasi everywhere in $\mathbb{R}^N \backslash \Omega$.

Since $W^{1,\vec{p}}(\mathbb{R}^N)$ is a Banach space, there is a function u such that $\tilde{u_n} \to u$ in $W^{1,\vec{p}}(\mathbb{R}^N)$ as $n \to +\infty$. By applying Theorem 3.3, we deduce that u is $C_{\vec{p}}-$ quasi-continuous and by Proposition 3.3, we have $u = 0\ \ C_{\vec{p}}$-q.e in $\mathbb{R}^N \backslash \Omega$. Consequently, $u \in B_0^{1,\vec{p}}(\Omega)$ and we conclude that the spaces $B_0^{1,\vec{p}}(\Omega)$ are complete.

Corollary 3.2 *The space $B_0^{1,\vec{p}}(\Omega)$ is reflexive.*

Proof. The space $W^{1,\vec{p}}(\mathbb{R}^N)$ is a reflexive Banach space, by applying Theorem 3.5, we deduce the space $B_0^{1,\vec{p}}(\Omega)$ is closed in $W^{1,\vec{p}}(\mathbb{R}^N)$ and therefore $B_0^{1,\vec{p}}(\Omega)$ is reflexive.

Corollary 3.3 *We have $W_0^{1,\vec{p}}(\Omega) \subset B_0^{1,\vec{p}}(\Omega) \subset W^{1,\vec{p}}(\Omega)$.*

Proof. Since $D(\Omega) \subset B_0^{1,\vec{p}}(\Omega)$ and by applying Theorem 3.5, we obtain the first inclusion. The second inclusion follows directly from the definition of the space $B_0^{1,\vec{p}}(\Omega)$.

Proposition 3.4 *Let $u \in B_0^{1,\vec{p}}(\Omega)$ and $v \in W^{1,\vec{p}}(\mathbb{R}^N)$ be bounded functions. If v is $C_{\vec{p}}$- quasicontinuous, then $uv \in B_0^{1,\vec{p}}(\Omega)$.*

Proof. Let $\tilde{u} \in W^{1,\vec{p}}(\mathbb{R}^N)$ be a $C_{\vec{p}}$- quasicontinuous representative function of u. $\tilde{u}v$ is $C_{\vec{p}}$- quasicontinuous in $\mathbb{R}^N$. Let $D = \{x \in \mathbb{R}^N \backslash \Omega : \tilde{u}v \neq 0\}$, $D = G \cup H$, where $G = \{x \in \mathbb{R}^N \backslash \Omega : \tilde{u} \neq 0\}$ and $H = \{x \in \mathbb{R}^N \backslash \Omega : v = \infty\}$. It is obvious that $C_{\vec{p}}(G) = 0$ and by Proposition 3.2, we have $C_{\vec{p}}(H) = 0$, thus $C_{\vec{p}}(D) = 0$. Therefore, $\tilde{u}v = 0\ C_{\vec{p}}$ -quasi everywhere in Ω. Since $\tilde{u}v = uv$ a.e in Ω, we get $uv \in B_0^{1,\vec{p}}(\Omega)$.

Theorem 3.6 *Let $O \subset \Omega$ be such that $C_{\vec{p}}(O) = 0$, we have*

$$B_0^{1,\vec{p}}(\Omega) = B_0^{1,\vec{p}}(\Omega \backslash O).$$

Proof. It is obvious that $B_0^{1,\vec{p}}(\Omega \backslash O) \subset B_0^{1,\vec{p}}(\Omega)$.

Let $u \in B_0^{1,\vec{p}}(\Omega)$, then there is a $C_{\vec{p}}$- quasicontinuous function $\tilde{u} \in W^{1,\vec{p}}(\mathbb{R}^N)$ such that $\tilde{u} = u$ a.e in Ω and $\tilde{u} = 0\ C_{\vec{p}}$- quasi everywhere in $\mathbb{R}^N \backslash \Omega$. Since $C_{\vec{p}}(O) = 0$, we have $\tilde{u} = 0\ C_{\vec{p}}$ - quasi everywhere in $\mathbb{R}^N \backslash (\Omega \backslash O)$. Thus $u \in B_0^{1,\vec{p}}(\Omega \backslash O)$.

