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Comprehensive Description of Solutions to Semilinear Sectorial Equations: an Overview

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Abstract: Description of all possible types of behavior, or evolution, of solutions to a semilinear sectorial equation is given. The phase space is divided into separate regions containing bounded solutions, grow-up solutions and those which blow up in a finite time. An overview of results concerning the typical situation when solutions of various types of behavior coexist is given and illustrated by chosen examples of reaction-diffusion equations.

Keywords: *parabolic equation; sectorial equation; Cauchy problem; global solutions; grow-up solutions; blow-up solutions; comprehensive description.*

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1 Introduction

This paper is devoted to the fundamental question connected with solutions of semilinear sectorial equations (1) being generalizations of parabolic equations: *Provided that a local in time solution exists, what is the expected future for the rest of its existence?*

It is known from the classical references, such as [20, Chapter I], that, in general, there are three potential forms of the further evolution of such solutions:
– the local solution may *blow up*, which means that its phase space norm becomes unbounded in a finite time; in general, it can be a consequence of unboundedness of the values of the solution or the values of some of its derivatives, even though the solution itself may stay bounded in the $L^\infty-$norm,
– the local solution may *grow up*, that is, it will exist for all positive times, while some of its norms will become unbounded as $t \to \infty$,

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– the local solution will be extended globally in time with the required norms being bounded for all $t \geq 0$. This is a particularly interesting form of behavior, including the possibility that the equation generates a dynamical system possessing a *global attractor*.

Throughout the last 70 years, plenty of results appeared in the literature concerning qualitative behavior of solutions and describing separately the blow-up phenomenon, less known case of grow-up solutions and, finally, well studied globally bounded in time solutions, the latter including the particular case of semigroups with global attractors. From the abundance of references, we distinguish [20] and [15] for the local solvability of parabolic and sectorial equations, [24] for the issues regarding the blow-up, and [4, 12, 19, 31] for the existence of the global attractors for dissipative semigroups.

Usually, the authors study the above types of behavior as if these types would exist apart. The reason is perhaps connected with the fact that even the description of one kind is complicated enough. However, the situation we face in practice is the coexistence of all these three types of behavior for a single evolution equation.

Our aim in this paper is thus to describe such a general typical situation for the Cauchy problem for the semilinear sectorial equation

$$u_t + Au = F(u),\ t > 0, \quad u(0) = u_0, \tag{1}$$

where A is a sectorial positive operator and F stands for the nonlinear term. It is well-known that many ordinary and partial differential equations or systems from the Applied Sciences can be investigated within the approach of (1). This includes the heat propagation equation, reaction-diffusion systems, Fitzhugh-Nagumo equation, pattern formation models like the Cahn-Hilliard equation or viscous Cahn-Hilliard equation, models of fluid flows like the Burgers equation or the celebrated Navier-Stokes system and many others (see e.g. [4, 31, 33]). Wherever possible, we illustrate the discussed type of behavior of solutions using particular examples, mostly of ordinary differential equations or parabolic second order equations, that allow a more detailed description. Of course, the questions studied in this paper are much more involved for real world systems. Nevertheless, our paper may serve as a guide for the future application to the above mentioned problems.

The contents of this paper are as follows. In Section 2, we formulate the basic Assumption 2.1 on A and F in (1) and recall in Corollary 2.1, following [4, 15], the local existence of X^α solutions of (1) under this assumption (X^α stands for the phase space). In Definition 2.2, we introduce the partition of X^α according to the above-mentioned three types of behavior, introducing the subsets X^α_D, X^α_G and X^α_B. Moreover, we briefly describe consequences of their coexistence and mention some previous results from works where asymptotics of equations with solutions of different behavior was investigated.

In Section 3, we present a simple introductory example of a scalar reaction-diffusion problem (7), (8) exhibiting the coexistence of all three ways solutions may evolve.

In Section 4, we show in Theorem 4.1 that the life time of an X^α solution of (1) is a lower semicontinuous function of the initial data u_0. As Example 4.1 shows, in general, this function is not continuous, which makes it hard to characterize the components of the partition of X^α from Definition 2.2. Nevertheless, the *subordination condition* (18) together with an appropriate a priori estimate (17) allows to estimate the life time from below (see Theorem 4.2).

In Section 5, we present a range of examples of parabolic equations which possess, among others, solutions which grow up. The first example (20) shows that a linear reaction term leads to the existence of grow-up solutions. However, for the Neumann problem of the form (19), this observation can be generalized to nonlinearities with the divergent integral (21). Of course, this property still holds if we perturb the linear reaction term by a bounded nonlinearity. In this case, except for the grow-up solutions, all other solutions are globally bounded. As the example of (19), (25) exhibits, not only sub-linear nonlinearities lead to grow-up solutions. Furthermore, as seen in problems (26) and (27), reaction-diffusion equations with gradient-dependent nonlinearities may also possess grow-up solutions. In certain cases, the asymptotics of equations with grow-up solutions can be described in terms of non-compact attractors (see [22, 23]).

In Section 6, we briefly explain the reasons of appearance of blow-up solutions for parabolic equations and provide further examples of equations with solutions which become unbounded in finite time.

If the problem under consideration manifests at least two different kinds of behavior of solutions, there cannot exist a global attractor in the whole phase space in the sense of Definition 2.4. Nevertheless, there may be determined *local attractors*, like stable stationary solutions, and their *basins of attraction* can be considered. In Section 7, we discuss these notions and relate them with the existence of a Lyapunov function. In particular, a Lyapunov function on X^α_D for the problem (1) with A having compact resolvent guarantees that solutions which stay bounded must approach the set of equilibria, although the other solutions may become unbounded in a finite or infinite time (see Corollary 7.1).

For completeness of the presentation, we gather in the Appendix results concerning the existence of sufficiently regular solutions and their global extendibility in time for the homogeneous Neumann boundary problem for a reaction-diffusion equation with a gradient-dependent nonlinearity.

2 Setting of the Problem

Our purpose is to examine the behavior of solutions of evolution equations, which can be treated as autonomous abstract parabolic equations. To this end, consider an abstract Cauchy problem (1) under the following assumptions.

Assumption 2.1 *(i)* $-A\colon X \supset D(A) \to X$ *generates a strongly continuous analytic linear semigroup* $\{e^{-At}\colon t \geq 0\}$ *in a Banach space* X *and* $\operatorname{Re}\sigma(A) > 0$*,*
(ii) $F\colon X^\alpha \to X$ *is Lipschitz continuous on the bounded subsets of* $X^\alpha = D(A^\alpha)$ *for some* $\alpha \in [0,1)$.

Remark 2.1 Note that the generation of a strongly continuous analytic semigroup by $-A$ is equivalent to the sectoriality of the operator A (see e.g. [4, 15]). If A is merely sectorial, the condition $\operatorname{Re}\sigma(A) > 0$ of positivity of its spectrum can always be achieved by adding a term cu to both sides of the differential equation in (1) with a sufficiently large constant c. Then we define fractional power spaces X^β, $\beta \in \mathbb{R}$, connected with the domains of the operators A^β (see also [4, 15]) and the semigroup $\{e^{-At}\colon t \geq 0\}$ satisfies

$$\left\|e^{-At}x\right\|_X \leq C_0 e^{-at}\left\|x\right\|_X,\ t \geq 0,\ \left\|e^{-At}x\right\|_{X^\beta} \leq C_\beta t^{-\beta} e^{-at}\left\|x\right\|_X,\ t > 0,\ x \in X, \quad (2)$$

for any $\beta > 0$ with some $a > 0$ and $C_0, C_\beta \geq 1$.

Following the formalism of Dan Henry, we introduce a *local* X^α *solution* of (1).

Definition 2.1 Let $u_0 \in X^\alpha$. A function u is called a *local* X^α *solution* of (1) if, for some $\tau > 0$, u belongs to $C([0,\tau); X^\alpha) \cap C((0,\tau); X^1) \cap C^1((0,\tau); X)$, $u(0) = u_0$ and the first equation in (1) holds in X for all $t \in (0,\tau)$.

Below we quote a general theorem devoted to the local in time solvability of abstract Cauchy problems even for nonautonomous equations. This theorem is a straightforward generalization of the well-known results from [15] or [4].

Theorem 2.1 *Let $A\colon X \supset D(A) \to X$ satisfy (i) of Assumption 2.1. Assume also that $G\colon [t_0,T_0)\times X^\alpha \to X$, where $-\infty < t_0 < T_0 \le \infty$, is a continuous function satisfying for compact sets $K_1 \subset [t_0,T_0)$, $K_2 \subset (t_0,T_0)$ and each bounded set $B \subset X^\alpha$*

$$\|G(s,w_1)-G(s,w_2)\|_X \le M_{K_1,B}\,\|w_1-w_2\|_{X^\alpha}\,,\ s\in K_1,\ w_1,w_2\in B,$$

$$\|G(s_1,w_1)-G(s_2,w_2)\|_X \le M_{K_2,B}\big(|s_1-s_2|^\theta+\|w_1-w_2\|_{X^\alpha}\big),\ s_1,s_2\in K_2,\ w_1,w_2\in B$$

with some positive $M_{K_1,B}$, $M_{K_2,B}$ and $0<\theta\le 1$. Then, for any $w_0\in X^\alpha$, there exists a unique local X^α solution of the problem

$$w_t + Aw = G(t,w),\ t_0<t<T_0,\quad w(t_0)=w_0, \tag{3}$$

i.e., $w \in C([t_0,\tau);X^\alpha)\cap C((t_0,\tau);X^1)\cap C^1((t_0,\tau);X)$ and satisfies (3) in X on $[t_0,\tau)$. Under the above assumptions, this X^α solution is equivalently a function $w\in C([t_0,\tau);X^\alpha)$ satisfying the variation of constants formula

$$w(t)=e^{-A(t-t_0)}w_0+\int_{t_0}^t e^{-A(t-s)}G(s,w(s))ds,\ t\in[t_0,\tau).$$

Moreover, the local X^α solution can be extended to the maximal interval of existence $[0,\tau_{w_0})$, which means that either $\tau_{w_0}=T_0$ or $\tau_{w_0}<T_0$ and $\limsup_{t\to\tau_{w_0}^-}\|w(t)\|_{X^\alpha}=\infty$.

Henceforth, we understand a solution as an X^α solution defined on the maximal interval of existence. If $T_0=\infty$ and the life time $\tau_{w_0}=\infty$, then we call such a solution *global*.

For our problem (1), we thus have the following existence result.

Corollary 2.1 *Under Assumption 2.1, for each $u_0\in X^\alpha$, there exists a unique X^α solution $u=u(t,u_0)$ of (1) defined on its maximal interval of existence $[0,\tau_{u_0})$, i.e.,*

$$\text{either } \tau_{u_0}=\infty,\quad \text{or if }\ \tau_{u_0}<\infty,\ \text{ then }\ \limsup_{t\to\tau_{u_0}^-}\|u(t,u_0)\|_{X^\alpha}=\infty. \tag{4}$$

According to the alternative (4), we define a partition of X^α into three disjoint parts, which distinguish the behavior of a particular solution of (1).

