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On the Equivalence of Lorenz System and Li System

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Abstract: The question of the equivalence of various Lorenz-like systems has been recently discussed, it has been found that with the help of various transformations it is possible to reduce such systems to the same form. In this paper, we show that the Lorenz system and the Li system are topologically equivalent. However, in a recent work it was shown that there is a homothetic transformation which converts the Li system into the Lorenz system and, therefore, all the dynamical behavior exhibited by the Li system is also present in the Lorenz system. Consequently, the results obtained in the papers devoted to the study of the Li system unnecessarily duplicate the scientific literature, while it can be trivially derived from the corresponding results on the Lorenz system.

Keywords: *Lorenz system; Li system; homothetic transformation; topological equivalence.*

Mathematics Subject Classification (2010): 93C10, 34C41, 34C20, 37C15.

1 Introduction

In 1963, E.N. Lorenz [9] discovered chaos in a simple system of three autonomous ordinary differential equations

$$\begin{cases} X' = \sigma(Y - X), \\ Y' = \rho X - Y - XZ, \\ Z' = -\beta Z + XY, \end{cases} \tag{1}$$

where σ, ρ and β are real parameters, the system is chaotic on a small subset $\{\sigma, \rho, \beta\} = \{10, 28, \frac{8}{3}\}$. The Lorenz system is the first mathematical and physical model of chaos. Since the introduction of the Lorenz system, which attracted much attention from research teams, many other chaotic systems (generally called Lorenz-like systems) have

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been analyzed (see, for instance, [13–18]). Among them, we mainly focus on the autonomous chaotic system proposed by X.F. Li et al. [15], which has been the subject of some study (see, for example, [20–22]).

For several years great effort has been devoted to the study of the question of the equivalence of various Lorenz-like systems, as discussed in [1–8], thus, with the help of various transformations, it is possible to reduce such systems to the same form. However, Algaba et al. showed that the Chen and the Lü systems are a special case of the Lorenz system [1, 2].

The purpose of this paper is to show that the Lorenz system and the Li system are topologically equivalent. Moreover, in [8] it was shown that there is a homothetic transformation which converts the Li system into the Lorenz system.

The Li system has the following form [15]:

$$\begin{cases} x' = -ax + ay, \\ y' = -y + xz, \\ z' = b - cz - xy, \end{cases} \tag{2}$$

where a, b and c are positive real parameters. With the following transformation:

$$x = X, \ y = Y, \ z = -Z + \frac{b}{c}, \ \ (c \neq 0), \tag{3}$$

the system (2) becomes

$$\begin{cases} X' = a\,(Y - X)\,, \\ Y' = \frac{b}{c}X - Y - XZ, \\ Z' = -cZ + XY. \end{cases} \tag{4}$$

Note that system (4) corresponds to the Lorenz system with parameters

$$\sigma = a, \ \rho = \frac{b}{c}, \ \beta = c. \tag{5}$$

Therefore, if $c \neq 0$, the Li system is equivalent to the Lorenz system. Thus, for each Lorenz system, there are infinitely many Li systems, parameterized by c. In this case, the two systems are homothetic copies, i.e., all the dynamics found in the Li system with $c \neq 0$ is also present in the Lorenz system.

Moreover, if $c = 0$ and $a \neq 0$ (for $a = 0$, the Li system is linear and then trivially solvable), with the linear scaling

$$x = aX, \ y = aY, \ z = -aZ, \ \tau = at,$$

the Li system is transformed into the system

$$\begin{cases} X' = -X + Y, \\ Y' = -\frac{1}{a}Y - XZ, \\ Z' = -\frac{b}{a^2} + XY. \end{cases} \tag{6}$$

Consequently, the system (6) is a particular case of a system, which has been proposed and analysed by Pehlivan and Uyaroğlu [19].

2 Dynamics Found in the Lorenz and the Li Systems

In this section, we give some examples to illustrate how we can trivially deduce the dynamics that appears in the Li system from the dynamics found in the Lorenz system.