Remark 3.1 *If $C_{\vec{p}}(\partial\Omega) = 0$, then $B_0^{1,\vec{p}}(\mathring{\Omega}) = B_0^{1,\vec{p}}(\overline{\Omega})$.*

4 Application

4.1 The Dirichlet energy integral minimisers

Definition 4.1 Let $w \in W^{1,\vec{p}}(\Omega)$. For all $u \in B_0^{1,\vec{p}}(\Omega)$, we define $I_{\Omega,w}^{\vec{p}}(u)$ by

$$I_{\Omega,w}^{\vec{p}}(u) = \int_\Omega \mid w \mid dx + \sum_{i=0}^{N} \int_\Omega \left(\mid \frac{\partial u}{\partial x_i} \mid^{p_i} + \mid \frac{\partial w}{\partial x_i} \mid^{p_i} \right) dx.$$

$I_{\Omega,w}^{\vec{p}}$ is called the energy operator corresponding to the boundary value function w.

Lemma 4.1 *[17] Let H be a reflexive Banach space. If $I : H \to \mathbb{R}$ is a convex, lower semi-continuous and coercive operator, then there is an element in H that minimizes I.*

Theorem 4.1 *Let $B_0^{1,\vec{p}}(\Omega)$ be the anisotropic Sobolev space with zero boundary values. Then there exists a function $u \in B_0^{1,\vec{p}}(\Omega)$ such that*

$$I^{\vec{p}}_{\Omega,w}(u) = \inf_{v \in B_0^{1,\vec{p}}(\Omega)} I^{\vec{p}}_{\Omega,w}(v).$$

Proof. It follows from Theorem 3.5 and Corollary 3.2 that $B_0^{1,\vec{p}}(\Omega)$ is a reflexive Banach space. Since the function $x \to x^p$ is convex for every fixed $1 < p < \infty$, we deduce that $I^{\vec{p}}_{\Omega,w}$ is convex. Moreover, $I^{\vec{p}}_{\Omega,w}$ is lower semi-continuous and coercive, hence all assumptions of Lemma 4.1 are satisfied.

4.2 Conclusion

In this work, we first show that the anisotropic Sobolev $\vec{p}$-capacity $C_{\vec{p}}$ is an outer capacity and we give sufficient conditions ensuring that $C_{\vec{p}}(E) = 0$ whenever E is a subset of Ω. Then, we discuss the convergence of a sequence in $C_{\vec{p}}$-capacity. This allows us to show that the anisotropic Sobolev space with zero boundary values $B_0^{1,\vec{p}}(\Omega)$ is a reflexive Banach space. We also prove that $B_0^{1,\vec{p}}(\Omega)$ coincides with $B_0^{1,\vec{p}}(\Omega \backslash E)$ for all $E \subset \Omega$ satisfying $C_{\vec{p}}(E) = 0$. Finaly, we apply our results to show that the Dirichlet energy has a minimizer in anisotropic Sobolev spaces with zero boundary values.

References

[1] R. Adams. *Sobolev Space.* New York, Academic Press, 1975.

[2] N. Aissaoui, Y. Akdim, A. Benkirane and M. C. Hassib. Capacity and Non-linear Potential in Musielak-Orlicz Spaces. *Nonlinear Dynamics and Systems Theory* **16** (3) (2016) 276–293.

[3] N. Aissaoui, Y. Akdim, A. Benkirane and M. C. Hassib. Capacity, Theorem of H. Brezis and F.E. Browder type in Musielak-Orlicz-Sobolev Spaces and Application. *Nonlinear Dynamics and Systems Theory* **17** (2) (2017) 175–192.