Definition 2.2 We have $X^\alpha = X^\alpha_D\cup X^\alpha_G\cup X^\alpha_B$, where

• X^α_D denotes the set of initial data u_0 in X^α corresponding to global in time and *globally bounded solutions* for $t\ge 0$, that is, $\tau_{u_0}=\infty$ and the norm $\|u(t,u_0)\|_{X^\alpha}$ stays bounded as $t\to\infty$,

• X^α_G denotes the set of initial data u_0 in X^α corresponding to *global solutions which are unbounded* as $t\to\infty$, that is, $\tau_{u_0}=\infty$ and $\limsup_{t\to\infty}\|u(t,u_0)\|_{X^\alpha}=\infty$,

• X^α_B denotes the set of initial data $u_0\in X^\alpha$ corresponding to solutions that *blow up in a finite time*, that is, $u(t,u_0)$ exists for $t>0$ near 0, but there exists $\tau_{u_0}>0$ such that $\limsup_{t\to\tau_{u_0}^-}\|u(t,u_0)\|_{X^\alpha}=\infty$.

Thus, the solutions starting from $u_0\in X^\alpha_D$ are the *global bounded solutions*, the solutions originating from $u_0\in X^\alpha_G$ are the *grow-up solutions* and those starting from $u_0\in X^\alpha_B$ are called the *blow-up solutions.*

Clearly, knowledge of interiors and boundaries of the above-introduced sets would be vital for understanding the global dynamics of the problem under consideration on the entire phase space. Unfortunately, for many models arising from the Applied Sciences, global in time solvability is limited only to small initial data (see e.g. [17]).

In the last decades, we observed among scientists a kind of specialization in a specific behavior of solutions. The group being focused on global bounded solutions treated the other admissible behavior as non-existent and considered only equations for which $X^\alpha = X^\alpha_D$. This approach concentrated on the theory of *dissipative semigroups* and the description of asymptotic behavior of solutions using the notion of a *global attractor* (see [4, 12, 15, 19, 25, 33] among many others). Let us recall these notions.

Definition 2.3 A *semigroup* $\{S(t)\colon t \geq 0\}$ on a metric space M is a continuous mapping $S\colon \mathbb{R}^+ \times M \to M$, which satisfies

$$S(0, u_0) = u_0, \quad S(t + s, u_0) = S(t, S(s, u_0)) \text{ for all } t, s \geq 0 \text{ and all } u_0 \in M.$$

Henceforth, we will write $S(t)u_0 = S(t, u_0)$.

Definition 2.4 Let $\{S(t)\colon t \geq 0\}$ be a semigroup on a metric space (M, d). We say that a set $A \subset M$ *attracts* a set $B \subset M$ if for any $\varepsilon > 0$, there exists $T > 0$ such that

$$\operatorname{dist}(S(t)B, A) := \sup_{u_0 \in B} \inf_{v \in A} d(S(t)u_0, v) < \varepsilon \text{ whenever } t \geq T.$$

A nonempty compact set $\mathcal{A} \subset M$ is said to be a *global attractor* for $\{S(t)\colon t \geq 0\}$ if it is invariant, i.e., $S(t)\mathcal{A} = \mathcal{A}$ for all $t \geq 0$, and it attracts each bounded subset of M.

Definition 2.5 A semigroup $\{S(t)\colon t \geq 0\}$ on a metric space M is called *asymptotically compact* if for arbitrary sequences $t_n \to \infty$ and $\{u_n\} \subset M$ bounded, the sequence $\{S(t_n)u_n\}$ has a convergent subsequence in M. We say that $\{S(t)\colon t \geq 0\}$ is *dissipative* if there exists a bounded set $B_0 \subset M$ which attracts each bounded subset of M.

In the case of $X^\alpha = X^\alpha_D$, the solutions of (1) form a semigroup $\{S(t)\colon t \geq 0\}$ on X^α,

$$S(t)u_0 = u(t, u_0),\ t \geq 0,\ u_0 \in X^\alpha.$$

If there exists a bounded absorbing set $B_0 \subset X^\alpha$ for this semigroup, that is, for any bounded subset B of X^α there exists $t_B > 0$ such that $S(t)B \subset B_0$ for $t \geq t_B$ and the semigroup is asymptotically compact (or asymptotically smooth in the sense of [12]), then the semigroup is dissipative and it possesses a global attractor in X^α. This compact maximal invariant set $\mathcal{A}$ attracting all bounded subsets of X^α determines then all possible long-time dynamics of solutions (cp. e.g. [25, Proposition 10.14]). The global attractor contains, in particular, all stationary solutions, all periodic solutions (if they exist) and all bounded invariant complete orbits connecting them. Recently, much effort is put to thoroughly describe the structure of a global attractor (see for instance [8, 9] and references therein) for particular classes of equations.

Observe that the notion of a semigroup for (1) can be defined just on a subset M of X^α such that solutions originating from M exist globally in time and do not leave M. Such a general approach was presented in the introductory part of the monograph [27, Sections 2.1-2.3]. For instance, one can take $M = X^\alpha_D \cup X^\alpha_G$ or $M = X^\alpha_D$. Note, however, that in general, we do not know in advance whether M is a closed subset of X^α.

The case of a semigroup on $M = X^\alpha_D$ in the admissible presence of other behavior of solutions was considered, e.g. in [6]. Besides Assumption 2.1, it was required there the following.

Assumption 2.2 *The resolvent of the operator A is compact.*

In this setting, it was shown in [6] that the semigroup on $M = X^\alpha_D \neq \emptyset$ is asymptotically smooth. Hence the point dissipativity of $\{S(t) \colon t \geq 0\}$ on M implies that for any $u_0 \in X^\alpha$ the solution $u(\cdot, u_0)$ of (1) either blows up in a finite time, or grows up, or approaches a nonempty compact invariant set. Moreover, if all bounded complete orbits of points are uniformly bounded in X^α, then the solutions that stay bounded approach a maximal compact invariant set, which plays the role of the global attractor in this setting.

Note that the presence of grow-up solutions forbids that they approach a maximal invariant set which is bounded in X^α. In [3], the authors introduced a concept of an unbounded attractor, where the boundedness of the attractor was substituted by the minimality property. The asymptotic behavior of grow-up solutions was studied, for example, in [1,22,23] for 'slowly non-dissipative reaction-diffusion equations' of the form

$$\begin{cases} u_t = u_{xx} + bu + g(x, u, u_x),\ x \in (0,\pi),\ t > 0, \\ u_x(t,0) = u_x(t,\pi) = 0,\ t > 0, \quad u(0,x) = u_0(x),\ x \in (0,\pi), \end{cases} \tag{5}$$

with $b > 0$ and g being a C^2 bounded function. Such a problem defines a semigroup on $X^\alpha = X^\alpha_D \cup X^\alpha_G$ with $\alpha \in (\frac{3}{4}, 1)$ and with nonempty X^α_G. Then any solution to (5) converges either to a bounded stationary solution or a certain object called an equilibrium at infinity. For a characterization of the structure of a non-compact global attractor, see [23].

As regards the blow-up solutions, there exists a vast literature investigating the rates and the profiles of blow-up solutions to particular differential equations, but the notion, which would encompass the dynamics of the problem and include blow-up solutions, has not been formulated yet.

The aim of this paper is to emphasize that a typical situation is the coexistence of various types of behavior of solutions, formulate common properties of solutions, characterize their three classes, and indicate open problems connected with that partition.

3 Introductory Example

It is easy to find examples of systems allowing only for a limited set of behavior of solutions. In particular, if there is a global attractor for the system in a phase space, then all solutions need to exist globally and be bounded in the phase space. Many examples of such systems coming from the Applied Sciences are available, see e.g. [4,15,19,31].

It is also simple to find a system having only blow-up solutions. For instance, the ODE problem

$$y' = y^2 + 1, \quad y(0) = y_0, \tag{6}$$

has an explicit solution

$$y(t) = \tan(t + \arctan(y_0)) \text{ defined for } t \in \big(-\pi/2 - \arctan(y_0), \pi/2 - \arctan(y_0)\big),$$

which blows up at the finite life time $\tau_{y_0} = \pi/2 - \arctan(y_0)$ for each $y_0 \in \mathbb{R}$.

We will now present a fairly complete analysis of a 1-D scalar parabolic equation, which exhibits the coexistence of all the three types of behavior: the blow-up solutions, the grow-up solutions, the bounded solutions approaching a certain local attractor as well as the bounded solutions being unstable equilibria.

Consider a 1-D Neumann semilinear parabolic problem of the form

$$\begin{cases} u_t = u_{xx} + f(u),\ t > 0,\ x \in (0,\pi), \\ u_x(t,0) = u_x(t,\pi) = 0,\ t > 0, \quad u(0,x) = u_0(x),\ x \in (0,\pi), \end{cases} \tag{7}$$

with the nonlinearity f given by

$$f(y) = \frac{\mu}{2}(y^3 - y) \text{ for } y < 1 \text{ and } f(y) = \mu(y-1) \text{ for } y \geq 1, \tag{8}$$

with $\mu > 0$. The polynomial occurring in the nonlinear term f in $(-\infty, 1)$ is *opposite* to the well-known 'bi-stable nonlinearity' as in the Chafee-Infante problem.

The existence of X^α solutions u to (7) as well as the subordination condition follow from a more general Example 4.2 below.

Now we analyze an ordinary differential equation connected with the parabolic problem (7) satisfied by the x-independent solutions $y = y(t)$ of (7), that means

$$y' = f(y),\ t > 0, \quad y(0) = y_0. \tag{9}$$

The equation in (9) is of separable variables and can be explicitly solved. Except for three equilibria: the asymptotically stable $y_0 = 0$, unstable $y_0 = -1$ and $y_0 = 1$, we have other bounded globally defined solutions

$$y(t) = \operatorname{sgn}(y_0)\left(1 - \left(1 - y_0^{-2}\right)e^{\mu t}\right)^{-1/2},\ t \in \mathbb{R} \text{ for } y_0 \in (-1,0) \cup (0,1).$$

For $y_0 > 1$, the solutions $y(t) = (y_0 - 1)e^{\mu t} + 1$ are also globally defined for $t \in \mathbb{R}$, but they are unbounded as $t \to \infty$. Finally, the solutions for $y_0 < -1$ are given by

$$y(t) = -\left(1 - \left(1 - y_0^{-2}\right)e^{\mu t}\right)^{-1/2},\ t \in \left(-\infty, -\mu^{-1}\ln\left(1 - y_0^{-2}\right)\right),$$

and blow up in a finite time.

Using the explicit form of solutions of the ordinary differential equation (9), we are able to give a description of solutions to (7) based on the *Comparison Principle* (see [28, Theorem 10.1]). We recall that theorem for completeness.

Proposition 3.1 *Consider a uniformly parabolic linear operator in divergence form in a bounded domain $\Omega \subset \mathbb{R}^N$ with regular boundary $\partial\Omega$:*

$$\mathcal{P}u := u_t - Au = u_t - \sum_{i,j=1}^{N} \left(a_{ij}(t,x)u_{x_i}\right)_{x_j}, \quad (t,x) \in (0,T) \times \Omega,$$

where $\{a_{ij}\}$ is a symmetric matrix with bounded coefficients. Let $g = g(t,x,u)$ be C^1 in u and Hölder continuous in t and x. Assume that u and v are C^1 functions of t in $[0,T]$ and C^2 functions in x in Ω, which satisfy the following three inequalities:

$$\begin{aligned} &\mathcal{P}u - g(t,x,u) \geq \mathcal{P}v - g(t,x,v),\ (t,x) \in (0,T) \times \Omega, \\ &u(0,x) \geq v(0,x),\ x \in \Omega, \\ &\frac{\partial u}{\partial \nu} + \beta u \geq \frac{\partial v}{\partial \nu} + \beta v,\ (t,x) \in (0,T) \times \partial\Omega, \end{aligned}$$

where $\beta = \beta(t,x) \geq 0$ on $(0,T) \times \partial\Omega$. Then $u(t,x) \geq v(t,x)$ for all $(t,x) \in [0,T] \times \overline{\Omega}$. Moreover, if, in addition, $u(0,x) > v(0,x)$ for x in an open subset $\Omega_1 \subset \Omega$, then we have $u(t,x) > v(t,x)$ in $[0,T] \times \overline{\Omega_1}$.

