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Finite Element Solution to the Strongly Reaction-Diffusion System

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Abstract: This research is devoted to demonstrating a numerical solution that adopts the cubic Hermite finite element method for a strongly reaction-diffusion system. L_2 and L_∞ error norms computed at varying time points are employed to draw comparisons between the numerical solutions attained by virtue of the presented technique and both the exact solutions and the analogous numerical ones already available in the literature. Evaluating the accuracy and efficacy of the technique utilized in this study, a perfect agreement with the exact solution is concluded.

Keywords: *finite element method; strongly reaction-diffusion system; cubic Hermite element.*

Mathematics Subject Classification (2010): 74S05, 76M10, 35K57, 70K99.

1 Introduction

The reaction diffusion system occurs in multifarious physical, biological and chemical problems. Numerous numerical techniques such as a cubic B-spline method [1], linearized finite difference scheme based upon the order reduction method [2], exponential cubic B-spline collocation algorithms [3], and trigonometric quintic B-spline collocation method [4] have been used to solve the strongly reaction-diffusion system. On the other hand, global solutions for this system have been addressed in [5]– [9]. The finite element method is one of the most accurate, flexible, and powerful techniques for approximating the solution to a wide range of linear and nonlinear partial differential equations. Examples of its implementation include: the Rosenau-RLW equation by Atouani and Omrani [10], fourth order parabolic equation by Chai et al. [11], biharmonic equation by

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Mu et al. [12], Cahn-Hilliard equation by Wang et al. [13], coupled bulk-surface problems by Burman et al. [14], fracture model in porous media by Capatina et al. [15], Stokes-Darcy coupling by Camano et al. [16], Cahn-Hilliard-Navier-Stokes-Darcy phase field model by Gao et al. [17], and Navier-Stokes/Darcy coupled problem by Discacciati and Oyarza [18], nonlinear nonstandard Volterra integral equations by Khumalo and Dlamini [19], and higher order fractional boundary value problems by Darweesh and Al-Khaled [20]. This paper is organized as follows: In Section 2, the application of the finite element method to the strongly reaction-diffusion system is presented; in Section 3, numerical results are illustrated and discussed. Finally, the paper ends with conclusions in Section 4.

2 Finite Element Solution to the Strongly Reaction-Diffusion System

Consider the strongly reaction-diffusion system as follows:

$$u_t = u_{xx} + (2\pi^2 - 1)u - 2\pi^2 v\,, 0 < x < 1,\, 0 < t < T, \tag{1}$$

$$v_t = u_{xx} + v_{xx} - v\,, 0 < x < 1,\, 0 < t < T \tag{2}$$

with the following boundary and initial conditions:

$$u_x(0,t) = 0\,, u_x(1,t) = 0\,, v_x(0,t) = 0\,, v_x(1,t) = 0\,, 0 < t < T, \tag{3}$$

$$u(x,0) = sin^2\pi x, \quad v(x,0) = cos^2\pi x, \quad 0 < x < 1\,, \tag{4}$$

where $u = u(x,t)$ and $v = v(x,t)$ are two substances of interacting concentrations. The exact solution of the system is [2]

$$u(x,t) = e^{-t} sin^2\pi x, \quad v(x,t) = e^{-t} cos^2\pi x\,.$$

Multiplying equations (1) and (2) by a test function, $w \in W(\Omega)$, where $\Omega = (a,b)$, $a, b \in \Re$, and conducting integration over the finite element (x_e, x_{e+1}) with the length h, we obtain the following equations:

$$\int_{x_e}^{x_{e+1}} (wu_t - wu_{xx} + (1 - 2\pi^2)wu + 2\pi^2 wv)dx = 0, \tag{5}$$

$$\int_{x_e}^{x_{e+1}} (wv_t - wu_{xx} - wv_{xx} + wv)dx = 0\,, \tag{6}$$

which give

$$\int_{x_e}^{x_{e+1}} (wu_t + w_x u_x + (1 - 2\pi^2)wu + 2\pi^2 wv)dx = 0, \tag{7}$$

$$\int_{x_e}^{x_{e+1}} (wv_t + w_x u_x + w_x v_x + wv)dx = 0\,. \tag{8}$$

Owing to the test function w, which satisfies the essential boundary condition, the boundary terms vanish when performing integration by parts. Then the acquired solution that is an approximation to the exact solution can be written as

$$\begin{aligned} u(x,t) &= \sum_{s=1}^{n_e} u_s(t) H_s(x)\,, \\ v(x,t) &= \sum_{s=1}^{n_e} v_s(t) H_s(x)\,, \\ w(x) &= H_i(x)\,, \qquad i = 1, ..., n_e. \end{aligned} \tag{9}$$