2.1 Equilibria and local bifurcations

It is clear that the Lorenz system has three equilibrium points if $\beta(\rho-1)>0$, i.e.,

$$P_1(0,0,0),\ P_2(-\sqrt{\beta(\rho-1)},-\sqrt{\beta(\rho-1)},\rho-1),$$

$$P_3(\sqrt{\beta(\rho-1)},\sqrt{\beta(\rho-1)},\rho-1).$$

Simply use equations (3) and (5) to obtain that the corresponding equilibrium points of the Li system are

$$Q_1(0,0,b/c),\ Q_2(\sqrt{b-c},\sqrt{b-c},1),$$

$$Q_3(-\sqrt{b-c},-\sqrt{b-c},1),$$

when $b-c>0$.

Denote the vector fields on the right-hand sides of (1) and (2) by $\overrightarrow{U}(X,Y,Z)$ and $\overrightarrow{V}(x,y,z)$, respectively. It is clear that the Jacobian of (1) is

$$D\overrightarrow{U}(X,Y,Z)=\begin{pmatrix} -\sigma & \sigma & 0 \\ \rho-Z & -1 & -X \\ Y & X & -\beta \end{pmatrix}.$$

Simply use equations (3) and (5) to obtain that the corresponding Jacobian of the system (2) is

$$D\overrightarrow{V}(x,y,z)=\begin{pmatrix} -a & a & 0 \\ z & -1 & -x \\ y & x & -c \end{pmatrix}.$$

For $\rho>1$, the origin is unstable. A pitchfork bifurcation of equilibria in the Lorenz system appears for $\beta(\rho-1)=0$, and, consequently, a pitchfork bifurcation in the Li system occurs when $b=c$. The Hopf bifurcation of the nontrivial equilibria occurs in the Lorenz system at

$$\rho=\frac{\sigma(\sigma+\beta+3)}{\sigma-\beta-1}\equiv\rho_h>1,\qquad \sigma-\beta-1>0, \tag{7}$$

using equations (5), that corresponds to the Hopf bifurcation of the nontrivial equilibria in the Li system [15]

$$b_h=\frac{ac(a+c+3)}{a-c-1},\qquad a>c+1.$$

In [15], the following statement appears: "If we fix $c=1$ and vary a and b, we can observe a continuous Hopf bifurcation, as shown in Fig.1. It is similar to that of the Lorenz and Chen systems, all of them have quadratic functions of parameter a". This fact is very easy to obtain in the dynamic of the Lorenz system: if we use equations (5) in the expression (7) with $\beta=1$.

2.2 Invariant algebraic surfaces

Invariant algebraic surfaces in the Lorenz system were discussed in [10–12]. From the invariant algebraic surfaces of the Lorenz system, using equations (3) and (5), the invariant algebraic surfaces of the Li system for $c\neq 0$ are trivially obtained.

The Lorenz system, when $\beta = 2\sigma$, has the invariant algebraic surface

$$X^2 - 2\sigma Z,$$

using equations (3) and (5) we get, when $c = 2a$, the invariant algebraic surface of the Li system is written as

$$2ax^2 + 4a^2 z - 2ab.$$

The Lorenz system, when $\beta = 6\sigma - 2$ and $\rho = 2\sigma - 1$, has the invariant algebraic surface

$$X^4 - 4\sigma X^2 Z - 4\sigma^2 Y^2 + 8\rho\sigma XY + 4\rho^2 X^2,$$

using equations (3) and (5) we get, when $c = 6a - 2$ and $b = 2ac - c$, the invariant algebraic surface of the Li system is written as

$$c^2x^4 + 4ac^2x^2z + \left(4b^2 - 4abc\right)x^2 - 4a^2c^2y^2 + 8abcxy.$$

The Lorenz system, when $\beta = 1$ and $\rho = 0$, has the invariant algebraic surface

$$Y^2 + Z^2,$$

using equations (3) and (5) we get, when $b = 0$ and $c = 1$, the Li system has the invariant algebraic surface

