Nonlinear Dynamics and Systems Theory, **22** (1) (2022) 105–116

Global Stability and Optimal Harvesting of Predator-Prey Model with Holling Response Function of Type II and Harvesting in Free Area of Capture

Syamsuddin Toaha *, Firman and Agustinus Ribal

Department of Mathematics, Faculty of Mathematics and Natural Sciences, Hasanuddin University, Indonesia

Received: May 31, 2021; Revised: December 26, 2021

Abstract: This paper analyzes the dynamical behavior of predator and prey in free and forbidden areas of capture. The dynamics of the populations are expressed in the form of equations system. The predator and prey in the free area are exploited with fixed efforts. The presence of an interior fixed point and its stability is studied. Harvesting efforts as control variables in the model are discussed. The interior fixed point is connected to the problems of maximizing the profit and present value. Local stability of the fixed point is analyzed via linearization and global stability in terms of the Lyapunov function. A critical value of fixed efforts is found, maximizing the profit function and the fixed point remains stable. According to Pontryagin's maximum principle, there exists an optimal path for the harvesting efforts that maximizes the present value of revenues. The predator and prey populations are possibly living together for a certain span of time even though the predator and prey populations are harvested with efforts as control variables. From simulation, the control variables can reduce the predator population and increase the prey population.

Keywords: *predator-prey model; free area of capture; global stability; maximum profit; present value.*

Mathematics Subject Classification (2010): 34Dxx, 49-XX .

* Corresponding author: mailto:syamsuddint@gmail.com

1 Introduction

In various fields, studies on nonlinear dynamics of systems include the behavior and stability of the systems, both local and global stability, for example, see [1], [2], and [3]. The dynamics of population is one of interesting objects of research in the field of mathematical ecology. Interaction between some populations such as predation, competition, and mutualism has ecological consequences. Population as a useful stock also has social and economics consequences. The study of dynamical behavior of population becomes complex and comprehensive because population as a stock should be managed well to protect the population from extinction, besides, the population also gives more benefits for a certain span of time.

Modeling in predator and prey populations involves many factors such as harvesting, tax, migration, diffusion, and stage structure, which have been widely studied by many researchers. Some of them considered the dynamics of one predator with two preys or two predators with one prey in the population behavior. The authors in [4] studied the dynamics of population with a reserve area and imposed tax to control the overexploitation of the populations. In [5], the authors also studied the dynamics of populations in the reserve area with harvesting and considered the problem on maximizing the present value. The behavior of the stage structure of predator and prey model in the two areas of environment with harvesting in the free area of capture was discussed in [6] and a certain condition was obtained to get an optimal value of harvesting.

The effect of selective harvesting in predator and prey populations has been observed in some purposes. Some researchers have examined only the prey being harvested, see for instance [4], [7], and [8]. The studies of predator and prey models when only the predator is harvested, can be seen in [9], [10], [11], and [12]. Some other researchers have studied predator prey models by considering both populations being harvested, the examples can be seen in [10] and [13]. Predator and prey models with exploitation were often associated with the economic point of view including maximum profit and present value problems, some examples can be found in [4], [5], and [13].

In Malili Lake Complex, South Sulawesi, Indonesia, butani fish (Glossogobius matanansis) which lives at the bottom of the lakes and its predator nile tilapia fish (Oreochromis nilotichus) are sources of food for the surrounding community. The dynamics of butani fish as an endemic and its predator must be managed properly to prevent the fish from the extinction. Based on the findings of the researchers above and as a strategy to manage the endemic butani fish and its predator, we consider the dynamical behaviors of both predator and prey populations, where the prey lives in two areas, one of which is a free area of capture and another area is a forbidden area of capture. The economically valuable predator and prey in the free area are exploited with fixed efforts. We study the presence of an interior fixed point and its local and global stability.

