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First Integral of a Class of Two Dimensional Kolmogorov Systems

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Abstract: In this paper, we are interested in studying the existence of a first integral and the curves which are formed by the trajectories of the autonomous planar Kolmogorov systems. Concrete examples exhibiting the applicability of our result are introduced.

Keywords: *dynamical system; Kolmogorov system; first integral; periodic orbits; limit cycle.*

Mathematics Subject Classification (2010): 34C05, 34C07, 37C27, 37K10.

1 Introduction

By definition, an autonomous planar Kolmogorov system is a system of the form

$$\begin{cases} x' = \dfrac{dx}{dt} = xF\left(x,y\right), \\ y' = \dfrac{dy}{dt} = yG\left(x,y\right), \end{cases} \tag{1}$$

these equations are equivalent to the differential equation

$$\frac{dy}{dx} = \frac{yQ\left(x,y\right)}{xP\left(x,y\right)},$$

where F , G are two functions in the variables x and y and the derivatives are taken with respect to the time variable. The theory of differential equations is one of the basic tools of mathematical science [1–3, 20]. System (1) is frequently used to model the iteration of two species occupying the same ecological niche [14, 16]. There are many

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natural phenomena which can be modeled by the Kolmogorov systems, for example, in mathematical ecology and population dynamics [11,15,17,18], chemical reactions, plasma physics [13], hydrodynamics [7], etc. We remind that in the phase plane, a limit cycle of system (1) is an isolated periodic orbit in the set of all its periodic orbits. In the qualitative theory of planar dynamical systems [9, 19], one of the most important topics is related to the second part of the unsolved Hilbert 16th problem [12]. There is a huge literature about limit cycles, most of it deal essentially with their detection, their number and their stability and rare are papers concerned with giving them explicitly [4, 5].

System (1) is integrable on an open set Ω of $\mathbb{R}^2$ if there exists a non constant C^1 function $H : \Omega \to \mathbb{R}$, called a first integral of the system on Ω, which is constant on the trajectories of the system (1) contained in Ω, i.e., if

$$\frac{dH(x,y)}{dt} = \frac{\partial H(x,y)}{\partial x} xF(x,y) + \frac{\partial H(x,y)}{\partial y} yG(x,y) \equiv 0 \text{ in the points of } \Omega.$$

Moreover, $H = h$ is the general solution of this equation, where h is an arbitrary constant. It is well known that for differential systems defined on the plane $\mathbb{R}^2$, the existence of a first integral determines their phase portrait [8], and one of the classical tools in the classification of all trajectories of a dynamical system is to find first integrals, for more details about the first integral, see for instance [6, 10].

In this paper, we are interested in studying the existence of a first integral and the curves which are formed by the trajectories of the autonomous planar Kolmogorov systems of the form

$$\begin{cases} x' = x\left(B_1(x,y)\sin\left(\frac{A_3(x,y)}{A_4(x,y)}\right) + B_3(x,y)\sin\left(\frac{A_1(x,y)}{A_2(x,y)}\right)\right), \\ y' = y\left(B_2(x,y)\sin\left(\frac{A_5(x,y)}{A_6(x,y)}\right) + B_3(x,y)\sin\left(\frac{A_1(x,y)}{A_2(x,y)}\right)\right), \end{cases} \tag{2}$$

where $A_1(x,y)$, $A_2(x,y)$, $A_3(x,y)$, $A_4(x,y)$, $A_5(x,y)$, $A_6(x,y)$, $B_1(x,y)$, $B_2(x,y)$ and $B_3(x,y)$ are homogeneous polynomials of degree a, a, b, b, c, c, n, n, m, respectively.