$$y^2 + z^2.$$

The Lorenz system, when $\beta = 4$ and $\sigma = 1$, has the invariant algebraic surface

$$X^4 - 4X^2Z - 4Y^2 - 8XY + 4\rho X^2 - 16\left(1 - \rho\right)Z,$$

using equations (3) and (5) we get, when $c = 4$ and $a = 1$, the Li system has the invariant algebraic surface

$$x^4 + 4x^2z - 4y^2 - 8xy - (4 - b)(-4z + b).$$

The Lorenz system, when $\beta = 1$ and $\sigma = 1$, has the invariant algebraic surface

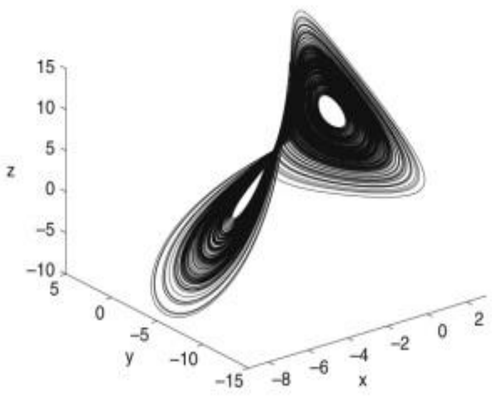

Figure 1: A chaotic attractor that exists in the Li system for $a = 5,\ b = 16,\ c = 1$.

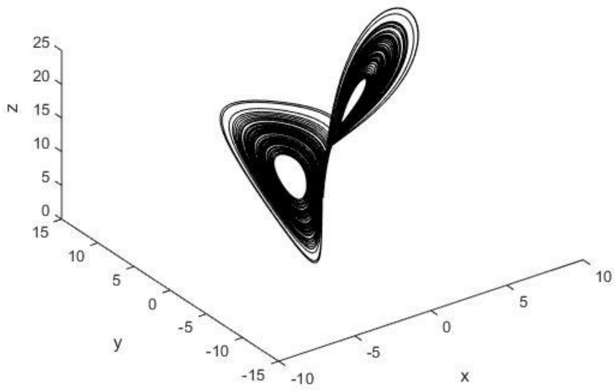

Figure 2: A chaotic attractor that exists in the Lorenz system for $\sigma = 5,\ \rho = 16,\ \beta = 1$.

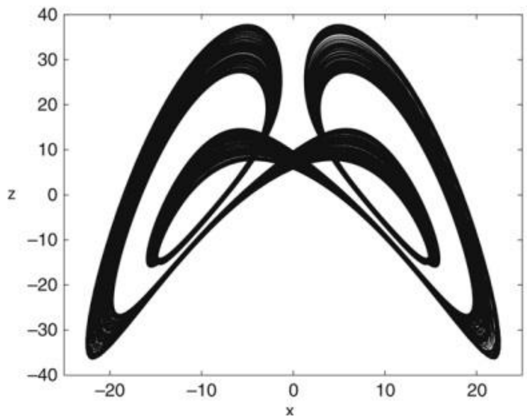

Figure 3: Projections onto two coordinate planes of a chaotic attractor that exists in the Li system for $a = 5,\ b = 115,\ c = 1$.

$$Y^2 + Z^2 - \rho X^2,$$

using equations (3) and (5) we get, when $c = 1$ and $a = 1$, the Li system has the invariant algebraic surface

$$y^2 + z^2 - bx^2 - 2bz + b^2.$$

The case when $\beta = 0$ and $\sigma = \frac{1}{3}$ has no companion case in the Li system, is the case when $c = 0$.