2 The Dynamical Behavior of Predator and Prey Populations

We consider predator and prey populations in an environment involving two areas, namely, forbidden and free areas of capture, when no fishing is allowed in the forbidden area. Both areas are considered to have the same conditions. The prey population can move in these areas freely. The prey populations grow in both areas when no predators are assumed to follow the logistic equation. The predator population is assumed to only eat the prey in the free area of capture. The behavior of predator and prey populations are stated in the form of the equations system as follows:

$$\frac{dx}{dt} = rx\left(1-\frac{x}{K}\right) - \tau_1 x + \tau_2 y - \frac{axz}{a+x}, \quad (1)$$

$$\frac{dy}{dt} = sy\left(1-\frac{y}{L}\right) + \tau_1 x - \tau_2 y, \quad (2)$$

$$\frac{dz}{dt} = \frac{\beta\alpha xz}{a+x} - kz. \quad (3)$$

From ecological point of view, we simply consider the model (1)-(3) in $R^3_+ = (x,y,z) \in R^3 \mid x,y,z > 0$ or in R^3_+. The variables x and y as the functions of time t denote the population sizes of prey in the free area of capture and in the forbidden area, respectively. The variable z as a function of t denotes the population size of the predator in the free area of capture. The growth rate of populations x and y is denoted by r and s, respectively. Carrying capacity of the environment for populations x and y is denoted by K and L, respectively. The predation rate is denoted by α, and the value of β $(0 < \beta < 1)$ is the predation scale. Parameter τ_1 denotes the movement rate for the prey from the free area to the forbidden area. Parameter τ_2 denotes the movement rate for the prey from the forbidden area to the free area. Parameter k is the mortality rate for the predator in the free area of capture.

The populations are assumed as beneficial stocks, then the predator and the prey populations in the free area of capture are harvested with fixed efforts. The dynamical behavior of predator and prey populations is developed and stated as follows:

$$\frac{dx}{dt} = rx\left(1-\frac{x}{K}\right) - \tau_1 x + \tau_2 y - \frac{axz}{a+x} - q_1 E_1 x, \quad (4)$$

$$\frac{dy}{dt} = sy\left(1-\frac{y}{L}\right) + \tau_1 x - \tau_2 y, \quad (5)$$

$$\frac{dz}{dt} = \frac{\beta\alpha xz}{a+x} - kz - q_2 E_2 z. \quad (6)$$

In the model (4)-(6), parameters q_1 and q_2 denote the catchability levels for the prey and predator populations, respectively. The symbols E_1 and E_2 denote the fixed efforts of harvesting satisfying $0 \le E_i \le E_{imax}$ for $i = 1,2$ and some given value of E_{imax}.

3 Local and Global Stability of Interior Fixed Point

The interior fixed point for model (4)–(6) may exist as long as a certain condition is satisfied. The fixed point of model (4)–(6) is found by equating the equations of the system to zero and solving them. The interior fixed point for the model is $EQ = (x_1, y_1, z_1)$, where
$x_1 = \frac{a(k+q_2E_2)}{\alpha\beta-k-q_2E_2}$, $y_1 = \frac{L(s-\tau_2)+\sqrt{L^2(s-\tau_2)^2+4s\tau_1 L x_1}}{2s}$, and
$z_1 = \frac{\left(rKx_1 - rx_1^2 - \tau_1 K x_1 + \tau_2 K y_1 - q_1 K E_1 x_1\right)(a+x_1)}{K\alpha x_1}$.

From model (4)–(6), we get the Jacobian matrix evaluated at the fixed point $EQ = (x_1, y_1, z_1)$ as

$$J_E = \begin{pmatrix} d_1 & \tau_2 & -d_2 \\ \tau_1 & d_3 & 0 \\ d_4 & 0 & d_5 \end{pmatrix},$$

where $d_1 = r - \frac{2rx_1}{K} - \tau_1 - \frac{\alpha az}{(a+x_1)^2} - q_1E_1$, $d_2 = \frac{\alpha x_1}{a+x_1}$, $d_3 = s - \frac{2sy}{L} - \tau_2$, $d_4 = \frac{\alpha\beta az}{(a+x_1)^2}$, and $d_5 = \frac{\alpha\beta x_1}{a+x_1} - k - q_2E_2$.