Here, $u_s(t)$ and $v_s(t)$, $s = 1, ..., n_e$, are undetermined time dependent quantities and $H_s(x)$ are the interpolation functions. By substituting (9) into (7) and (8), we obtain

$$\sum_{s=1}^{n_e} \int_0^h (H_i H_s \dot{u}_s + H_i' H_s' u_s + (1-2\pi^2) H_i H_s u_s + 2\pi^2 H_i H_s v_s) dx = 0, \tag{10}$$

$$\sum_{s=1}^{n_e} \int_0^h (H_i H_s \dot{v}_s + H_i' H_s' u_s + H_i' H_s' v_s + H_i H_s v_s) dx = 0\,, \tag{11}$$

where $\cdot$ denotes the derivative with respect to time. Rewriting the equations (10) and (11) in a matrix form, we get

$$A^e \dot{u}^e + (B^e + (1-2\pi^2)A^e) u^e + 2\pi^2 A^e v^e = 0, \tag{12}$$

$$A^e \dot{v}^e + B^e u^e + (B^e + A^e) v^e = 0\,. \tag{13}$$

For the cubic Hermite element, the matrices A^e_{is} and B^e_{is} are given as follows:

$$A^e_{is} = \int_0^h H_i H_s dx = \begin{bmatrix} \frac{13h}{35} & -\frac{11h^2}{210} & \frac{9h}{70} & \frac{13h^2}{420} \\ -\frac{11h^2}{210} & \frac{h^3}{105} & -\frac{13h^2}{420} & -\frac{h^3}{140} \\ \frac{9h}{70} & -\frac{13h^2}{420} & \frac{13h}{35} & \frac{11h^2}{210} \\ \frac{13h^2}{420} & -\frac{h^3}{140} & \frac{11h^2}{210} & \frac{h^3}{105} \end{bmatrix},$$

$$B^e_{is} = \int_0^h H_i' H_s' dx = \begin{bmatrix} \frac{6}{5h} & -\frac{1}{10} & -\frac{6}{5h} & -\frac{1}{10} \\ -\frac{1}{10} & \frac{2h}{15} & \frac{1}{10} & -\frac{h}{30} \\ -\frac{6}{5h} & \frac{1}{10} & \frac{6}{5h} & \frac{1}{10} \\ -\frac{1}{10} & -\frac{h}{30} & \frac{1}{10} & \frac{2h}{15} \end{bmatrix}.$$

The global matrix equation resulted in assembling the element matrices for every included element is formulated by

$$A\dot{u} + (B + (1-2\pi^2)A)u + 2\pi^2 A v = 0, \tag{14}$$

$$A\dot{v} + Bu + (B + A)v = 0\,. \tag{15}$$

For the cubic Hermite elements, let $v_{2k-1} = v_x(x_{k-1})$, $u_{2k-1} = u_x(x_{k-1})$, $k = 1, ..., n$, $v_{2k} = v(x_k)$, $u_{2k} = u(x_k)$, $k = 1, ..., n-1$, $v_{2n} = v_x(x_n)$, $u_{2n} = u_x(x_n)$. Afterwards, the formula of the forward finite difference and the Crank-Nicolson scheme are employed to discretize time derivatives $\dot{u}$, $\dot{v}$ and the time dependent quantities $u(t)$, $v(t)$ in equations (14) and (15), respectively:

$$\dot{u} = \frac{u^{j+1} - u^j}{\triangle t}, \dot{v} = \frac{v^{j+1} - v^j}{\triangle t}, u = \frac{u^{j+1} + u^j}{2}, v = \frac{v^{j+1} + v^j}{2}.$$

This leads to

$$\left[\left(1 + \frac{k}{2} - k\pi^2\right) A + \frac{k}{2} B\right] u^{n+1} + k\pi^2 A v^{n+1} = \left[\left(1 - \frac{k}{2} + k\pi^2\right) A - \frac{k}{2} B\right] u^n - k\pi^2 A v^n,$$

$$\left[\left(1 + \frac{k}{2}\right) A + \frac{k}{2} B\right] v^{n+1} + \frac{k}{2} B u^{n+1} = \left[\left(1 - \frac{k}{2}\right) A - \frac{k}{2} B\right] v^n - \frac{k}{2} B u^n,$$

where $k = \triangle t, \{u\} = \{u_x(x_0), u_x(x_1), u_x(x_2), ..., u_x(x_{n-1}),\, u(x_{n-1}), u_x(x_n)\}^T$, and similarity holds for $\{v\}$.