Using Proposition 3.1, we will compare the solutions of (7) and (9), and knowing the behavior of solutions to (9), we get the corresponding information for certain solutions of the parabolic problem (7). More precisely, having solutions u of (7) and y of (9), we see that $u_t - u_{xx} - f(u) = 0 = y' - f(y)$ as long as both solutions exist and $x \in (0, \pi)$. Moreover, $u_x(t, 0) = u_x(t, \pi) = 0$ and the same is true for the x-independent solution $y(t)$. Thus we can compare u with the solution y starting from $y_0 = \min_{x\in[0,\pi]} u_0(x)$ or $y_0 = \max_{x\in[0,\pi]} u_0(x)$. We introduce *the range* of initial data

$$R_{u_0} = \big[\min_{x\in[0,\pi]} u_0(x), \max_{x\in[0,\pi]} u_0(x)\big].$$

The following characterization is then a consequence of Proposition 3.1 (compare Theorem 4.2 and Proposition 4.1 below to get the estimates of the life time of solution u).

(i) Whenever $R_{u_0} \subset (-\infty, -1)$, the corresponding to u_0 solution of (7) blows up in a finite time τ_{u_0}. Moreover, τ_{u_0} is estimated from above by the blow-up time of the solution to (9) with $y_0 = \max_{x\in[0,\pi]} u_0(x)$, and estimated from below by the blow-up time of the solution to (9) with initial data $y_0 = \min_{x\in[0,\pi]} u_0(x)$.

(ii) If $R_{u_0} \subset (-1, 1)$, then the solution $u(\cdot, u_0)$ of (7) tends to zero as $t \to \infty$.

(iii) Whenever $R_{u_0} \subset (1, \infty)$, the corresponding solution grows up as $t \to \infty$.

Evidently, there are many initial data u_0 outside of the above three classes; then the situation is more delicate and requires further studies using more sophisticated tools. Nevertheless, the three types of behavior of solutions are present among the solutions of (7).

4 Life Time of Solutions

We have seen in problem (6) possessing only blow-up solutions that the life time was a continuous function of the initial data. However, we show below that, in general, the life time of a solution to a sectorial equation need not be upper semicontinuous, but certainly is a lower semicontinuous function.

Theorem 4.1 *Under Assumption 2.1, consider the X^α solution $u(t, u_0)$ of*

$$u_t + Au = F(u),\ t > 0, \tag{10}$$

satisfying the initial condition $u(0) = u_0 \in X^\alpha$. Then the life time τ_{u_0} is a lower semicontinuous function *of u_0. More precisely, we have*

$$\forall_{0<T<\tau_{u_0}} \exists_{\delta>0} \forall_{v_0\in X^\alpha}\ \|v_0 - u_0\|_{X^\alpha} < \delta \Rightarrow \tau_{v_0} > T,$$

where τ_{v_0} is the life time of the X^α solution of (10) *starting from v_0.*

Moreover, the solutions depend continuously on the initial data; for $0 < T < \tau_{u_0}$, there exists $\delta > 0$ and $L \geq 1$ such that if $\|v_0 - u_0\|_{X^\alpha} < \delta$, then we have

$$\|u(t, v_0) - u(t, u_0)\|_{X^\alpha} \leq L \|v_0 - u_0\|_{X^\alpha},\ t \in [0, T]. \tag{11}$$

Proof. Let $u(t)$ be the solution of (10) corresponding to the initial data u_0 and let $v(t)$ be its 'perturbation', that is, the solution of (10) corresponding to the initial data v_0 (eventually close to u_0). Setting $w(t) := v(t) - u(t)$, we see that w is a solution of

$$w_t + Aw = F(w + u(t)) - F(u(t)),\ 0 < t < \tau_{u_0}, \quad w(0) = w_0, \tag{12}$$

with $w_0 = v_0 - u_0 \in X^\alpha$. Observe that

$$G(t,w) = F(w+u(t)) - F(u(t)),\ (t,w) \in [0,\tau_{u_0}) \times X^\alpha,$$

satisfies Theorem 2.1 with $\theta = 1$ since $u_t \in C((0,\tau_{u_0}), X^\alpha)$ (see [4, Corollary 2.3.1]). Thus, for any $w_0 \in X^\alpha$, we have a unique solution of (12) with the life time τ_{w_0}.

Let $h\colon \mathbb{R} \to [0,1]$ be of class C^1 such that $h(s) = 1$ for $s \le 1$ and $h(s) = 0$ for $s \ge 2$. We fix an arbitrary $T \in (0,\tau_{u_0})$. We define a function $H(t,z) = G(t, zh(\|z\|_{X^\alpha}))$, $(t,z) \in [0,T] \times X^\alpha$. Note that H is continuous, $H(t,0) = G(t,0) = 0$ and there exists $L_H > 0$ depending on F, T and u_0 such that

$$\|H(t,z_1) - H(t,z_2)\|_X \le L_H \|z_1 - z_2\|_{X^\alpha},\ t \in [0,T],\ z_1,z_2 \in X^\alpha, \tag{13}$$

since $\|zh(\|z\|_{X^\alpha})\|_{X^\alpha} \le 2$ for any $z \in X^\alpha$.

Let $E = C([0,T], X^\alpha)$ be equipped with *equivalent Bielecki's norm*

$$\|z\|_E = \max\{\|z(s)\|_{X^\alpha} e^{-\xi s} \colon s \in [0,T]\},$$

where $\xi > 0$ is so large that $C_\alpha L_H \Gamma(1-\alpha)\frac{1}{(a+\xi)^{1-\alpha}} < 1$. Let $z_0 \in X^\alpha$ and define the transformation $\Phi\colon E \to E$ by

$$\Phi(z)(t) = e^{-At}z_0 + \int_0^t e^{-A(t-s)} H(s,z(s))ds,\ t \in [0,T],\ z \in E.$$

Note that for $z_1, z_2 \in E$ and $t \in [0,T]$, using estimates (2), we get

$$\|\Phi(z_1)(t) - \Phi(z_2)(t)\|_{X^\alpha} \le C_\alpha L_H \int_0^t \frac{e^{-a(t-s)}}{(t-s)^\alpha} \|z_1(s) - z_2(s)\|_{X^\alpha} ds$$

$$\le C_\alpha L_H \|z_1 - z_2\|_E \int_0^t \frac{e^{-a(t-s)}}{(t-s)^\alpha} e^{\xi s} ds = C_\alpha L_H \|z_1 - z_2\|_E \frac{e^{\xi t}}{(a+\xi)^{1-\alpha}} \int_0^{(a+\xi)t} r^{-\alpha} e^{-r} dr.$$

Thus we obtain

$$\|\Phi(z_1) - \Phi(z_2)\|_E \le C_\alpha L_H \Gamma(1-\alpha) \frac{1}{(a+\xi)^{1-\alpha}} \|z_1 - z_2\|_E,\ z_1,z_2 \in E,$$

and Φ is a contraction on E. By the Banach Fixed Point Theorem, for any $z_0 \in X^\alpha$, there exists a unique $z \in C([0,T], X^\alpha)$, which satisfies

$$z(t) = e^{-At}z_0 + \int_0^t e^{-A(t-s)} H(s,z(s))ds,\ t \in [0,T]. \tag{14}$$

Take $z_1, z_2 \in X^\alpha$ and let $z(t,z_1)$, $z(t,z_2)$ be the corresponding solutions of (14) starting from z_1 and z_2, respectively. Let $y(t) = \|z(t,z_1) - z(t,z_2)\|_{X^\alpha}$ for $t \in [0,T]$ and note that by (2) and (13)

$$y(t) \le C_0 e^{-at} \|z_1 - z_2\|_{X^\alpha} + C_\alpha L_H \int_0^t \frac{e^{-a(t-s)}}{(t-s)^\alpha} y(s) ds,\ t \in [0,T].$$

By the Volterra type inequality (see e.g. [4, Lemma 1.2.9]) there exists a constant $L \ge 1$ such that the following Lipschitz condition holds:

$$\|z(t,z_1) - z(t,z_2)\|_{X^\alpha} \le L \|z_1 - z_2\|_{X^\alpha},\ t \in [0,T]. \tag{15}$$

Since $H(t,0)=0$, $t\in[0,T]$, we also have $z(t,0)=0$, $t\in[0,T]$. Take any $w_0\in X^\alpha$ such that $\|w_0\|_{X^\alpha}\le\frac{1}{L}$. By (15) we obtain $\|z(t,w_0)\|\le 1$ for $t\in[0,T]$. Since $H(t,z)=G(t,z)$ for $t\in[0,T]$ and $z\in X^\alpha$ such that $\|z\|_{X^\alpha}\le 1$, we obtain from (14)

$$z(t,w_0)=e^{-At}w_0+\int_0^t e^{-A(t-s)}G(s,z(s,w_0))ds,\ t\in[0,T].$$

Thus $z(t,w_0)$ is an X^α solution of (12) on $[0,T]$. By the uniqueness of solutions of (12), we see that $\tau_{w_0}>T$ for $w_0\in X^\alpha$ such that $\|w_0\|_{X^\alpha}\le\frac{1}{L}$. Set $\delta=\frac{1}{L}$ and take $v_0\in X^\alpha$ such that $\|v_0-u_0\|_{X^\alpha}<\delta$. Then the solution $w(t,w_0)$ of (12) with $w_0=v_0-u_0$ exists at least on the interval $[0,T]$. Hence $w(t,w_0)+u(t,u_0)$, $t\in[0,T]$, is an X^α solution of (10) on $[0,T]$ starting from v_0, which shows that $\tau_{v_0}>T$. Moreover, we have (11), which ends the proof. □

In general, the life time τ_{u_0} need not be upper semicontinuous with respect to u_0 as the following example shows.