The characteristic polynomial corresponds to the Jacobian matrix J_E and is expressed as $f(\lambda) = det(\lambda I - J_E)$, i.e.,

$$f(\lambda) = \lambda^3 + b_2\lambda^2 + b_1\lambda + b_0, \tag{7}$$

where $b_2 = -(d_1 + d_3 + d_5)$, $b_1 = -\tau_1\tau_2 + d_1d_3 + d_1d_5 + d_2d_4 + d_3d_5$, and $b_0 = \tau_1\tau_2d_5 - d_1d_3d_5 - d_2d_3d_4$. From equation (7) and according to the Routh-Hurwitz criteria of stability [14], the interior fixed point $EQ = (x_1, y_1, z_1)$ is locally and asymptotically stable provided the conditions $b_0 > 0, b_2 > 0$, and $b_2b_1 - b_0 > 0$ are satisfied. Global stability of the interior fixed point $EQ = (x_1, y_1, z_1)$ is analyzed using the Lyapunov function. We suppose that the conditions for the presence of the interior fixed point are satisfied. Consider a Lyapunov function

$$V(x,y,z) = \beta\left(x - x_1 - x_1 ln\frac{x}{x_1}\right) + \left(y - y_1 - y_1 ln\frac{y}{y_1}\right) + \left(z - z_1 - z_1 ln\frac{z}{z_1}\right). \tag{8}$$

It is clear that $V(x,y,z)$ is defined and also continuous for all x, y, and $z > 0$. Differentiate the Lyapunov function (8) with respect to t to get

$$\begin{aligned}
\frac{dV}{dt} &= \beta\left(\frac{dx}{dt} - \frac{x_1}{x}\frac{dx}{dt}\right) + \left(\frac{dy}{dt} - \frac{y_1}{y}\frac{dy}{dt}\right) + \left(\frac{dz}{dt} - \frac{z_1}{z}\frac{dz}{dt}\right) \\
&= \beta\,(x - x_1)\left(r - \frac{rx}{K} - \tau_1 + \tau_2\frac{y}{x} - \frac{\alpha z}{a+x}\right) \\
&+ (y - y_1)\left(s - \frac{sy}{L} + \tau_1\frac{x}{y} - \tau_2\right) + (z - z_1)\left(\frac{\alpha\beta x}{a+x} - k\right).
\end{aligned} \tag{9}$$

Since $EQ = (x_1, y_1, z_1)$ is an interior fixed point, it follows that $rx_1 - \frac{rx_1^2}{K} - \tau_1x_1 + \tau_2y_1 - \frac{\alpha z_1x_1}{a+x_1} = 0$, $sy_1 - \frac{sy_1^2}{L} + \tau_1x_1 - \tau_2y_1 = 0$, and $\frac{\alpha\beta x_1z_1}{a+x_1} - kz_1 = 0$. Then the equation (9) can be rewritten as

$$\begin{aligned}
\frac{dV}{dt} &= \beta\,(x - x_1)\left(\left[r - \frac{rx}{K} - \tau_1 + \tau_2\frac{y}{x} - \frac{\alpha z}{a+x}\right] - \left[r - \frac{rx_1}{K} - \tau_1 + \tau_2\frac{y_1}{x_1} - \frac{\alpha z_1}{a+x_1}\right]\right) \\
&+ (y - y_1)\left(\left[s - \frac{sy}{L} + \tau_1\frac{x}{y} - \tau_2\right] - \left[s - \frac{sy_1}{L} + \tau_1\frac{x_1}{y_1} - \tau_2\right]\right) \\
&+ (z - z_1)\left(\left[\frac{\alpha\beta x}{a+x} - k\right] - \left[\frac{\alpha\beta x_1}{a+x_1} - k\right]\right) \\
&= -\frac{r\beta}{K}(x - x_1)^2 - \frac{s}{L}(y - y_1)^2 + P + Q,
\end{aligned} \tag{10}$$

where $P = \left(\beta\tau_2\frac{(x-x_1)}{xx_1} - \tau_1\frac{(y-y_1)}{yy_1}\right)(x_1y - xy_1)$ and $Q = \left(\alpha\beta(xz_1 - zx_1)\frac{(x-x_1)}{(a+x)(a+x_1)}\right)$. If $P \le 0$ and $Q \le 0$, then the equation (10) becomes non-positive.