We define the trigonometric functions

$$\begin{aligned}
&f_1(\theta) = B_1(\cos\theta, \sin\theta)\left(\cos^2\theta\right)\sin\left(\frac{A_3(\cos\theta,\sin\theta)}{A_4(\cos\theta,\sin\theta)}\right) + \\
&B_2(\cos\theta,\sin\theta)\left(\sin^2\theta\right)\sin\left(\frac{A_5(\cos\theta,\sin\theta)}{A_6(\cos\theta,\sin\theta)}\right), \\
&f_2(\theta) = B_3(\cos\theta,\sin\theta)\sin\left(\frac{A_1(\cos\theta,\sin\theta)}{A_2(\cos\theta,\sin\theta)}\right), \\
&f_3(\theta) = (\cos\theta\sin\theta)\, B_2(\cos\theta,\sin\theta)\sin\left(\frac{A_5(\cos\theta,\sin\theta)}{A_6(\cos\theta,\sin\theta)}\right) - \\
&(\cos\theta\sin\theta)\, B_1(\cos\theta,\sin\theta)\sin\left(\frac{A_3(\cos\theta,\sin\theta)}{A_4(\cos\theta,\sin\theta)}\right).
\end{aligned}$$

2 Main Result

Our main result on the integrability and the periodic orbits of the Kolmogorov system (2) is as follows.

Theorem 2.1 *Consider the Komogorov system* (2), *then the following statements hold.*

(1) *If* $f_3(\theta) \neq 0$, $A_i(\cos\theta, \sin\theta) \neq 0$ *for* $i = 2, 4, 6$ *and* $n \neq m$, *then system* (2) *has the first integral*

$$H(x, y) = \left(x^2 + y^2\right)^{\frac{n-m}{2}} \exp\left((m-n)\int_0^{\arctan\frac{y}{x}} M(s)\, ds\right) - (n-m) F\left(\arctan\tfrac{y}{x}\right),$$

where $M(\theta) = \frac{f_1(\theta)}{f_3(\theta)}$, $N(\theta) = \frac{f_2(\theta)}{f_3(\theta)}$ *and* $F(\theta) = \int_0^\theta \exp\left((m-n)\int_0^w M(s)\, ds\right) N(w)\, dw$.

The curves which are formed by the trajectories of the differential system (2), *in Cartesian coordinates are written as*

$$x^2 + y^2 = \left[\left(h + (n-m) F\left(\arctan\frac{y}{x}\right)\right) \exp\left((n-m)\int_0^{\arctan\frac{y}{x}} M(s)\, ds\right)\right]^{\frac{2}{n-m}},$$

where $h \in \mathbb{R}$. *Moreover, the system* (2) *has no periodic orbits.*

(2) *If* $f_3(\theta) \neq 0$, $A_i(\cos\theta, \sin\theta) \neq 0$ *for* $i = 2, 4, 6$ *and* $n = m$, *then system* (2) *has the first integral*

$$H(x, y) = \left(x^2 + y^2\right)^{\frac{1}{2}} \exp\left(-\int_0^{\arctan\frac{y}{x}} (M(s) + N(s))\, ds\right),$$

and the curves which are formed by the trajectories of the differential system (2), *in Cartesian coordinates are written as*

$$\left(x^2 + y^2\right)^{\frac{1}{2}} - h \exp\left(\int_0^{\arctan\frac{y}{x}} (M(s) + N(s))\, ds\right) = 0,$$

where $h \in \mathbb{R}$. *Moreover, the system* (2) *has no periodic orbits.*

(3) *If* $f_3(\theta) = 0$ *for all* $\theta \in \mathbb{R}$, *then system* (2) *has the first integral* $H = \frac{y}{x}$, *and the curves which are formed by the trajectories of the differential system* (2), *in Cartesian coordinates are written as* $y - hx = 0$, *where* $h \in \mathbb{R}$. *Moreover, the system* (2) *has no periodic orbits.*

Proof. In order to prove our results, we write the differential system (2) in polar coordinates (r, θ), defined by $x = r\cos\theta$ and $y = r\sin\theta$, then system (2) becomes

$$\begin{cases} r' = f_1(\theta) r^{n+1} + f_2(\theta) r^{m+1}, \\ \theta' = f_3(\theta) r^n, \end{cases} \tag{3}$$

where the trigonometric functions $f_1(\theta)$, $f_2(\theta)$, $f_3(\theta)$ are given in the Introduction, $r' = \frac{dr}{dt}$ and $\theta' = \frac{d\theta}{dt}$.