2.3 Chaotic attractors

The celebrated method developed by Tucker to demonstrate the existence of Lorenz's attractor can also be used to prove the existence of Li's attractor. We illustrate now the equivalence between both dynamical systems drawing a chaotic attractor. Thus, in

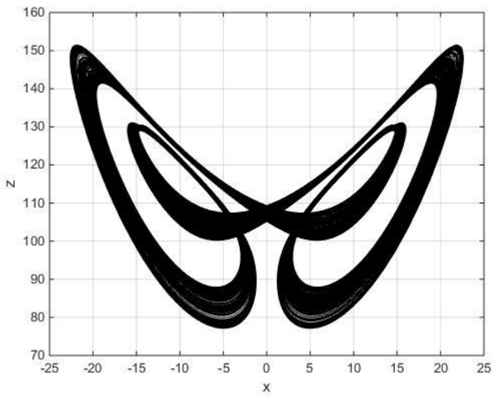

Figure 4: Projections onto two coordinate planes of a chaotic attractor that exists in the Lorenz system for $\sigma = 5,\ \rho = 115,\ \beta = 1$.

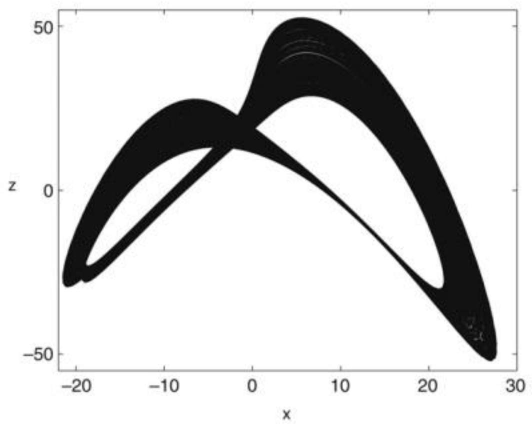

Figure 5: Projections onto two coordinate planes of a chaotic attractor that exists in the Li system for $a = 5,\ b = 167,\ c = 1$.

Figure 1, the chaotic attractor of the Li system is shown for the typical values $a = 5,\ b = 16,\ c = 1$ (Fig.2, [15]), in Figure 2, the companion chaotic attractor is presented that exists in the Lorenz system for $\sigma = 5,\ \rho = 16,\ \beta = 1$.

In Figure 3, we have a projection of the chaotic attractors of the Li system for the parameter values $a = 5,\ b = 115,\ c = 1$ (Fig.4(c), [15]), Figure 4, demonstrates the projections onto two coordinate planes of the companion chaotic attractor that exists in the Lorenz system for the parameter values $\sigma = 5,\ \rho = 16,\ \beta = 1$. In Figure 5, we have a projection of the chaotic attractors of the Li system for the parameter values $a = 5,\ b = 167,\ c = 1$ (Fig.4(h), [15]), Figure 6, displays the projections onto two coordinate planes of the companion chaotic attractor that exists in the Lorenz system for the parameter values $\sigma = 5,\ \rho = 167,\ \beta = 1$.

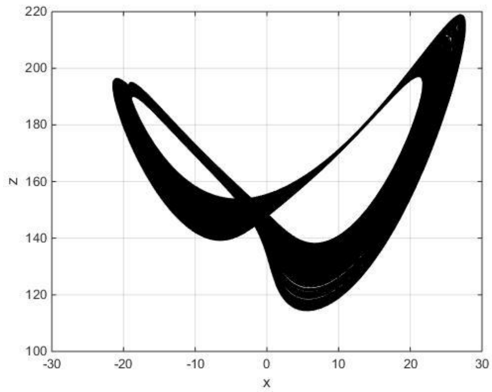

Figure 6: Projections onto two coordinate planes of a chaotic attractor that exists in the Lorenz system for $\sigma = 5,\ \rho = 167,\ \beta = 1$.

3 Conclusion

In conclusion, this study has shown with the help of a coordinate transform that the Li system is only a particular case of the Lorenz system from the dynamical point of view. Therefore, all the dynamical behavior exhibited by the Li system is present in the Lorenz system. From this, we conclude that most results obtained in the previous studies of the Li system (equilibria, bifurcations, periodic orbits, chaotic attractors, etc.) are a duplicate of the corresponding literatures on the Lorenz system.

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