Obviously, the solutions $x(t)$, $y(t)$, and $z(t)$ of model (4)–(6) with the initial conditions $x(0)$, $y(0)$, and $z(0)$ are positive for every time $t \ge 0$. From equations (4)-(5), we have

$$\begin{aligned}
\frac{d}{dt}(x+y) &= \frac{dx}{dt} + \frac{dy}{dt} = rx\left(1 - \frac{x}{K}\right) + sy\left(1 - \frac{y}{L}\right) - \frac{\alpha xz}{a+x} \\
&\le rx\left(1 - \frac{x}{K}\right) + sy\left(1 - \frac{y}{L}\right).
\end{aligned} \tag{11}$$

Given any number $\epsilon > 0$ and following the lemma in [15], we get $x(t)+y(t) \leq K+L+\epsilon$ for time t being sufficiently large. This means that the size number of $x(t) + y(t)$ is bounded for every time $t \geq 0$. Further, there exist some points $(x_*, y_*) \in R^2_+$ which satisfy $A(x_*, y_*) = 0$, where $A(x, y) = rx\left(1 - \frac{x}{K}\right) + sy\left(1 - \frac{y}{L}\right)$. The inequality (11) implies the growth of $x(t) + y(t)$ becomes non-positive. From model (4)-(6), we also know that the populations $x(t)$ and $y(t)$ grow following the logistic equation when there is no interaction and influence from other population. This has the consequence that the populations $x(t)$ and $y(t)$ are bounded and there exist real positive numbers M_1 and M_2 such that $0 < x(t) \leq M_1$ and $0 < y(t) \leq M_2$.

From the three equations of model (4)-(6), we have

$$\begin{aligned}\frac{d}{dt}(x+y+z) &= \frac{dx}{dt}+\frac{dy}{dt}+\frac{dz}{dt} = rx\left(1-\frac{x}{K}\right)+sy\left(1-\frac{y}{L}\right)-\frac{\alpha(1-\beta)xz}{a+x}-kz\\ &\leq rx\left(1-\frac{x}{K}\right)+sy\left(1-\frac{y}{L}\right). \qquad (12)\end{aligned}$$

From the previous analysis, there exist real positive numbers M_3 such that $0 < z(t) \leq M_3$. Since $x(t)$, $y(t)$, and $z(t) \geq 0$ are bounded, and due to inequality (12), there exist M_1, M_2, and $M_3 > 0$ such that $0 < x(t) \leq M_1$, $0 < y(t) \leq M_2$, and $0 < z(t) \leq M_3$. The result of this analysis is summarized in Theorem 3.1.

Theorem 3.1 *Suppose that $EQ = (x_1, y_1, z_1)$ is the only interior fixed point for model (4)-(6). If the conditions $P \leq 0$ and $Q \leq 0$, with $0 < x(t) \leq M_1$, $0 < y(t) \leq M_2$, and $0 < z(t) \leq M_3$ are fulfilled, then the interior fixed point $EQ = (x_1, y_1, z_1)$ is globally and asymptotically stable via the Lyapunov function (8).*

4 Maximum Profit Problems

The interior fixed point EQ of the model (4)-(6) is connected with an economic problem. The predator and prey populations in the free area of capture are assumed as profitable stocks. The populations are then harvested with fixed efforts. The economic activities require operating costs and provide beneficial results. For this purpose, a function of total cost is defined as $TC = cE$, where c states the cost of exploitation and E is the fixed effort of harvesting. A function of total revenue is defined as $TR = pY(E)$, where p denotes the price of profitable stock (N). The result of exploitation is stated as $Y(E, N) = qEN$, where q is the catchability level. Further we also define the profit function as $\pi = TR - TC$. Since the interior fixed point $EQ = (x_1, y_1, z_1)$ leans on the fixed efforts, the profit function also leans on the fixed efforts. Therefore the profit function is stated as $\pi(E) = TR(E) - TC(E)$.