If $f_3(\theta) \neq 0$, $A_i(\cos\theta, \sin\theta) \neq 0$ for $i = 2, 4, 6$ and $n \neq m$, we take as a new independent variable the coordinate θ, then the differential system (3) becomes the differential equation

$$\frac{dr}{d\theta} = M(\theta) r + N(\theta) r^{1+m-n}, \tag{4}$$

where $M(\theta) = \frac{f_1(\theta)}{f_3(\theta)}$ and $N(\theta) = \frac{f_2(\theta)}{f_3(\theta)}$, which is a Bernoulli equation. By introducing the standard change of variables $\rho = r^{n-m}$, we obtain the linear equation

$$\frac{d\rho}{d\theta} = (n-m)(M(\theta)\rho + N(\theta)). \tag{5}$$

The general solution of linear equation (5) is

$$\rho(\theta) = \exp\left((n-m)\int_0^\theta M(s)\,ds\right)(\mu + (n-m)F(\theta)),$$

where $\mu \in \mathbb{R}$.

From the expression of the constant μ, we deduce the first integral of system (2) as

$$H(x,y) = \left(x^2+y^2\right)^{\frac{n-m}{2}} \exp\left((m-n)\int_0^{\arctan\frac{y}{x}} M(s)\,ds\right) + (m-n)F\left(\arctan\tfrac{y}{x}\right).$$

Let Γ be a periodic orbit surrounding an equilibrium located in one of the open quadrants, and let $h_\Gamma = H(\Gamma)$.

The curves $H = h$ with $h \in \mathbb{R}$, which are formed by the trajectories of the differential system (2), in Cartesian coordinates are written as

$$x^2+y^2 = \left[\left(h + (n-m)F\left(\arctan\frac{y}{x}\right)\right)\exp\left((n-m)\int_0^{\arctan\frac{y}{x}} M(s)\,ds\right)\right]^{\frac{2}{n-m}},$$

where $h \in \mathbb{R}$.

Therefore the periodic orbit Γ is contained in the curve

$$x^2+y^2 = \left[\left(h_\Gamma + (n-m)F\left(\arctan\frac{y}{x}\right)\right)\exp\left((n-m)\int_0^{\arctan\frac{y}{x}} M(s)\,ds\right)\right]^{\frac{2}{n-m}}.$$

But this curve cannot contain the periodic orbit Γ in the realistic quadrant ($x > 0, y > 0$), because this curve in the realistic quadrant has at most a unique point on every straight line $y = \eta x$ for all $\eta \in]0,+\infty[$.

To be convinced by this fact, one has to compute the abscissa points of the intersection of this curve with the straight line $y = \eta x$ for all $\eta \in]0,+\infty[$, the abscissa is given by

$$\begin{aligned} x &= \frac{1}{\sqrt{1+\eta^2}}\left[(h_\Gamma + (n-m)F(\arctan\eta))\exp\left((n-m)\int_0^{\arctan\eta} M(s)\,ds\right)\right]^{\frac{1}{n-m}} \\ &= f(\eta). \end{aligned}$$

Since f is a function (of η), there exists at most one value of x on the half-line OX^+. Consequently, at most one point in the realistic quadrant ($x > 0, y > 0$) exists. So, this curve cannot contain the periodic orbit.

Hence statement (1) of Theorem 1 is proved.

Suppose now that $f_3(\theta) \neq 0$, $A_i(\cos\theta, \sin\theta) \neq 0$ for $i = 2,4,6$ and $n = m$.

Taking as the independent variable the coordinate θ, this differential system (3) is written as

$$\frac{dr}{d\theta} = (M(\theta) + N(\theta))\,r. \tag{6}$$

The general solution of equation (6) is

$$r(\theta) = \mu\exp\left(\int_0^\theta (M(s) + N(s))\,ds\right),$$

where $\mu \in \mathbb{R}$.