Example 4.1 Consider the planar system of ordinary differential equations

$$\begin{cases} x'=1,\\ y'=e^y\sin x,\end{cases}\tag{16}$$

with the initial condition $u(0)=(x(0),y(0))=u_0\in\mathbb{R}^2$. If $u_0=(0,-\ln 2)$, then the solution $u(t)=(x(t),y(t))$ of (16) is $u(t,u_0)=(t,-\ln(\cos t+1))$ for $t\in(-\pi,\pi)$, if $u_n=(0,-\ln 2-\frac{1}{n})$, $n\in\mathbb{N}$, then the solution of (16) is $u(t,u_n)=(t,-\ln(\cos t+2e^{\frac{1}{n}}-1))$ for $t\in\mathbb{R}$, whereas if $\hat{u}_n=(0,-\ln 2+\frac{1}{n})$, $n\in\mathbb{N}$, then the solution of (16) is

$$u(t,\hat{u}_n)=(t,-\ln(\cos t+2e^{-\frac{1}{n}}-1)),\ t\in(-\arccos(1-2e^{-\frac{1}{n}}),\arccos(1-2e^{-\frac{1}{n}})).$$

Observe that $u_n\to u_0$, $\hat{u}_n\to u_0$ in $\mathbb{R}^2$ and $\tau_{u_n}=\infty$, $\tau_{\hat{u}_n}=\arccos(1-2e^{-\frac{1}{n}})$ for $n\in\mathbb{N}$. Thus, using the lower semicontinuity of τ_{u_0}, we obtain in this case

$$\pi=\tau_{u_0}=\liminf_{v_0\to u_0}\tau_{v_0}<\limsup_{v_0\to u_0}\tau_{v_0}=\infty.$$

It is of interest to estimate the life time τ_{u_0} of a solution u to (10). Note that it is typical for mathematical models of phenomena in the Applied Sciences that certain natural a priori estimates of solutions are available, for example, energy decay, conservation of mass, etc. Below we present a technique to estimate τ_{u_0} based on such an appropriate a priori estimate combined with a *subordination condition* for the nonlinearity due to Wolf von Wahl (see [32]). This condition (see (18) below) allows to translate, or sharpen, that natural a priori estimate into a form suitable to control the nonlinear term.

Theorem 4.2 *Assume that the following a priori estimate for the solution $u(t)$ of (10) satisfying $u(0)=u_0\in X^\alpha$ holds in a normed space $Y\supset X^\alpha$, that is, there exists a function $c\colon[0,T)\to[0,\infty)$, $0<T\le\infty$, bounded on compact intervals and such that*

$$\|u(t)\|_Y\le c(t),\ t\in(0,\min\{\tau_{u_0},T\}),\tag{17}$$

where τ_{u_0} denotes the life time of the solution. Furthermore, assume that the following subordination condition holds for the nonlinearity, that is, there exist a nondecreasing function $g\colon[0,\infty)\to[0,\infty)$ and a constant $\theta\in[0,1)$ such that

$$\|F(u(t))\|_X\le g(\|u(t)\|_Y)\left(1+\|u(t)\|^\theta_{X^\alpha}\right),\ t\in(0,\tau_{u_0}).\tag{18}$$

Then we have $\tau_{u_0}\ge T$.

Proof. On the contrary, suppose that $\tau_{u_0} < T$. The variation of constants formula

$$u(t) = e^{-At}u_0 + \int_0^t e^{-A(t-s)}F(u(s))ds,\ t \in (0, \tau_{u_0}),$$

the subordination condition (18) and the estimates (2) yield

$$\|u(t)\|_{X^\alpha} \le C_0 e^{-at}\|u_0\|_{X^\alpha} + \int_0^t C_\alpha \frac{e^{-a(t-s)}}{(t-s)^\alpha} g(\|u(s)\|_Y)\big(1 + \|u(s)\|^\theta_{X^\alpha}\big)ds.$$

Applying the a priori estimate (17), we obtain

$$\|u(t)\|_{X^\alpha} \le C_0\|u_0\|_{X^\alpha} + C_\alpha g\big(\sup_{s\in[0,\tau_{u_0}]} c(s)\big)\Big(1 + \Big(\sup_{s\in[0,t]}\|u(s)\|_{X^\alpha}\Big)^\theta\Big)a^{\alpha-1}\Gamma(1-\alpha).$$

Thus, setting

$$b(u_0) = C_0\|u_0\|_{X^\alpha} + C_\alpha a^{\alpha-1}\Gamma(1-\alpha)g\big(\sup_{s\in[0,\tau_{u_0}]} c(s)\big),$$

we get

$$\sup_{\tau\in[0,t]}\|u(\tau)\|_{X^\alpha} \le b(u_0)\Big(1 + \Big(\sup_{\tau\in[0,t]}\|u(\tau)\|_{X^\alpha}\Big)^\theta\Big),\ t \in [0, \tau_{u_0}).$$

Therefore, $\sup_{\tau\in[0,t]}\|u(\tau)\|_{X^\alpha}$ is estimated above by the non-negative root $z_0(u_0)$ of the algebraic equation $b(u_0)(1 + z^\theta) - z = 0$. Hence we obtain

$$\|u(t)\|_{X^\alpha} \le z_0(u_0),\ t \in [0, \tau_{u_0}),$$

which contradicts the maximality of τ_{u_0}. □

Remark 4.1 If $T = \infty$ in the a priori estimate (17), then the solution of (10) exists globally in time. Moreover, the argument of the above proof shows that if $T = \infty$ in (17) and the function $c(t)$ is bounded on $[0, \infty)$ by some constant $\hat{c}$, then the solution of (10) exists globally in time and is bounded by the non-negative root $\hat{z}_0(u_0)$ of the algebraic equation $\hat{b}(u_0)(1 + z^\theta) - z = 0$ with

$$\hat{b}(u_0) = C_0\|u_0\|_{X^\alpha} + C_\alpha a^{\alpha-1}\Gamma(1-\alpha)g(\hat{c}).$$

We also state a simple observation to estimate the life time τ_{u_0} from above.

Proposition 4.1 *Let $u(t)$ be a solution of* (10) *satisfying $u(0) = u_0 \in X^\alpha$ with the life time τ_{u_0}. Assume there exists a normed space Y such that X^α is continuously embedded into Y, and a function $\bar{c}\colon [0,T) \to [0,\infty)$, $0 < T < \infty$, such that $\limsup_{t\to T^-}\bar{c}(t) = \infty$ and $\|u(t)\|_Y \ge \bar{c}(t)$ for $t \in (0, \min\{\tau_{u_0}, T\})$. Then we have $\tau_{u_0} \le T$.*

For other results based on this technique, including the existence of a semigroup of global solutions of (10) with bounded orbits of bounded sets, dissipativity of this semigroup and the existence of its global attractor, we refer the reader to [4, Chapters 3 and 4].

Example 4.2 In a bounded domain $\Omega \subset \mathbb{R}^N$ of class C^2 (if $N \geq 2$) consider the Neumann boundary value problem

$$\begin{cases} u_t = \Delta u + f(u), \ t > 0, \ x \in \Omega, \\ \frac{\partial u}{\partial \nu} = 0, \ x \in \partial\Omega, \quad u(0,x) = u_0(x), \ x \in \Omega, \end{cases} \tag{19}$$

together with the corresponding to it ODE Cauchy problem (9). For $f\colon \mathbb{R} \to \mathbb{R}$ locally Lipschitz continuous, working in a base space $X = L^p(\Omega), p > N$, we consider the sectorial operator $A = -\Delta + I$ with the domain $D(A) = \{\phi \in W^{2,p}(\Omega)\colon \frac{\partial \phi}{\partial \nu} = 0 \text{ at } \partial\Omega\}$ (compare [33, Chapter 16]). Then, for a bounded subset B of $W^{1,p}(\Omega)$, we have

$$\|f(u) - f(v)\|_{L^p(\Omega)} \leq c\|f(u) - f(v)\|_{L^\infty(\Omega)} \leq c(B)\|u - v\|_{W^{1,p}(\Omega)}, \ u, v \in B.$$

By Corollary 2.1 local X^α solutions to (19) exist for any $\alpha \in [\frac{1}{2}, 1)$ since $X^{\frac{1}{2}} = D(A^{\frac{1}{2}}) = W^{1,p}(\Omega)$ (see [33, Theorem 16.10]) and $W^{1,p}(\Omega)$ is continuously embedded into $L^\infty(\Omega)$.

Furthermore, we have

$$\|f(u)\|_{L^p(\Omega)} \leq c\|f(u)\|_{L^\infty(\Omega)} \leq g(\|u\|_{L^\infty(\Omega)})(1 + \|u\|_{W^{1,p}(\Omega)})$$

with some nondecreasing function $g\colon [0,\infty) \to [0,\infty)$. For $\alpha \in (\frac{1}{2}, 1)$ the moments inequality

$$\|u\|_{X^{\frac{1}{2}}} \leq c\,\|u\|_X^{1-\frac{1}{2\alpha}} \|u\|_{X^\alpha}^{\frac{1}{2\alpha}}, \ u \in X^\alpha,$$

and the embedding $L^\infty(\Omega) \subset L^p(\Omega) = X$ imply the subordination condition (18).

This, together with an a priori estimate in $L^\infty(\Omega)$, allows to estimate the life time τ_{u_0} of solutions or extend the local solution globally in time (see Theorem 4.2 and Remark 4.1).

5 Grow-up Solutions

An interesting class of solutions that are global in time consists of the so-called *grow-up solutions*. Although these solutions exist globally, they have unbounded norms (usually the $L^\infty-$norm) when time t tends to infinity. As a prototype example of this type of behavior, consider the following 1-D problem:

$$\begin{cases} u_t = u_{xx} + \gamma u, \ t > 0, \ x \in (0,\pi), \\ u(t,0) = u(t,\pi) = 0, \ t > 0, \quad u(0,x) = u_0(x), \ x \in [0,\pi], \end{cases} \tag{20}$$

with $\gamma > 1$. For $u_0(x) = \sin x$, the problem (20) has an explicit solution of the form

$$u(t,x) = \sin x e^{(\gamma-1)t}, \ (t,x) \in [0,\infty) \times [0,\pi],$$

which grows up. We extend the analysis of the problem (20). The key point is the relation between the coefficient γ and the squares of natural numbers. Assume that $\gamma \in ((n-1)^2, n^2)$ for some $n \in \mathbb{N}$ and consider explicit solutions of the above problem corresponding to the initial data $u_0(x) = \sin(kx), k \in \mathbb{N}$, having the form $u(t,x) = \sin(kx)e^{(\gamma-k^2)t}$. When $k \leq n-1$, these are the grow-up solutions. Conversely, if $k \geq n$, then these solutions will decay to zero as $t \to \infty$. Therefore, for the problem (20) with a large positive number γ, we have simultaneous existence of grow-up solutions and solutions decaying to zero.

Moreover, if we let $\gamma = n^2, n \in \mathbb{N}$, we have also a stationary solution $u(t,x) = \sin(nx)$.