In order to get the fixed point $EQ = (x_1, y_1, z_1)$ lying in the first octant, the condition $\alpha\,\beta - k - q_2E > 0$, i.e. $E < \frac{\alpha\beta - k}{q_2}$ must be satisfied. Under the assumption that the value of effort is non-negative, the values of parameter must satisfy the conditions $\alpha\beta - k > 0$ and $0 \leq E_2 < \frac{\alpha\beta-k}{q_2}$. Besides, we also have to assume that $rkx_1 - rx_1^2 - \tau_1 kx_1 + \tau_2 ky_1 - q_1KEx_1 > 0$. By taking $E_{1max} = 1$ and $E_{2max} = 1$, the fixed point EQ becomes an interior fixed point when $(E_1,\ E_2) \in D$, where $D = \{(E_1, E_2) : 0 \leq E_2 \leq A,\ 0 \leq E_1 \leq B\}$, $A = min\{1, \frac{\alpha\beta-k}{q_2}, f(0, E_2) = 0\}$, and $B = min\{1, E_{12} = f(E_2)\}$. The function $E_{12} = f(E_2)$ is found from the implicit function $f(E_1, E_2) = rkx_1 - rx_1^2 - \tau_1 kx_1 + \tau_2 ky_1 - q_1KEx_1 = 0$. Moreover, we assume that $E_1 < E_{12}$. The profit function associated with the fixed point $EQ = (x_1, y_1, z_1)$ is given by $\pi(E_1, E_2) = (p_1 q_1 x_1)E_1 + (p_2 q_2 z_1)E_2 - (c_1 E_1 + c_2 E_2)$.

Example 4.1 Suppose that the hypothetical values of the paramaters of the model are given as $r = 1.5$, $s = 1.5$, $a = 100$, $K = 1000$, $L = 1000$, $\tau_1 = 0.25$, $\tau_2 = 0.25$, $\alpha = 0.5$, $\beta = 0.5$, $k = 0.1$, $q_1 = 1$, $q_2 = 1$, $p_1 = 10$, $p_2 = 12$, $c_1 = 5$, and $c_2 = 6$ with appropriate units. We get the fixed point $EQ = (x_1, y_1, z_1)$, where $x_1 = \frac{100(E_2+1)}{0.15-E_2}$, $y_1 = 416.6667 + 0.3334\sqrt{1562500 + 1500x_1}$, $z_1 = 0.00200\frac{\left(1250x_1-1.5x_1^2+250y_1-1000E_1x_1\right)}{x_1}$. The fixed point becomes an interior fixed point when the conditions $0 \leq E < 0.15$ and $1250x_1 - 1.5x_1^2 + 250y_1 - 1000E_1x_1 > 0$ are satisfied. The positive solutions of $f(E_2) = 0$ are $E_2 = 0.1273$, $E_2 = 0.1500$, and $E_2 = 0.8607$. Therefore, we get $D = \{(E_1, E_2) : 0 \leq E_2 \leq 0.1273,\ 0 \leq E_1 \leq min\{1, f(E_2)\}\}$, where

$$\begin{aligned} f(E_2) &= \frac{2.5.10^{-13}}{(3-20E_2)(1+10E_2)}\left[-2.24.10^{14}E_2 - 2.867.10^{14}E_2^2 + 3.255.10^{13}\right. \\ &+ 1.500.10^{10}\sqrt{1.562.10^6 + \frac{1.500 - 10^5(E_2+0.100)}{0.150 - E_2}} \\ &- \left(2.000.10^{11}\sqrt{1.562.10^6 - \frac{1.500 - 10^5(E_2+0.100)}{0.150 - E_2}}\right)E_2 \\ &+ \left.\left(6.667.10^{11}\sqrt{1.562.10^6 - \frac{1.500 - 10^5(E_2+0.100)}{0.150 - E_2}}\right)E_2^2\right]. \end{aligned} \tag{13}$$