From the expression of the constant μ, we deduce the first integral of system (2) as

$$H(x,y) = \left(x^2+y^2\right)^{\frac{1}{2}} \exp\left(-\int_0^{\arctan \frac{y}{x}} (M(s)+N(s))\,ds\right).$$

Let Γ be a periodic orbit surrounding an equilibrium located in one of the realistic quadrants $(x>0, y>0)$, and let $h_\Gamma = H(\Gamma)$.

The curves $H=h$ with $h \in \mathbb{R}$, which are formed by the trajectories of the differential system (2), in Cartesian coordinates are written as

$$\left(x^2+y^2\right)^{\frac{1}{2}} - h\exp\left(\int_0^{\arctan \frac{y}{x}} (M(s)+N(s))\,ds\right) = 0,$$

where $h \in \mathbb{R}$.

Therefore the periodic orbit Γ is contained in the curve

$$\left(x^2+y^2\right)^{\frac{1}{2}} = h_\Gamma \exp\left(\int_0^{\arctan \frac{y}{x}} (M(s)+N(s))\,ds\right).$$

Again, this curve cannot contain the periodic orbit Γ in the realistic quadrant $(x > 0, y > 0)$, for the same reason as in the previous case.

To be convinced by this fact, one has to compute the abscissa points of the intersection of this curve with the straight line $y = \eta x$ for all $\eta \in \left]0,+\infty\right[$, the abscissa is given by

$$x = \frac{h_\Gamma}{\sqrt{(1+\eta^2)}} \exp\left(\int_0^{\arctan \eta} (M(s)+N(s))\,ds\right) = f(\eta).$$

Since f is a function (of η), there exists at most one value of x on the half-line OX^+. Consequently, at most one point in the realistic quadrant $(x > 0, y > 0)$ exists. So, this curve cannot contain the periodic orbit.

Hence statement (2) of Theorem 1 is proved.

Assume now that $f_3(\theta) = 0$ for all $\theta \in \mathbb{R}$. Then from system (3) it follows that $\theta' = 0$. So, the straight lines through the origin of coordinates of the differential system (2) are invariant by the flow of this system. Hence, $\frac{y}{x}$ is a first integral of the system, then curves which are formed by the trajectories of the differential system (2), in Cartesian coordinates are written as $y - hx = 0$, where $h \in \mathbb{R}$, since all straight lines through the origin are formed by the trajectories, clearly, the system has no periodic orbits.

This completes the proof of statement (3) of Theorem 1.

2.1 Examples

The following examples are given to illustrate our result.

Example 1 If we take $A_1(x,y) = 5x^2+4y^2$, $A_2(x,y) = x^2+y^2$, $A_3(x,y) = \frac{\pi}{2}A_4(x,y)$, $A_5(x,y) = \frac{\pi}{2}A_6(x,y)$, $B_1(x,y) = x^4+x^3y+2x^2y^2+xy^3+y^4$, $B_2(x,y) = x^4+2x^3y+2x^2y^2+2xy^3+y^4$ and $B_3(x,y) = 3x^2-xy+3y^2$, then system (2) reads

$$\begin{cases} x' = x\left(\left(x^4 + x^3y + 2x^2y^2 + xy^3 + y^4\right) + \left(3x^2 - xy + 3y^2\right)\sin\left(\frac{5x^2+4y^2}{x^2+y^2}\right)\right), \\ y' = y\left(\left(x^4 + 2x^3y + 2x^2y^2 + 2xy^3 + y^4\right) + \left(3x^2 - xy + 3y^2\right)\sin\left(\frac{5x^2+4y^2}{x^2+y^2}\right)\right). \end{cases} \quad (7)$$