The solutions which grow up seem not to form a very large subclass of all the solutions. Anyway, generalizing the latter example, we return to the semilinear Neumann problem (19) under the assumptions of Example 4.2 having local in time solutions corresponding to the initial data $u_0 \in X^\alpha \subset W^{1,p}(\Omega)$ with $\alpha > \frac{1}{2}$, $p > N$. Following the idea known from the Hartman-Wintner theorem (see e.g. [11,14,16]), we are able to verify the global existence of a solution of (19) due to the corresponding properties of solutions to (9). The main assumption is the divergence of an integral

$$\int_a^\infty \frac{ds}{f(s)} = \infty. \tag{21}$$

Lemma 5.1 *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz function and assume that $f([a,\infty)) \subset (0,\infty)$ and condition (21) hold for some $a \in \mathbb{R}$. Then all the local solutions to (19), as described above, corresponding to the initial data u_0 having values in the interval $[\inf u_0, \sup u_0] \subset [a,\infty)$, possess an a priori estimate in $L^\infty(\Omega)$ by the corresponding solutions of (9). Moreover, each such solution $u(t,u_0)$ can be extended globally in time and is a grow-up solution.*

Proof. First, note that due to the assumption (21), solutions $y(t) = y(t,y_0)$ to the ODE Cauchy problem (9) with $y_0 \geq a$ exist for all $t \geq 0$. Indeed, we have

$$t = \int_{y_0}^{y(t)} \frac{ds}{f(s)} \quad \text{as long as} \quad y(t) \quad \text{exists.} \tag{22}$$

Suppose contrary to the claim that y does not exist for all $t \geq 0$. Thus there must be a finite $\tau > 0$ and a sequence $t_n \to \tau$ such that $y(t_n) \to \infty$ as $n \to \infty$. From (21) and (22), we get $\tau = \infty$, which gives a contradiction. For u_0 such that $[\inf u_0, \sup u_0] \subset [a,\infty)$, a simple comparison argument of Proposition 3.1 and global existence of y yield

$$y(t, \inf_{x\in\Omega} u_0(x)) \leq u(t,x) \leq y(t, \sup_{x\in\Omega} u_0(x)),\ t \in [0,\tau_{u_0}),\ x \in \Omega. \tag{23}$$

Since the left-hand side of (23) is increasing to ∞ and is greater than or equal to $\inf_{x\in\Omega} u_0(x) \geq a$ and both sides are globally defined in time, it yields the $L^\infty(\Omega)$ a priori estimate for the solution of (19). Hence u is global in time by Theorem 4.2 via the subordination condition. □

Remark 5.1 A result similar to Lemma 5.1 holds if $f((-\infty,a]) \subset (-\infty,0)$ and

$$\int_{-\infty}^a \frac{ds}{f(s)} = -\infty$$

hold for some $a \in \mathbb{R}$. Then all solutions $u(t,u_0)$ to (19) with the initial data u_0 having values in $[\inf u_0, \sup u_0] \subset (-\infty,a]$ can be extended globally in time and are grow-up solutions.

Remaining inside the framework of (19), following [1], consider the Neumann problem

$$\begin{cases} u_t = \Delta u + bu + g(u),\ t > 0,\ x \in \Omega, \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega, \quad u(0,x) = u_0(x), \end{cases} \tag{24}$$

with $b > 0$ and g being a bounded C^1 function ($|g| \leq M$). As a consequence of the considerations of Example 4.2, local solutions to (24) exist in the phase space $D((-\Delta_{\mathcal{N},p})^\alpha)$ with $\alpha > \frac{1}{2}$, $p > N$ and $p \geq 2$. Moreover, for the nonlinearity $f(s) := bs + g(s)$, $s \in \mathbb{R}$, the condition (21) is satisfied with $a = \frac{M+\varepsilon}{b}$, $\varepsilon > 0$ and, consequently, *all solutions* fulfilling the condition $\inf_{x\in\Omega} u_0(x) \geq \frac{M+\varepsilon}{b}$ are extended globally in time and are grow-up solutions. Moreover, none of the solutions of (24) blows up.

We further observe that a faster than linear growth of nonlinearity does not exclude the existence of the grow-up solutions. Consider, namely, problem (19) with the nonlinearity

$$f(s) = s \ln s \text{ for } s > 1 \quad \text{and} \quad f(s) = 0 \text{ for } s \leq 1. \tag{25}$$

Evidently, condition (21) is now satisfied with $a = e$, the base of the natural logarithm. Hence, whenever $\inf_{x\in\Omega} u_0(x) \geq e$, the corresponding solution of (19), (25) exists globally in time and grows up. The phenomenon of grow-up is thus not limited to the equations in which nonlinear terms are sub-linear.

It is easy to find more complicated parabolic equations (with gradient-dependent nonlinearity) having grow-up solutions. Consider, for example, the 1-D Neumann problem

$$\begin{cases} u_t = u_{xx} + u_x^3 + 1 \equiv (u_{xx} - u) + u + u_x^3 + 1, \ t > 0, \ x \in (0,1), \\ u_x = 0 \text{ for } x = 0, 1, \quad u(0,x) = u_0(x), \ x \in [0,1], \end{cases} \tag{26}$$

admitting, in particular, the x-independent solutions of the ODE $z'(t) = 1$.

We will consider problem (26) in the phase space $H_{\mathcal{N}}^{\frac{3}{2}+\varepsilon}(0,1)$ with $\varepsilon \in (0, \frac{1}{4})$. Indeed, when noting the embeddings $H^{\frac{3}{2}}(0,1) \subset W^{1,6}(0,1)$ and $H^{\frac{3}{2}+\varepsilon}(0,1) \subset W^{1,\infty}(0,1)$, the main component of the nonlinearity will satisfy

$$\begin{aligned} &\|(\phi_x)^3\|_{L^2(0,1)} = \|\phi_x\|^3_{L^6(0,1)} \leq c\|\phi\|^3_{W^{1,6}(0,1)} \leq c'\|\phi\|^3_{H^{\frac{3}{2}+\varepsilon}(0,1)}, \\ &\|(\phi_x)^3 - (\psi_x)^3\|_{L^2(0,1)} \leq \|((\phi_x) - (\psi_x))(\phi_x^2 + \phi_x\psi_x + \psi_x^2)\|_{L^2(0,1)} \\ &\leq c'(\|\phi\|_{H^{\frac{3}{2}+\varepsilon}(0,1)}, \|\psi\|_{H^{\frac{3}{2}+\varepsilon}(0,1)})\|\phi - \psi\|_{H^{\frac{3}{2}+\varepsilon}(0,1)}, \end{aligned}$$

and, consequently, the whole nonlinearity $f(u) = (u + u_x^3 + 1)$ defines a Lipschitz continuous on bounded sets Nemytskii operator acting from $H_{\mathcal{N}}^{\frac{3}{2}+\varepsilon}(0,1)$ into $L^2(0,1)$. Moreover, note that the operator $(-u_{xx} + u)$ with a Neumann boundary condition is *sectorial and positive* in $L^2(0,1)$. Thus, Corollary 2.1 establishes the local existence of solutions.

Note also that after changing the unknown function to $\overline{u}(t,x) = u(t,x) - t$, the new unknown will satisfy the problem

$$\begin{cases} \overline{u}_t = \overline{u}_{xx} + \overline{u}_x^3, \ t > 0, \ x \in (0,1), \\ \overline{u}_x = 0 \text{ for } x = 0, 1, \quad \overline{u}(0,x) = u_0(x), \ x \in [0,1]. \end{cases} \tag{27}$$

Despite the violation of the *sub-quadratic growth condition* (see the Appendix) in (26), the derivative $v := u_x$ is bounded and fulfills the maximum principle since it solves

$$\begin{cases} v_t = v_{xx} + 3v^2 v_x, \ t > 0, \ x \in (0,1) \\ v = 0 \text{ for } x = 0, 1, \quad v(0,x) = u_{0x}(x), \ x \in [0,1]. \end{cases} \tag{28}$$

We will justify shortly the last claim. Multiplying the first equation in (28) by $v^{2k-1}, k = 1, 2, \ldots$, and integrating, we obtain

$$\frac{1}{2k}\frac{d}{dt}\int_0^1 v^{2k}dx = -\frac{2k-1}{k^2}\int_0^1 [(v^k)_x]^2 dx \leq -\pi^2\frac{2k-1}{k^2}\int_0^1 v^{2k}dx,$$

where we used the fact that the function $v^k = (u_x)^k$, vanishing at $x = 0, 1$, fulfills the Poincaré inequality. Solving the differential inequality and taking the $2k$-roots, we get

$$\|u_x(t,\cdot)\|_{L^{2k}(0,1)} \leq \|u_{0x}\|_{L^{2k}(0,1)} \exp\Big(-\pi^2\frac{2k-1}{k^2}t\Big).$$

Letting $k \to \infty$, we obtain

$$\|u_x(t,\cdot)\|_{L^\infty(0,1)} \leq \|u_{0x}\|_{L^\infty(0,1)}. \tag{29}$$

Note that the *sub-quadratic growth condition* (cp. (41)) is not violated in the case of equation (28) for the derivative u_x. Having already the last estimate, we return to (26) and multiply the first equation by u, obtaining

$$\frac{1}{2}\frac{d}{dt}\int_0^1 u^2 dx = -\int_0^1 u_x^2 dx + \int_0^1 (u_x^3 + 1)u dx \leq (\|u_x\|^3_{L^\infty(0,1)} + 1)\|u\|_{L^2(0,1)},$$

and, consequently,

$$\|u(t,\cdot)\|_{L^2(0,1)} \leq \|u_0\|_{L^2(0,1)} + (\|u_{0x}\|^3_{L^\infty(0,1)} + 1)t. \tag{30}$$

As a result of the a priori estimates (29) and (30), the local solutions to (26) will be extended globally in time due to the following subordination condition:

$$\|u + u_x^3 + 1\|_{L^2(0,1)} \leq \|u_0\|_{L^2(0,1)} + (\|u_{0x}\|^3_{L^\infty(0,1)} + 1)(t+1).$$

As a consequence of the above considerations, we get the existence of grow-up solutions for *at least one* of the problems (26) or (27). Indeed, for the arbitrary initial data $u_0 \in H^{\frac{3}{2}+\varepsilon}(0,1)$ with $\varepsilon > 0$, there exist global in time solutions to both these problems. But the difference of their global solutions, $u(t, u_0)$ and $\bar{u}(t, u_0)$, corresponding to the initial data u_0, is equal to t. Consequently, at least one of them must grow up as $t \to \infty$.

The phenomenon of solutions that grow up can be also viewed in another way. Unboundedness of a norm, as $t \to \infty$, will be seen as a convergence 'to an equilibrium at infinity' (see e.g. [3]). The authors introduce there a modification of the notion of a global attractor replacing it with their *maximal attractor* for a semigroup $\{S(t) \colon t \geq 0\}$ generated by the equation (10) on a Banach space E (cp. [3, Definition 1.2]).

Definition 5.1 A closed set $\mathcal{U} \subset E$ is called a *maximal attractor* if $S(t)\mathcal{U} = \mathcal{U}$ for all $t \geq 0$, $\mathrm{dist}(S(t)K, \mathcal{U}) \to 0$ as $t \to \infty$, for any bounded set $K \subset E$, and there is no proper closed subset $\mathcal{U}' \subset \mathcal{U}$ having the above two properties.

Such maximal attractor can, however, be unbounded and not unique. Moreover, the existence of the semigroup excludes the blow-up of solutions starting from E. Also, the growth condition of the nonlinearity imposed there (in the case of the Hilbert space H) is rather restrictive (see [3, Property IV, p. 89]): $\|F(u)\|_H \leq \varepsilon\|u\|_H + C$ for some $\varepsilon, C > 0$.

The non-compact global attractors for slowly non-dissipative scalar reaction-diffusion equations of the form

$$\begin{cases} u_t = u_{xx} + bu + g(x, u, u_x),\ t > 0,\ x \in (0, \pi), \\ u_x = 0 \text{ for } x = 0, \pi, \quad u(0, x) = u_0(x), \end{cases} \tag{31}$$

were also investigated in [1, 22]. It turns out that a noncompact global attractor $\mathcal{U}$ can be decomposed as

$$\mathcal{U} = \mathcal{E}^c \cup \mathcal{E}^\infty \cup \mathcal{H},$$

where $\mathcal{E}^c$ denotes the set of bounded hyperbolic equilibria of (31), $\mathcal{E}^\infty$ is the set of 'equilibria at infinity' and $\mathcal{H}$ consists of heteroclinic connections between equilibria. A thorough study of this structure, using the zero number properties of solutions, was carried out in [23], where we refer the reader for details.