The profit function is now written as

$$\begin{aligned} \pi(E_1, E_2) &= \left(\frac{1000(E_2+0.1)}{0.15-E_2}\right)E_1 \\ &+ \left(\frac{0.00024E_2}{0.1+E_2}\left(\frac{1.2500.10^5(E_2+0.1)}{-0.15+E_2} + \frac{15000(E+0.1)^2}{(-0.15+E_2)^2}\right.\right. \\ &- 1.0417.10^5 - 83.3333\sqrt{1.5625.10^6 - \frac{1.5.10^5(E_2+0.1)}{-0.15+E_2}} \\ &+ \left.\left.\frac{100000E_1(E_2+0.1)}{(0.15-E_2)}\right)(-0.15+E_2)\left(100 + \frac{100(E_2+0.1)}{0.15-E_2}\right) - 6\right). \end{aligned}$$

By observing the critical values of the profit function in the feasible region D and equation (13), a pair of fixed efforts $(E_1^*, E_2^*) = (1,\ 0.10718)$ is found, which maximizes the profit function of $\pi\left(E_1^*, E_2^*\right) = 4833.0425$. The pair of the fixed efforts lies in the curve $f(E_1, E_2) = 0$ which is the boundary of the feasible region D. The critical value of fixed efforts $(E_1^*, E_2^*) = (1,\ 0.10718)$ gives the fixed point $EQ = (483.8687,\ 920.9046,\ 0)$. This condition leads the predator population towards extinction when the fixed point is asymptotically stable.

We consider that there exists a minimum number of predators in the free area of capture, for example, we may assume that the allowed minimum number of the prey population is $z_{1E} = z_{min} = 200$. Then we get a new constrain function $g(E_1, E_2) = 0$, where

$$\begin{aligned} g(E_1,E_2) &= \frac{1}{0.1+E_2}\left(0.00002\left(\frac{125.10^5(E_2+0.1)}{-0.15+E_2}+\frac{15000.(E_2+0.1)^2}{(-0.15+E_2)^2}\right.\right.\\ &- 1.0416.10^5-83.3333\sqrt{1.5625.10^6-\frac{1.5.10^5(E_2+0.1)}{-0.15+E_2}}\\ &- \left.\left.\frac{100000E_1(E_2+0.1)}{-0.15+E_2}\right)(E_2-0.15)\left(100-\frac{100(E_2+0.1)}{-0.15+E_2}\right)\right)-200. \end{aligned}$$

The problem now becomes to maximize the profit function (E_1^*, E_2^*) subject to $g(E_1, E_2) = 0$. Solving the equations $\nabla\pi(E_1, E_2) = \mu\nabla g(E_1, E_2) = 0$ and $g(E_1, E_2) = 0$ simultaneously, where μ is the Lagrange multiplication, we get $E_1^* = 0.95507$ and $E_2^* = 0.10249$. By applying the value of the pairs of efforts $(E_1^*, E_2^*) = (0.95507,\ 0.10249)$, we obtain an interior fixed point $EQ = (426.2563,\ 911.2916,\ 200)$. From the Jacobian matrix evaluated at the interior fixed point, we get the eigenvalues $-0.9041,\ -1.5922$, and -0.0075. The maximum profit now becomes $\pi\,(E_1^*, E_2^*) = 4,311.6345$. In this case, if we apply the value of efforts at the level of $E_1^* = 0.95507$ and $E_2^* = 0.10249$, then both populations will live together for a certain span of time even though the populations in the free area of capture are harvested with fixed efforts of harvesting. Besides, the harvested populations also maximize the profit function.