[4] N. Aissaoui and A. Benkirane. Capacités dans les epaces d'Orlicz. *Ann. Sci. Math. Quèbec.* **18** (1) (1994) 1–23.

[5] Y. Akdim, R. Elharch, M. C Hassib and S. Lalaoui Rhali. Capacity in anisotropic Sobolev spaces. *Nonlinear Dynamics and Systems Theory* **21** (2) (2021) 115–125.

[6] L. Boccardo, T.Gallouët and L. Orsina. Existence and non existence of solutions for some nonlinear elliptic equations. *Journal d'Analyse Mathématique* **73** (1) (1997) 203–223.

[7] H. Brezis. *Analyse Fonctionnelle; Théorie et Applications.* Masson, Paris, 1983.

[8] L. C. Evans and R. F. Gariepy. *Measure Theory and Fine Properties of Functions CRC Press.* Boca Raton, 1992.

[9] P. Harjulehto, P. Hästö, M. Koskenoja and S. Varonen. Sobolev capacity on the space $W^{1,p(.)}(\mathbb{R}^N)$. *Journal of Function Spaces and Applications* **1** (1) (2003) 17–33.

[10] P. Harjulehto, P. Hästö, M. Koskenoja, S. Varonen. The Dirichlet energy integral and variable exponent Sobolev spaces with zero boundary values. *Potential Analysis* **25** (3) (2006) 205–222.

[11] M. C. Hassib, Y. Akdim. Weighted Variable Exponent Sobolev spaces on metric measure spaces. *Moroccan Journal of Pure and Applied Analysis* **4** (2) (2018) 62–76.

[12] J. Heinonen, T. Kilpeläinen and O. Martio. *Nonlinear Potential Theory of Degenerate Elliptic Equations.* New York: Oxford University Press, 1993.

[13] V. P. Khavin and V. G. Maz'ya. Nonlinear potential theory. *Uspekhi Math. Nauk* **27** (1972) 71–148.

[14] T. Kilpeläinen. Weighted Sobolev spaces and capacity. *Annales Academi Scientiarium Fennic. Mathematica* **19** (1) (1994) 95–113.

[15] T. Kilpeläinen. A remark on the uniqueness of quasi continuous functions. *Ann.Acad. Sci. Fenn. Math.* **23** (1998) 261–262.

[16] T. Kilpeläinen, J. Kinnunen, and O. Martio. *Sobolev spaces with zero boundary values on metric spaces. Potential Analysis.* **12** (3) (2000) 233–247.

[17] D. Kinderlehrer and G. Stampacchia. *An Introduction to Variational Inequalities and Their Applications.* Academic Press, London, 1980.

[18] J. Kinnunen. *Sobolev spaces. Department of Mathematics and Systems Analysis.* Aalto University, 2017.

[19] V. G. Maz'ya. *Sobolev spaces.* Translated from the Russian by TO Shaposhnikova. Springer Ser. Soviet Math., Springer, 1985.

[20] N. G. Meyers. A theory of capacities for potentials of functions in Lebesgue classes. *Math., Scand.* **26** (2) (1970) 255–292.

[21] S. M. Nikolski. On imbedding, continuation and approximation theorems for differentialble functions of several variables. *Russian Math. Surveys* **16** (5) (1961) 55–104.

[22] T. Ohno and T. Shimomura. Musielak-Orlicz-Sobolev spaces with zero boundary values on metric measure spaces. *Czechoslovak Mathematical Journal* **66** (2) (2016) 371–394.

[23] J. Rakosnik. Some remarks to anisotropic Sobolev spaces I. *Beiträge zur Analysis.* **13** (1979) 55–68.

[24] J. Rakosnik. Some remarks to anisotropic Sobolev spaces II. *Beiträge zur Analysis.* **15** (1981) 127–140.

[25] M. Troisi. Teoremi di inclusione per spazi di Sobolev non isotropi. *Ricerche Mat.* **18** (1969) 3–24.