6 Blow-up Solutions

The blow-up of solutions in a finite time is a frequent form of behavior for evolution equations, taking its origins from the simple problem

$$y'(t) = y^2(t), \quad y(0) = y_0,$$

with a stationary zero solution and other solutions of the explicit form $y(t) = \frac{1}{y_0^{-1} - t}$ for $y_0 \neq 0$. Evidently, this fraction becomes unbounded in a finite time $\tau_{y_0} = y_0^{-1}$ provided that $y_0 > 0$. Thus, when using the notation of Section 2, the phase space $X^\alpha = \mathbb{R}$ decomposes into open $X_B^\alpha = (0, \infty)$, closed $X_D^\alpha = (-\infty, 0]$ and empty X_G^α. Detecting the blow-up solutions of more complicated equations and characterizing the decomposition of the phase space is, in general, much harder. Without explicit formulas for solutions, the best available tools are the comparison techniques, which eventually provide us sufficient conditions for justifying the occurrence of blow-up. However, the assumptions on nonlinear terms allowing to use the comparison techniques are limited to particular equations only and cannot be applied to most cases.

A similar type of behavior is observed for semilinear parabolic equations of the form

$$u_t = \Delta u + f(u, \nabla u), \tag{32}$$

though in that case there are more reasons for the finite life time of solutions. A simpler possibility is that the $L^\infty(\Omega)-$norm of the solution grows to infinity in a finite time (cp. Proposition 4.1). We can also face the phenomenon of the *gradient blow-up.* Recall that a gradient blow-up occurs when the solution u stays L^∞ bounded but it does not exist globally in time because some of the derivatives of u blow-up in a finite time. Let us shed some more light on the background of this case.

It is not easy to formulate a sufficient condition for the blow-up of the gradient of a solution; see, however, [7, 24, 29] and Proposition 6.1. Easier is to find hypotheses allowing to limit its growth. In a bounded domain $\Omega \subset \mathbb{R}^N$ with $\partial\Omega \in C^2$, consider the homogeneous Dirichlet boundary value problem for (32), assuming that $f(0, \nabla u) = 0$ and $\left|\frac{\partial f}{\partial u}\right| \leq L_1$, $\left|\frac{\partial f}{\partial u_{x_i}}\right| \leq L_\nabla$, with certain positive constants L_1, L_∇.

Multiplying equation (32) by Δu and integrating over Ω, we obtain

$$-\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2 dx = \int_\Omega (\Delta u)^2 dx + \int_\Omega f(u, \nabla u)\Delta u dx, \tag{33}$$

and further

$$\int_\Omega f(u,\nabla u)\Delta u dx = \int_{\partial\Omega} f(u,\nabla u)\frac{\partial u}{\partial \nu}dS - \int_\Omega \sum_i \Big(\frac{\partial f}{\partial u}u_{x_i} + \sum_j \frac{\partial f}{\partial u_{x_j}} u_{x_i x_j}\Big)u_{x_i}dx,$$

where the boundary integral vanishes due to the assumption $f(0,\nabla u) = 0$. Then the boundedness of the derivatives of f and the Cauchy inequality imply that

$$\left|\int_\Omega f(u,\nabla u)\Delta u dx\right| \le L_1 \int_\Omega |\nabla u|^2 dx + L_\nabla \int_\Omega \sum_{i,j}(\varepsilon |u_{x_i x_j}|^2 + \frac{1}{4\varepsilon}|u_{x_i}|^2)dx \tag{34}$$

with an arbitrary $\varepsilon > 0$. Note that $\sum\limits_{i,j} \|\phi_{x_i x_j}\|^2_{L^2(\Omega)} = \|\Delta\phi\|^2_{L^2(\Omega)}$ for $\phi \in H^2_0(\Omega)$ (see e.g. [10, (9.34)]). Combining (33) and (34), we choose a sufficiently small $\varepsilon > 0$ to obtain

$$\frac{d}{dt}\int_\Omega |\nabla u|^2 dx \le C(L_1, L_\nabla)\int_\Omega |\nabla u|^2 dx$$

and, consequently, an exponential bound for the spatial gradient of the solution.

As we discuss in the Appendix, for L^∞ bounded solutions, even the *sub-quadratic growth* of $f(u,\nabla u)$ with respect to the gradient is allowed, not leading to their blow-up. But a higher than quadratic growth of $f(u,\nabla u)$ with respect to ∇u leads, in general, to the blow-up of the spatial derivatives of the solution. Using the technique of sub-solutions, such form of behavior was studied in [7], where several examples of equations allowing the gradient blow-up were constructed. Different methods were used in [29] to formulate a sufficient condition for the gradient blow-up for a model Dirichlet problem

$$\begin{cases} u_t = \Delta u + |\nabla u|^p, \ t > 0, \ x \in \Omega, \\ u(t,x) = g(t,x), \ t > 0, \ x \in \partial\Omega, \quad u(0,x) = u_0(x), \ x \in \Omega, \end{cases} \tag{35}$$

with $g \in C([0,T] \times \partial\Omega)$ for all $T > 0$, and $u_0 \in C^1(\overline{\Omega})$ fulfilling the compatibility condition $u_0(x) = g(0,x)$ on $\partial\Omega$.

Denoting by $\lambda_1 > 0$ the first positive eigenvalue of $-\Delta$ in $H^1_0(\Omega)$, with the corresponding normalized eigenfunction $\phi_1 > 0$, we recall (see [29, Theorem 2.1]) the following result.

Proposition 6.1 *When $p > 2$, then there exists a positive $k_0 = k_0(\Omega, p, g)$ such that if $\int_\Omega u_0(x)\phi_1(x)dx > k_0$, then the gradient blow-up for solution of* (35) *occurs.*

Certain generalizations of the above-mentioned result can be found in [29, Theorem 2.2].

There exists quite a large literature devoted to the occurrence of *blow-up* (see e.g. [24], [26] for more references). Several properties including *blow-up sets, blow-up rates* and *profiles* characterizing closer this phenomenon have already been investigated, at least for the basic model problem

$$\begin{cases} u_t - \Delta u = \lambda u + u|u|^{p-1}, \ t > 0, \ x \in \Omega, \\ u = 0 \ \text{ on } \ \partial\Omega, \quad u(0,x) = u_0(x), \ x \in \Omega, \end{cases}$$

with $p > 1$ and $\lambda \in \mathbb{R}$, in a bounded regular domain $\Omega \subset \mathbb{R}^N$. Popular are also the studies of a more general problem (see [30])

$$\begin{cases} u_t - \Delta u = u^p + g(t,x,u,\nabla u), \ t > 0, \ x \in \Omega \subset \mathbb{R}^N, \\ u = 0 \text{ on } \partial\Omega, \quad u(0,x) = u_0(x), \ x \in \Omega, \end{cases} \tag{36}$$

with C^1 nonlinearity g satisfying $g(t,x,0,0) \geq 0$, the latter requirement being connected with the non-negativity of solutions. The solutions of (36) are searched in the set

$$X := \{0 \leq \phi \in C^1(\overline{\Omega}) \colon \phi, \nabla\phi \in L^\infty(\Omega), \phi = 0 \text{ on } \partial\Omega\}$$

subject to the norm $\|\phi\|_X = \|\phi\|_{L^\infty(\Omega)} + \|\nabla\phi\|_{L^\infty(\Omega)}$. Further developments concerning the blow-up of solutions can be found in the recent monograph [24].

7 Local Attractors and Lyapunov Functions

As regards the solutions which exist globally in time, it is interesting to investigate their long-time behavior. For solutions which stay bounded, one may try to find out sets they approach. Conversely, having a given subset of the phase space, one may look for solutions which are attracted by this set. This inspires the introduction of the following notion.

Definition 7.1 Let $\{S(t) \colon t \geq 0\}$ be a semigroup on a metric space (M,d). The *basin of attraction* of a set $A \subset M$ is defined as

$$\Omega(A) = \{u_0 \in M \colon \lim_{t\to\infty} \operatorname{dist}(S(t)u_0, A) = 0\},$$

where $\operatorname{dist}(S(t)u_0, A) = \inf_{v\in A} d(S(t)u_0, v)$.

Remark 7.1 It is easy to see that the basins of attraction of the two disjoint compact sets need to be disjoint. In particular, the basins of attraction of two separate stationary points are disjoint. Indeed, let A_1 and A_2, $A_1 \cap A_2 = \emptyset$, be two disjoint compact sets with their basins of attraction $\Omega(A_1)$ and $\Omega(A_2)$, respectively, and suppose that $u_0 \in \Omega(A_1) \cap \Omega(A_2)$. Then, taking successive subsequences, we find $v \in A_1$, $w \in A_2$, and a sequence $t_n \to \infty$ such that $S(t_n)u_0 \to v$ and $S(t_n)u_0 \to w$. Consequently, $v = w$ by the uniqueness of the limit, which is not possible.

A special role in dynamical systems is played by compact invariant subsets of the phase space. The simplest ones are stationary points or periodic orbits. Some of them may attract their neighborhoods.

Definition 7.2 A compact set $\mathcal{A} \subset M$ is said to be an *attractor* (or a *local attractor*) for a semigroup $\{S(t) \colon t \geq 0\}$ on M if it is invariant and attracts an open neighborhood U of itself.

Note that if $\mathcal{A}$ is an attractor, then $\Omega(\mathcal{A}) \supset U$ and $\mathcal{A}$ becomes a global attractor provided that it attracts each bounded subset of M. Moreover, if $\mathcal{A}$ is an attractor, then its basin of attraction $\Omega(\mathcal{A})$ is an open subset of M. We recall next a sufficient condition for the existence of an attractor, the result being taken from [27, Section 2.3.5].

Proposition 7.1 *Let $\{S(t) \colon t \geq 0\}$ be a semigroup on $M \subset X$, where X is a complete metric space. Assume there are a compact set $K \subset M$ and a neighborhood U of K in M having the property that K attracts all bounded sets in U. Then the semigroup $\{S(t) \colon t \geq 0\}$ has an attractor $\mathcal{A} = \omega(K) \subset K$, where $\omega(K)$ is the $\omega-$limit set of K.*

The following result justifies the existence of an attractor for asymptotically compact semigroups (cp. Definition 2.5).

Proposition 7.2 *Assume that there exists a bounded closed set $A \subset M$ that attracts a neighborhood of itself. If the semigroup $\{S(t)\colon t \geq 0\}$ on M is asymptotically compact, then there exists an attractor $\mathcal{A} \subset A$.*

Remark 7.2 It is easy to observe that a finite sum of attractors is an attractor itself. Indeed, if $\mathcal{A}_1, \mathcal{A}_2$ are two attractors, then their sum is evidently compact and invariant. Moreover, the sum of the neighborhoods of $\mathcal{A}_1$ and $\mathcal{A}_2$ will be attracted by $\mathcal{A}_1 \cup \mathcal{A}_2$.

Consequently, if there is a finite number of attractors in the system, we can always consider only their sum as a common attractor.

Solutions of some parabolic equations have a natural tendency to approach a stationary solution (see e.g. [13,21,34]), which is the simplest local attractor. Below we observe that if a global solution to a semilinear sectorial equation is convergent, then it must tend to an equilibrium.