5 Optimal Present Value of Net Revenue

The biological steady state is reached for the equations $\frac{dx}{dt} = 0$, $\frac{dy}{dt} = 0$, and $\frac{dz}{dt} = 0$. The economic steady state is found whenever the total revenue and total cost are at the same level. The profit function for the harvested populations is stated as $\pi(E_1, E_2) = p_1q_1xE_1 + p_2q_2zE_2 - c_1E_1 - c_2E_2$. Our goal is maximizing J as the present value of the net revenue for the problem of infinite horizon which is stated as

$$J = \int_0^\infty e^{-\delta t}\{(p_1q_1x - c_1)E_1(t) + (p_2q_2z - c_2)E_2(t)\}dt. \tag{14}$$

The discount rate of the net revenue is denoted by δ. The present value J subject to the equation (4)–(6) will be maximized using Pontryagin's maximum principle [16]. The control variables $E_1(t)$ and $E_2(t)$ are subject to the condition $0 \le E_i(t) \le E_{imax}$ for $i = 1, 2$. From equation (14), the Hamiltonian function is stated as

$$\begin{aligned} H &= e^{-\delta t}\{(p_1q_1x-c_1)E_1(t)+(p_2q_2z-c_2)E_2(t)\}+\tau_1\{rx\left(1-\frac{x}{K}\right)-\tau_1x\\ &+ \tau_2y-\frac{axz}{a+x}-q_1E_1x\}+\lambda_2\{sy\left(1-\frac{y}{L}\right)+\tau_1x-\tau_2y\}\\ &+ \lambda_3\{\frac{\beta\alpha xz}{a+x}-kz-q_2E_2z\}, \end{aligned} \tag{15}$$

where the adjoint variables are given by $\lambda_1(t)$, $\lambda_2(t)$, and $\lambda_3(t)$, respectively.

As the necessary conditions, we set $\frac{\partial H}{\partial E_1} = 0$ and $\frac{\partial H}{\partial E_2} = 0$ to get the control variables E_1 and E_2 to be optimal. From equation (15), we have $\frac{\partial H}{\partial E_1} = e^{-\delta t}(p_1q_1x - c_1) - \lambda_1q_1x = 0$ and $\frac{\partial H}{\partial E_2} = e^{-\delta t}(p_2q_2z - c_2) - \lambda_3q_2z = 0$. Then we get $\lambda_1 = \frac{e^{-\delta t}(p_1q_1x-c_1)}{q_1x}$ and $\lambda_3 =$

$\frac{e^{-\delta t}(p_2q_2z-c_2)}{q_2z}$. From equation (15), we also have

$$\begin{aligned}\frac{\partial H}{\partial x} &= e^{-\delta t}p_1q_1E_1 + \lambda_1\left(r - \frac{2r}{K}x - \tau_1 - \frac{\alpha x}{a+x} + \frac{\alpha z}{(a+x)^2} - q_1E_1\right)\\ &\quad + \tau_2\lambda_1 + \lambda_3\left(\frac{\beta\alpha z}{a+x} + \frac{\beta\alpha z}{(a+x)^2}\right),\\ \frac{\partial H}{\partial y} &= \lambda_1\tau_2 + \lambda_2\left(s_2 - \frac{2s}{L}y - \tau_2\right),\\ \frac{\partial H}{\partial z} &= e^{-\delta t}p_2q_2E_2 - \frac{\lambda_1\alpha x}{a+x} + \lambda_3\left(\frac{\beta\alpha x}{a+x} - k - q_2E_2\right).\end{aligned}$$