3 Numerical Results

Aiming at computing a numerical solution for a strongly reaction-diffusion system with the initial conditions (4) and boundary conditions (3), the proposed finite element solution with the cubic Hermite element is applied. Both the L_2 and L_∞ error norms defined by

$$L_2 = \|u^{exact} - u^{num}\|_2 = \sqrt{h \sum_{j=0}^{n} |u_j^{exact} - u_j^{num}|^2},$$

$$L_\infty = \max_j |u_j^{exact} - u_j^{num}|$$

are used as tools to measure the accuracy of the method under consideration. In Table 1 and Table 2, L_2 and L_∞ and error norms at different time levels and different number of partitions have been computed and compared with the errors obtained by [1]. The absolute errors of the proposed numerical solution at some points with $t = 1$ are evaluated and compared with the errors obtained by [1] and [2] and are reported in Tables 3 and 4. It can be seen from Tables 1 and 2 that the error norms obtained from the numerical results reduce with the increasing number of partitions. This indicates that the convergence towards the exact solution increases with the increasing number of partitions for different time levels. It is noted that the convergence towards the exact solution is achieved when $t = 1$ and with different values of x, as shown in Tables 3 and 4. From Tables $1 - 4$, we observe that our technique has yielded results that are very close to the exact solution. Moreover, in Figs. 1 and 2, the numerical solutions for $u(x,t)$ and $v(x,t)$ have been plotted with the exact solutions at different times. We notice that the plots of those solutions are indistinguishable.

4 Conclusions

A numerical scheme that involves the finite element method with the cubic Hermite element for solving the strongly reaction-diffusion system has been described. The accuracy and performance of the method has been measured using the L_2 and L_∞ error norms. We have illustrated that our numerical results are of higher accuracy than those produced by other methods [1, 2]. Furthermore, the proposed method shows perfect agreement with the exact solution for different values of time and step size. Here, we point out that the proposed method for dealing with this system is relatively new and more efficient than other methods that were used recently. Therefore, we recommend using this technique to solve different types of partial differential equations.

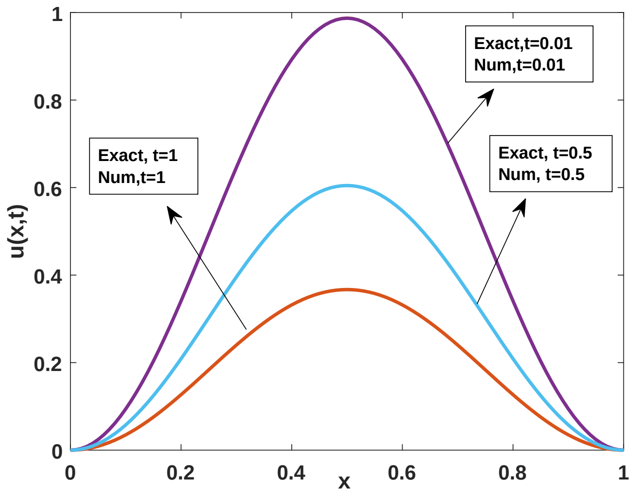

Figure 1: Comparison between numerical and exact solutions for $u(x, t)$ at different time levels.

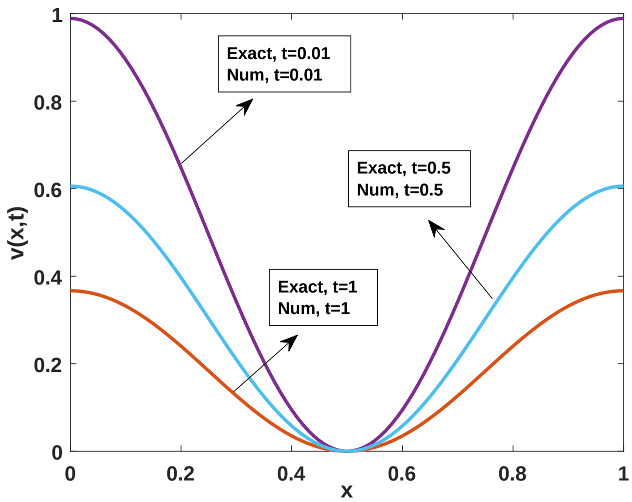

Figure 2: Comparison between numerical and exact solutions for $v(x, t)$ at different time levels.

	T	0.01	0.1	0.5	1
N=100	L_2	3.600E-6	2.731E-5	4.114E-4	5.763E-4
	L_2 [1]	5.90E-05	4.24E-04	1.33E-03	1.60E-03
	L_∞	3.900E-6	3.031E-5	7.233E-4	8.863E-4
	L_∞ [1]	6.95E-05	5.47E-04	2.56E-03	5.06E-03
N=300	L_2	4.231E-7	3.851E-6	3.577E-5	9.224E-5
	L_2 [1]	6.56E-06	4.71E-05	1.48E-04	1.81E-03
	L_∞	4.920E-7	4.371E-6	5.602E-5	8.011E-5
	L_∞ [1]	7.67E-06	6.04E-05	2.83E-04	5.61E-03
N=500	L_2	8.872E-8	6.655E-7	2.143E-6	5.341E-6
	L_2 [1]	2.36E-06	1.69E-05	5.34E-05	8.10E-05
	L_∞	9.774E-8	8.971E-7	9.033E-6	7.558E-6
	L_∞ [1]	2.76E-06	2.17E-05	1.01E-05	2.15E-05

Table 1: Errors at different times and different number of partition for $u(x,t)$ at $\triangle t = 0.001$.