Proposition 7.3 *Let Assumption 2.1 hold and assume that $u(t,u_0)$ is a global X^α solution of* (1) *and there exists $v \in X^\alpha$ such that $\lim_{t\to\infty} u(t,u_0) = v$ in X^α. Then v is a stationary solution of* (1)*, that is, $v \in X^1 \subset X^\alpha$ and $Av = F(v)$.*

Proof. By (11), for any $0 < T < \tau_v$, we have $u(T, u(t,u_0)) \to u(T,v)$ as $t \to \infty$. On the other hand, by assumption, $u(T,u(t,u_0)) = u(T+t,u_0)$ converges to v as $t \to \infty$. Thus $u(T,v) = v$ for any $0 < T < \tau_v$, so v is a stationary solution of (1). □

Therefore, under Assumption 2.1, the following alternative holds: either the solution u of (1) converges to a single stationary solution v or the solution u is not convergent in X^α as $t \to \infty$. In the second case, other forms of behavior are possible: the solution may grow up, blow up in a finite time, or eventually approach an attractor having more complicated structure (not reduced to a single equilibrium).

In literature, a common description of the behavior of dynamical systems generated by parabolic equations or systems was given using the notion of the *Lyapunov function*, see e.g. [12, 18, 34]. In the last two references, the semilinear and even fully nonlinear problems in one space dimension were analyzed within that approach. In Chapter 5 of [2], the connection of the existence of a global attractor and the Lyapunov function was described in the case of the so-called *gradient semigroups*.

Definition 7.3 A semigroup $\{S(t)\colon t \geq 0\}$ on a metric space (M,d) is called *gradient* if there exists a continuous function $V\colon M \to \mathbb{R}$ such that $V(S(t)u_0)$ is non-increasing along the trajectories of $u_0 \in M$ and, whenever $V(S(t)u_0) = V(u_0)$ for all $t \geq 0$, u_0 must be an equilibrium.

Note that in the above definition we do not require that V is bounded from below.

Remark 7.3 A semigroup $\{S(t)\colon t \geq 0\}$ on a metric space (M,d) is gradient if there exists a continuous function $V\colon M \to \mathbb{R}$ such that

$$\dot{V}(v) := \limsup_{t\to 0^+} \frac{V(S(t)v) - V(v)}{t} \leq 0, \ v \in M,$$

and for any $u_0 \in M$ if $V(v) = V(u_0)$, $v \in \gamma^+(u_0)$, then $u_0 \in \mathcal{E}$, where $\gamma^+(u_0) = \{S(t)u_0\colon t \geq 0\}$ and $\mathcal{E}$ denotes the set of equilibria in M. A function V having these properties or, equivalently, those from Definition 7.3, is called a *Lyapunov function*.

The following *LaSalle's Invariance Principle* holds, for the proof, see [15, Theorem 4.3.4].

Theorem 7.1 *Let $\{S(t)\colon t \geq 0\}$ be a gradient semigroup on a metric space M. If the positive orbit $\gamma^+(u_0) = \{S(t)u_0\colon t \geq 0\}$ of $u_0 \in M$ is a subset of a compact set K contained in M, then $\omega(u_0) \subset \mathcal{E}$ is a nonempty compact invariant subset of M, which attracts u_0, and $\operatorname{dist}(S(t)u_0, \mathcal{S}) \to 0$ as $t \to \infty$, where $\mathcal{S}$ is the maximal invariant subset of $\{v \in M\colon \dot{V}(v) = 0\}$.*

We apply LaSalle's Invariance Principle to the sectorial equation (1) (see [6]).

Corollary 7.1 *Consider the problem (1) under Assumptions 2.1 and 2.2 and let $\{S(t)\colon t \geq 0\}$ be the semigroup of X^α solutions on $M = X^\alpha_D \cup X^\alpha_G$, where we assume X^α_D to be nonempty (see Definition 2.2). Assume also that there exists a continuous function $V\colon X^\alpha_D \to \mathbb{R}$ such that*

$$\dot{V}(v) := \limsup_{t\to 0^+} \frac{V(S(t)v) - V(v)}{t} \leq 0, \ \ v \in X^\alpha_D,$$

and for any $u_0 \in X^\alpha_D$ if $V(v) = V(u_0)$, $v \in \gamma^+(u_0)$, then $u_0 \in \mathcal{E}$. Then, for any $u_0 \in X^\alpha_D$, the set $\operatorname{cl}_{X^\alpha} \gamma^+(u_0)$ is a compact subset of X^α_D and, by LaSalle's Invariance Principle, we obtain $\omega(u_0) \subset \mathcal{E}$. Thus the solutions $u(t, u_0)$ of (1) starting from $u_0 \in X^\alpha_D$ approach the set of equilibria $\mathcal{E}$ of (1).

Proof. Since X^α_D is positively invariant under the semigroup $\{S(t)\colon t \geq 0\}$, we may consider it only in the metric space X^α_D (with a metric inherited from X^α).

We first show that $\operatorname{cl}_{X^\alpha} \gamma^+(u_0)$ is a subset of X^α_D for any $u_0 \in X^\alpha_D$. Indeed, note that $B = \gamma^+(u_0)$ is a bounded subset of X^α_D. Let $v \in \operatorname{cl}_{X^\alpha} \gamma^+(u_0)$. Then there exists $t_n \geq 0$ such that $S(t_n)u_0 \to v$ in X^α. Since $\|u(t, S(t_n)u_0)\|_{X^\alpha} = \|u(t + t_n, u_0)\|_{X^\alpha} \leq R_B$ for all $t \geq 0$ and $u(t, S(t_n)u_0) \to u(t, v)$ in X^α for all $t \in [0, \tau_v)$ (see Theorem 4.1), it follows that the solution starting from v has an X^α norm bounded by R_B, hence $v \in X^\alpha_D$.

Observe also that the boundedness of B in X^α implies that $S(t)B$ with $t > 0$ is bounded in $X^{\alpha+\varepsilon}$ for $\alpha + \varepsilon < 1$. By Assumption 2.2, $X^{\alpha+\varepsilon}$ is compactly embedded in X^α, which yields the compactness of $\operatorname{cl}_{X^\alpha} S(t)\gamma^+(u_0)$ for any $t > 0$. Finally, we have

$$\operatorname{cl}_{X^\alpha} \gamma^+(u_0) = \operatorname{cl}_{X^\alpha} \bigcup_{s\in[0,1]} S(s)u_0 \cup \operatorname{cl}_{X^\alpha} S(1)\gamma^+(u_0),$$

which proves the compactness of $\operatorname{cl}_{X^\alpha} \gamma^+(u_0)$. □

Remark 7.4 A particularly complete description of Lyapunov functions is possible in one space dimension (see [18, 34, 35]). For a general quasi-linear problem of the type

$$\begin{cases} u_t = a(x, u, u_x)u_{xx} + b(x, u, u_x), \\ \alpha_i u_x(t, i) + \psi_i(u(t, i)) = 0, \ t > 0, \ i = 0, 1, \quad u(0, x) = u_0(x), \end{cases}$$

considered for $(t, x) \in [0, \infty) \times [0, 1]$ with $a, b, \psi_i \in C^3$, one constructs a pair of functions $\rho, \Phi(x, \xi, \eta)$ as in [35, Chapter 2, Theorem 1.1]. Then, after multiplying the first equation by $\rho(u, u, u_x)u_t$, they generate a Lyapunov function V through the relations

$$\frac{d}{dt}\int_0^1 \Phi(x, u, u_x)dx = -\int_0^1 \rho(x, u, u_x)u_t^2 dx, \quad V(u) = \int_0^1 \Phi(x, u, u_x)dx.$$

It is especially easy to indicate a Lyapunov function for the problem (24). Namely,

$$V(u) = \int_\Omega (|\nabla u|^2 - bu^2 - 2G(u))dx \quad \text{with} \quad G(u) = \int_0^u g(s)dx,$$

is a Lyapunov function for (24) on $X^\alpha = D((-\Delta_{\mathcal{N},p})^\alpha)$ with $\alpha > \frac{1}{2}$ and $p > N$ and $p \geq 2$. Considering constant functions $u_n \equiv n$, $n \in \mathbb{N}$, we see that $V(u_n) \to -\infty$ as $n \to \infty$. Hence V is not bounded from below. For (24), we have $X^\alpha = X^\alpha_D \cup X^\alpha_G$ and the resolvent of $-\Delta_{\mathcal{N},p}$ is compact for a sufficiently regular domain Ω. Thus, if the set of equilibria $\mathcal{E}$ of (24) is nonempty, then by Corollary 7.1, for any $u_0 \in X^\alpha_D$ we have $\omega(u_0) \subset \mathcal{E}$, whereas for $u_0 \in X^\alpha_G$ the solutions become unbounded in an infinite time.

8 Concluding Remarks

In summary, the picture sketched in this paper in the case of a semilinear parabolic problem, or even its generalization in the form of the abstract sectorial Cauchy problem (1) under Assumption 2.1, reveals that, typically, we have three potential forms of behavior of solutions as specified in Definition 2.2.

Considering a particular example of an abstract semilinear Cauchy problem (1), we first need to check which a priori estimates are available for its solutions in order to use them eventually in the subordination condition (see Theorem 4.2). More precisely, we shall find the strongest a priori estimate. In case this a priori estimate is too weak to guarantee the global in time extendibility of the local solutions, via the subordination condition (18), we need to find regions of the phase space (e.g. for small initial data) in which the existing a priori estimates are sufficient to extend solutions globally.

Furthermore, the stationary, time independent solutions should be detected and their (linearized) stability be determined. Local attractors will be next constructed for the stable stationary points, together with their basins of attraction.

For many dissipative equations, we can show the existence of a global attractor, that is, a compact maximal invariant subset of the phase space which attracts all bounded subsets. In the ideal situation, we will be even able to determine the structure of this object. However, generally, we should expect that some solutions run away to infinity. Some of them may grow up still being defined globally in time, whereas the rest of the phase space will be occupied by locally existing solutions, which blow up in a finite time.

The coexistence of at least two behavior types of solutions leads to the corresponding separation of the phase space, which is hard to be characterized in general. Moreover, most of the above procedures, while formally possible, still remain rather only theoretical for many practical problems arising from the Applied Sciences since, for instance, we cannot precisely locate all the stationary points or periodic solutions.

Nevertheless, the questions raised above should be addressed. In particular situations, they have already gained positive feedback. For example, the asymptotics of equations possessing grow-up solutions was described in terms of non-compact attractors for slowly non-dissipative reaction-diffusion equations. For specific equations, the profiles of blow-up solutions were determined via comparison techniques. We have also shown that the existence of a Lyapunov function for a general semilinear evolution equation with a main sectorial operator having compact resolvent guarantees the attraction of each bounded solution by the set of stationary solutions.

A Reaction-Diffusion Neumann Boundary Problem

We place here some auxiliary results that are connected with the applications presented in the paper, but do not belong to its main topic. They concern the existence of smooth solutions and their global extendibility in time to the reaction-diffusion Neumann boundary problem with gradient-dependent nonlinearity of the form

$$\begin{cases} u_t = \Delta u + f(u, \nabla u), & t > 0,\ x \in \Omega, \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega, & u(0,x) = u_0(x),\ x \in \Omega. \end{cases} \tag{37}$$

We start with sketching out the proof of existence of a smooth solution having bounded first derivatives ∇u. We will construct a local mild solution to (37) in the sense of [4,15] in the base space $D((-\Delta_{\mathcal{N},p})^{\frac{1}{2}}) \subset W^{1,p}(\Omega)$ with $p > N$ (here $\Delta_{\mathcal{N},p}$ denotes the Neumann Laplacian in $L^p(\Omega)$). The following proposition extends the result from [33, Section 11.10 (5)].