The Kolmogorov system (7) in polar coordinates (r, θ) becomes

$$\begin{cases} r' = \left(1 + \frac{3}{4}\sin 2\theta - \frac{1}{8}\sin 4\theta\right) r^5 + (3 - \cos\theta\sin\theta)\sin\left(\frac{9}{2} + \frac{1}{2}\cos 2\theta\right) r^3, \\ \theta' = \left(\cos^2\theta\sin^2\theta\right) r^4, \end{cases}$$

here $f_1(\theta) = 1+\frac{3}{4}\sin 2\theta - \frac{1}{8}\sin 4\theta$, $f_2(\theta) = (3 - \cos\theta\sin\theta)\sin\left(\frac{9}{2} + \frac{1}{2}\cos 2\theta\right)$ and $f_3(\theta) = \cos^2\theta\sin^2\theta$. In the realistic quadrant $(x > 0, y > 0)$ it is the case (1) of Theorem 1, then the Kolmogorov system (7) has the first integral

$$H(x,y) = \left(x^2 + y^2\right)\exp\left(-2\int_0^{\arctan\frac{y}{x}} M(s)\,ds\right) - 2\int_0^{\arctan\frac{y}{x}} \exp\left(-2\int_0^{w} M(s)\,ds\right) B(w)\,dw,$$

where $M(s) = \dfrac{1 + \frac{3}{4}\sin 2s - \frac{1}{8}\sin 4s}{\cos^2 s\sin^2 s}$, $N(w) = \dfrac{(3 - \cos w\sin w)\sin\left(\frac{9}{2} + \frac{1}{2}\cos 2w\right)}{\cos^2 w\sin^2 w}$.

The curves $H = h$ with $h \in \mathbb{R}$, which are formed by the trajectories of the differential system (7), in Cartesian coordinates are written as

$$x^2+y^2 = \left(h + 2\int_0^{\arctan\frac{y}{x}} \exp\left(-2\int_0^{w} N(s)\,ds\right) N(w)\,dw\right)\exp\left(2\int_0^{\arctan\frac{y}{x}} M(s)\,ds\right),$$

where $h \in \mathbb{R}$. Moreover, the system (7) has no periodic orbits.

Example 2 If we take $A_1(x,y) = \pi x^2 + \pi y^2$, $A_2(x,y) = 2x^2 + 2y^2$, $A_3(x,y) = A_5(x,y) = y$, $A_4(x,y) = A_6(x,y) = x$, $B_1(x,y) = -x^2+xy-y^2$, $B_2(x,y) = x^2+xy+y^2$ and $B_3(x,y) = x^2 + y^2$, then system (2) reads

$$\begin{cases} x' = x\left(\left(-x^2 + xy - y^2\right)\sin\left(\frac{y}{x}\right) + \left(x^2 + y^2\right)\sin\left(\frac{\pi x^2+\pi y^2}{2x^2+2y^2}\right)\right), \\ y' = y\left(\left(x^2 + xy + y^2\right)\sin\left(\frac{y}{x}\right) + \left(x^2 + y^2\right)\sin\left(\frac{\pi x^2+\pi y^2}{2x^2+2y^2}\right)\right). \end{cases} \quad (8)$$

The Kolmogorov system (8) in polar coordinates (r, θ) becomes

$$\begin{cases} r' = \left(1 + \left(\frac{1}{2}\sin 2\theta - \cos 2\theta\right)\sin(\tan\theta)\right) r^3, \\ \theta' = (\sin 2\theta)\sin(\tan\theta)\, r^2. \end{cases}$$

In the realistic quadrant $(x > 0, y > 0)$ it is the case (2) of Theorem 1, then the Kolmogorov system (8) has the first integral

$$H(x,y) = \left(x^2 + y^2\right)^{\frac{1}{2}}\exp\left(-\int_0^{\arctan\frac{y}{x}}\left(\frac{1 + \left(\frac{1}{2}\sin 2s - \cos 2s\right)\sin(\tan s)}{(\sin 2s)\sin(\tan s)}\right)ds\right).$$

The curves $H = h$ with $h \in \mathbb{R}$, which are formed by the trajectories of the differential system (8), in Cartesian coordinates are written as

$$\left(x^2 + y^2\right)^{\frac{1}{2}} - h \exp\left(\int_0^{\arctan \frac{y}{x}} \left(\frac{1 + \left(\frac{1}{2}\sin 2s - \cos 2s\right)\sin(\tan s)}{(\sin 2s)\sin(\tan s)}\right) ds\right) = 0,$$

where $h \in \mathbb{R}$. Moreover, the system (8) has no periodic orbits**.**

3 Conclusion

The elementary method used in this paper seems to be fruitful to investigate more general planar differential systems of ODEs in order to obtain an explicit expression for a first integral which characterizes its trajectories. This is one of the classical tools in the classification of all trajectories of dynamical systems.