	T	0.01	0.1	0.5	1
N=100	L_2	2.177E-06	2.897E-05	3.015E-04	2.882E-04
	L_2 [1]	1.05E-05	3.25E-04	1.26E-03	1.55E-03
	L_∞	2.683E-06	3.622E-05	5.223E-04	6.531E-04
	L_∞ [1]	1.22E-05	4.14E-04	2.40E-03	4.87E-03
N=300	L_2	1.899E-07	4.411E-06	2.774E-05	2.011E-05
	L_2 [1]	1.16E-06	3.62E-05	1.40E-04	1.74E-04
	L_∞	2.311E-07	5.102E-06	4.111E-05	5.978E-05
	L_∞ [1]	1.36E-06	4.61E-05	2.68E-04	5.45E-04
N=500	L_2	3.443E-08	1.899E-06	7.012E-07	6.767E-07
	L_2 [1]	4.21E-07	1.30E-05	5.07E-05	6.39E-05
	L_∞	3.895E-08	3.773E-06	9.887E-07	8.577E-07
	L_∞ [1]	4.90E-07	1.66E-05	9.65E-05	1.96E-05

Table 2: Errors at different times and different number of partition for $v(x,t)$ at $\triangle t = 0.001$.

n \ x	0.125	0.375	0.625	0.875
16	1.53213E-02	1.69554E-02	1.99778E-02	2.15531E-02
16[1]	3.7790E-02	3.7791E-02	3.7791E-02	3.7790E-02
16[2]	5.5265371E−2	3.1302542E−1	3.1302542E−1	5.5265370E−2
64	6.88663E-04	7.11213E-04	7.88990E-04	8.11892E-04
64[1]	2.7412E-03	2.7476E-03	2.7476E-03	2.7413E-03
64[2]	5.3961404E−2	3.1394376E−1	3.1394376E−1	5.3961404E−2
128	3.01234E-04	3.44501E-04	3.77006E-04	4.11002E-04
128[1]	6.9649E-04	6.8634E-04	6.8638E-04	6.9640E-04
128[2]	5.3896378E−2	3.1398950E−1	3.1398950E−1	5.3896368E−2
256	3.11332E-05	3.76654E-05	3.99815E-05	5.11234E-05
256[1]	1.9977E-04	1.4640E-04	1.4610E-04	2.0048E-04
256[2]	5.3880114E−2	3.1400093E−1	3.1400093E−1	5.3880114E−2
512	6.11889E-06	6.54321E-06	6.78802E-06	7.55001E-06
512[1]	5.1944E-05	4.5862E-05	4.6101E-05	5.2920E-05
512[2]	5.3876051E−2	3.1400379E−1	3.1400379E−1	5.3876051E−2

Table 3: The absolute errors between numerical and exact solution of $u(x,t)$ at $t = 1$.

n \ x	0.125	0.375	0.625	0.875
16	1.60011E-02	1.88555E-02	2.10044E-02	2.51110E-02
16[1]	3.6998E-02	3.6998E-02	3.6998E-02	3.6998E-02
16[2]	3.1290313E−1	5.5387659E−2	5.5387659E−2	3.1290313E−1
64	6.77134E-04	6.89891E-04	7.25778E-04	7.52211E-04
64[1]	2.6762E-03	2.6762E-03	2.6762E-03	2.6762E-03
64[2]	3.1393791E−1	5.3967251E−2	5.3967251E−2	3.1393791E−1
128	3.44899E-04	3.61122E-04	3.98877E-04	4.23321E-04
128[1]	6.7411E-04	6.7406E-04	6.7407E-04	6.7408E-04
128[2]	3.1398804E−1	5.3897831E−2	5.3897831E−2	3.1398804E−1
256	3.32211E-05	3.58876E-05	3.77521E-05	4.01903E-05
256[1]	1.6873E-04	1.6888E-04	1.6877E-04	1.6898E-04
256[2]	3.1400057E−1	5.3880480E−2	5.3880480E−2	3.1400057E−1
512	6.75665E-06	7.10099E-06	7.45565E-06	7.87890E-06
512[1]	4.2123E-05	4.2271E-05	4.2184E-05	4.2334E-05
512[2]	3.1400370E−1	5.3876143E−2	5.3876143E−2	3.1400370E−1

Table 4: The absolute errors between numerical and exact solution of $v(x,t)$ at $t = 1$.

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