Proposition A.1 *Assume that $f\colon \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a C^1 function and let $p > N$ and Ω be a bounded domain in $\mathbb{R}^N$ with the boundary $\partial\Omega$ of class C^1 if $N \geq 2$. Then the Nemytskii operator $u \mapsto f(u, \nabla u)$ acts from $W^{2,p}(\Omega)$ into $W^{1,p}(\Omega)$, and*

$$\|f(u, \nabla u)\|_{W^{1,p}(\Omega)} \leq q(\|u\|_{W^{2,p}(\Omega)}), \quad u \in W^{2,p}(\Omega), \tag{38}$$

with some non-decreasing function q. Moreover, if f is a C^2 function, then that Nemytskii operator is Lipschitz continuous on the bounded subsets B of $W^{2,p}(\Omega)$, i.e.,

$$\|f(u, \nabla u) - f(v, \nabla v)\|_{W^{1,p}(\Omega)} \leq C(B)\|u - v\|_{W^{2,p}(\Omega)},\ u, v \in B.$$

Proof. The key point for the first claim is the inclusion $W^{2,p}(\Omega) \subset C^1(\overline{\Omega})$, which holds since $p > N$. To shorten the calculation, we will show the estimate for one component of the $W^{1,p}(\Omega)$ norm only. We note that the argument $(u, \nabla u)$ of f and its partial derivatives is varying in a compact subset of $\mathbb{R}^{N+1}$, provided that $\|u\|_{W^{2,p}(\Omega)}$ is bounded. Also, the norms $\|\frac{\partial u}{\partial x_j}\|_{L^\infty(\Omega)}$ are bounded, so that we have an estimate

$$\|\frac{\partial}{\partial x_j} f(u, \nabla u)\|_{L^p(\Omega)} \leq c(\|u\|_{W^{2,p}(\Omega)}).$$

The proof of the second statement follows from (38) for the first derivatives of f and the fact that $W^{1,p}(\Omega)$ is a Banach algebra. □

The above proposition almost immediately translates into the local existence result; we only need to verify that the composite function $f(u, \nabla u) \in D((-\Delta_{\mathcal{N},p})^{\frac{1}{2}})$ whenever u varies in $D(-\Delta_{\mathcal{N},p})$, $p > N$. To this end, let us recall the *characterization of the fractional power spaces* connected with the Neumann Laplacian considered on $L^p(\Omega)$. Considering fractional powers up to the exponent $\theta = 1$, we will assume that $\partial\Omega \in C^2$ if $N \geq 2$. Using the description in [33, pp. 474, 554], for $1 < p < \infty$, we have

$$D((-\Delta_{\mathcal{N},p})^\theta) = \begin{cases} W^{2\theta,p}(\Omega) & \text{for} \quad 0 \leq \theta < \frac{1}{2} + \frac{1}{2p}, \\ W^{2\theta,p}_{\mathcal{N}}(\Omega) & \text{if} \quad \frac{1}{2} + \frac{1}{2p} < \theta < \frac{3}{2} + \frac{1}{2p}, \end{cases}$$

where we denote $W^{s,p}_{\mathcal{N}}(\Omega) := \{\phi \in W^{s,p}(\Omega)\colon \frac{\partial \phi}{\partial \nu} = 0 \text{ at } \partial\Omega\}$. Proposition A.1 together with the above characterization imply that the Nemytskii operator $u \mapsto f(u, \nabla u)$ from $D(-\Delta_{\mathcal{N},p})$ into $D((-\Delta_{\mathcal{N},p})^{\frac{1}{2}})$ is a Lipschitz continuous mapping on the bounded subsets of $D(-\Delta_{\mathcal{N},p})$. By the semigroup approach (see [4,5,15]), we obtain the local solutions.

Proposition A.2 *Let* $u_0 \in D(-\Delta_{\mathcal{N},p})$ *and* $f\colon \mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ *be* C^2. *Then there exists a unique local in time mild solution* u *to* (37) *having the following regularity properties:*

$$u \in C([0,\tau); D(-\Delta_{\mathcal{N},p})) \cap C((0,\tau); D((-\Delta_{\mathcal{N},p})^{\frac{3}{2}}), \ u_t \in C((0,\tau); D((-\Delta_{\mathcal{N},p})^{\frac{3}{2}-\varepsilon})),$$

with arbitrary $\varepsilon > 0$.

Following [20], we will now recall an a priori estimate of solutions to (37) leading to the global in time extendibility of the local solutions. Consider the Neumann semilinear problem (37) in a bounded domain $\Omega \subset \mathbb{R}^N$ with $\partial\Omega$ of class C^2 if $N \geq 2$, where the function $f\colon \mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ is C^2 and satisfies the following growth restriction:

$$f(v,q)v \leq c_0|q|^2 + c_1 v^2 + c_2, \ v \in \mathbb{R}, \ q \in \mathbb{R}^N \tag{39}$$

with non-negative constants c_0, c_1, c_2. Then, for classical solutions of (37) having continuous in $[0,T]\times\overline{\Omega}$ spatial derivatives ∇u, the following a priori estimate in $L^\infty(\Omega)$ is valid:

$$\max_{(t,x)\in[0,T]\times\overline{\Omega}} |u(t,x)| \leq \kappa e^{\lambda T} \max\{\sqrt{c_2}; \max_{x\in\overline{\Omega}} |u_0(x)|\}, \tag{40}$$

where $\kappa, \lambda > 0$ are constants dependent only on c_0, c_1 and the domain Ω. The proof of that estimate can be found in [20, Ch. V, Theorem 7.3].

Having an a priori $L^\infty(\Omega)$ estimate as in (40) for all classical solutions to (37) under the assumption (39), we can thus eliminate the possibility of the blow-up in that case whenever f grows less than quadratically with respect to ∇u. Indeed, assume that the *sub-quadratic growth condition* with respect to the gradient is satisfied (compare [20, Chapter I, (3.31)]):

$$\begin{aligned} &|f(u,\nabla u)| \leq c(|u|)(1+|\nabla u|^{2-\varepsilon}), \quad |D_1 f(u,\nabla u)| \leq c(|u|)(1+|\nabla u|^{2-\varepsilon}),\\ &|D_{i+1} f(u,\nabla u)| \leq c(|u|)(1+|\nabla u|^{1-\varepsilon}), \quad i=1,...,N, \end{aligned} \tag{41}$$

where $c\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a non-decreasing function and $\varepsilon \in (0,1)$.

Note first that whenever $\frac{(2-\varepsilon)p-N}{(2-\delta)p-N} < \theta(2-\varepsilon)$, where $0<\delta<\varepsilon$ and $0<\theta<1$, the Nirenberg-Gagliardo type estimate

$$\|\phi\|_{W^{1,(2-\varepsilon)p}(\Omega)} \leq c\|\phi\|^{\theta}_{W^{2-\delta,p}(\Omega)} \|\phi\|^{1-\theta}_{L^\infty(\Omega)}$$

holds. Further, since $\frac{(2-\varepsilon)p-N}{(2-\delta)p-N} < 1$, we can also assume that $\theta(2-\varepsilon) < 1$. Thus we get

$$\begin{aligned} &\|f(u,\nabla u)\|_{L^p(\Omega)} \leq c(\|u\|_{L^\infty(\Omega)})\|1+|\nabla u|^{2-\varepsilon}\|_{L^p(\Omega)}\\ &\leq c(\|u\|_{L^\infty(\Omega)})\big(|\Omega| + \|u\|^{2-\varepsilon}_{W^{1,(2-\varepsilon)p}(\Omega)}\big) \leq c'(\|u\|_{L^\infty(\Omega)})\big(1+\|u\|^{\theta(2-\varepsilon)}_{W^{2,p}(\Omega)}\big). \end{aligned}$$

We further consider the components of the norm $\|f(u,\nabla u)\|_{W^{1,p}(\Omega)}$:

$$\|\frac{\partial}{\partial x_j} f(u,\nabla u)\|_{L^p(\Omega)} \leq \|D_1 f \frac{\partial u}{\partial x_j}\|_{L^p(\Omega)} + \sum_{i=1}^{N} \|D_{i+1} f \frac{\partial^2 u}{\partial x_j \partial x_i}\|_{L^p(\Omega)}. \tag{42}$$

We will estimate the second component in (42), the first one can be treated analogously. Using the Hölder inequality (with $\frac{1}{r}+\frac{1}{s}=1$) and (41), we obtain

$$\begin{aligned} \|D_{i+1} f \frac{\partial^2 u}{\partial x_j \partial x_i}\|_{L^p(\Omega)} &\leq \|D_{i+1} f(u,\nabla u)\|_{L^{pr}(\Omega)} \|\frac{\partial^2 u}{\partial x_j \partial x_i}\|_{L^{ps}(\Omega)}\\ &\leq c'(\|u\|_{L^\infty(\Omega)})\big(1+\|\nabla u\|^{1-\varepsilon}_{L^{(1-\varepsilon)pr}(\Omega)}\big)\|u\|_{W^{2,ps}(\Omega)}. \end{aligned}$$

The last estimate, by the Nirenberg-Gagliardo type inequalities

$$\|\phi\|_{W^{1,(1-\varepsilon)pr}(\Omega)} \le c\|\phi\|^{\theta}_{W^{3-\delta,p}(\Omega)}\|\phi\|^{1-\theta}_{L^\infty(\Omega)}, \quad \|\phi\|_{W^{2,ps}(\Omega)} \le \bar{c}\|\phi\|^{\overline{\theta}}_{W^{3-\delta,p}(\Omega)}\|\phi\|^{1-\overline{\theta}}_{L^\infty(\Omega)},$$

extends to

$$\|D_{i+1}f\frac{\partial^2 u}{\partial x_j \partial x_i}\|_{L^p(\Omega)} \le C(\|u\|_{L^\infty(\Omega)})\big(1+\|u\|^{\overline{\theta}+(1-\varepsilon)\theta}_{W^{3-\delta,p}(\Omega)}\big),$$

where we need to fulfill two conditions

$$1-\frac{N}{(1-\varepsilon)pr} < \theta\big(3-\delta-\frac{N}{p}\big) \text{ and } 2-\frac{N}{ps} < \overline{\theta}\big(3-\delta-\frac{N}{p}\big),$$

or, jointly, $3-\varepsilon-\frac{N}{p} < \big(\overline{\theta}+(1-\varepsilon)\theta\big)\big(3-\delta-\frac{N}{p}\big)$. Note that, for a given $\varepsilon \in (0,1)$ and $0<\delta<\varepsilon$, the sum $(\overline{\theta}+(1-\varepsilon)\theta)$ will be made strictly less than 1. We thus obtained a *subordination type condition*

$$\|f(u,\nabla u)\|_{W^{1,p}(\Omega)} \le c(\|u\|_{L^\infty(\Omega)})\big(1+\|u\|^{\overline{\theta}+(1-\varepsilon)\theta}_{W^{3-\delta,p}(\Omega)}\big)$$

allowing to extend a local solution to (37), varying in the phase space $W^{3-\delta,p}_{\mathcal{N}}(\Omega)$ globally in time (see Theorem 4.2 and [4, Section 4.3] for details).

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