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References

[1] S. A. Alharbi, A. S. Rambely and A. O. Almatroud. Dynamic modelling the immune system and its functions by vitamins intervention. *Nonlinear Dynamics and Systems Theory* **19** (2) (2019) 263–273.

[2] M. De Angelis. Mathematical contributions to the dynamics of the josephson functions: State of the art and open problems. *Nonlinear Dynamics and Systems Theory* **15** (3) (2015) 231–241.

[3] C. H. D. Alliera. Existence of solutions for a biological model using topological degree theory. *Nonlinear Dynamics and Systems Theory* **19** (1) (2019) 1–9.

[4] A. Bendjeddou and R. Boukoucha. Explicit limit cycles of a cubic polynomial differential systems. *Stud. Univ. Babes-Bolyai Math.* **61** (1) (2016) 77–85.

[5] R. Boukoucha. Explicit limit cycles of a family of polynomial differential systems. *Electronic Journal of Differential Equations.* **2017** (217) (2017) 1–7.

[6] R. Boukoucha and A. Bendjeddou. On the dynamics of a class of rational Kolmogorov systems. *Journal of Nonlinear Mathematical Physics.* **23** (1) (2016) 21–27.

[7] F. H. Busse. *Transition to Turbulence Via the Statistical Limit Cycle Route.* Synergetics, Springer-Verlag, berlin, 1978.

[8] L. Cairó and J. Llibre. Phase portraits of cubic polynomial vector fields of Lotka–Volterra type having a rational first integral of degree 2. *J. Phys.* A **40** (2007) 6329–6348.

[9] F. Dumortier, J. Llibre and J. Artés. Qualitative Theory of Planar Differential Systems. Berlin, Springer, 2006.

[10] P. Gao. Hamiltonian structure and first integrals for the Lotka-Volterra systems. *Phys. Lett. A* **273** (2000) 85–96.

[11] X. Huang. Limit cycles in a Kolmogorov-type model. *Internat. J. Math. Math. Sci.* **13** (3) (1990) 555–566.

[12] D. Hilbert. Mathematatische probleme. *Lecture, Second Internat. Congr. Math.* (Paris, 1900), *Nachr. Ges. Wiss. Gttingen Math. Phys. Kl.* (1900) 253-297*, English transl, Bull. Amer. Math. Soc.* **8** (1902), 437-479.

[13] G. Lavel and R. Pellat. Plasma physics. In: *Proceedings of Summer School of Theoreal Physics*, Gordon and Breach. New York, 1975.

[14] C. Li and J. Llibre. The cyclicity of period annulus of a quadratic reversible Lotka–Volterra system. *Nonlinearity.* **22** (2009) 2971–2979.

[15] J. Llibre and T. Salhi. On the dynamics of class of Kolmogorov systems. *J. Appl. Math. Comput.* **225** (2013) 242–245.

[16] J. Llibre and C. Valls. Polynomial, rational and analytic first integrals for a family of 3-dimensional Lotka-Volterra systems. *Z. Angew. Math. Phys.* **62** (2011) 761–777.

[17] N. G. Llyod, J. M. Pearson, E. Saez and I. Szanto. Limit cycles of a cubic Kolmogorov system. *Appl. Math. Lett.* **9** (1) (1996) 15–18.

[18] R.M. May. *Stability and Complexity in Model Ecosystems*. Princeton, New Jersey, 1974.

[19] L. Perko. *Differential Equations and Dynamical Systems*. Third edition. Texts in Applied Mathematics. 7. Springer-Verlag, New York, 2001.

[20] R. V. Ruzich. Mathematical analysis in a model of primary succession. *Nonlinear Dynamics and Systems Theory* **15** (4) (2015) 418–427.