Following Pontryagin's maximum principle $\dot{\lambda_1} = -\frac{\partial H}{\partial x}$, $\dot{\lambda_2} = -\frac{\partial H}{\partial y}$, $\dot{\lambda_3} = -\frac{\partial H}{\partial z}$, and considering the transversality condition $\lambda_2(t) = 0$ as $t \to \infty$, we get $\lambda_1 = \frac{e^{-\delta t}(p_1q_1x-c_1)}{q_1x}$, $\lambda_2 = \frac{e^{-\delta t}\tau_2(-p_1q_1x+c_1)}{q_1x\left(-\delta+s-\frac{2s}{L}y-\tau_2\right)}$ and $\lambda_3 = \frac{e^{-\delta t}(p_2q_2z-c_2)}{q_2z}$. After substituting $\lambda_1 = \frac{e^{-\delta t}(p_1q_1x-c_1)}{q_1x}$, $\lambda_2 = \frac{e^{-\delta t}\tau_2(-p_1q_1x+c_1)}{q_1x\left(-\delta+s-\frac{2s}{L}y-\tau_2\right)}$ and $\lambda_3 = \frac{e^{-\delta t}(p_2q_2z-c_2)}{q_2z}$ into the equations $\dot{\lambda_1} = -\frac{\partial H}{\partial x}$, $\dot{\lambda_2} = -\frac{\partial H}{\partial y}$, and $\dot{\lambda_3} = -\frac{\partial H}{\partial z}$, we get E_1 and E_2. The optimal paths of E_1 and E_2 still depend on populations x, y, and z, i.e., $E_1 = E_1(x,y,z)$ and $E_2 = E_2(x,y,z)$. By substituting $x = x_1$, $y = y_1$, and $z = z_1$ into the implicit equations $E_1 = E_1(x,y,z)$ and $E_2 = E_2(x,y,z)$, we get the suitable values of control variables E_1 and E_2. The values of E_1, E_2, x_1, y_1, and z_1 give a maximum value of the present value J.

Example 5.1 Suppose that the hypothetical values of the paramaters of the model are given as $r = 1.5$, $s = 1.5$, $a = 100$, $K = 1000$, $L = 1000$, $\tau_1 = 0.25$, $\tau_2 = 0.25$, $\alpha = 0.5$, $\beta = 0.5$, $k = 0.1$, $q_1 = 1$, $q_2 = 1$ in appropriate units. Take $p_1 = 10$, $p_2 = 12$, $c_1 = 5$, $c_2 = 6$, and $\delta = 0.005$ in appropriate units. Further we have the fixed point $EQ = (x_1, y_1, z_1)$, where

$x_1 = \frac{100(E_2+0.1)}{0.15-E_2}$, $y_1 = 416.66667 + 0.33333\sqrt{1,562,500 + 1,500x_1}$, and

$z_1 = \frac{0.00200\left(1,250x_1-1.5x_1^2+250y_1-1000E_1x_1\right)(100+x_1)}{x_1}$.

The adjoint variables are $\lambda_1 = \frac{-e^{-0.005t}(10x_1-5)}{x_1}$, $\lambda_2 = \frac{0.25e^{-0.005t}(5-10x_1)}{(1.245-0.003y_1)x_1}$, and $\lambda_3 = \frac{-e^{-0.005t}(6-12z_1)}{x_1}$.

By solving the equations and then choosing the suitable values of fixed efforts of harvesting, we get $E_1 = 1.13676$ and $E_2 = 0.10278$. Further we get the fixed point $EQ = (429.40138,\ 911.82119,\ 0.01043)$ with the eigenvalues -1.0359, -1.62450, and -3.32935×10^{-7}. Under these conditions, the fixed point EQ is locally and asymptotically stable. The adjoint variables are denoted by $\lambda_1 = 9.988356e^{-0.005t}$, $\lambda_2 = 1.675377e^{-0.005t}$, and $\lambda_3 = -563.471151e^{-0.005t}$. Then we get the maximum value of the present value of the net revenue $J = \int_0^\infty 4,874.957199e^{-0.005t}dt = 9.749914 \times 10^5$.

We now continue the problem of maximizing the present value J of the net revenue for the problem of finite horizon which is stated as

$$J = \int_0^T e^{-\delta t}\{(p_1q_1x - c_1)E_1(t) + (p_2q_2z - c_2)E_2(t)\}dt. \tag{16}$$

The control variables $E_1(t)$ and $E_2(t)$ are subject to the condition $0 \le E_i(t) \le 1$ for

$i = 1, 2$. From equation (16), the Hamiltonian function is stated as

$$\begin{aligned} H &= e^{-\delta t}\{(p_1q_1x - c_1)E_1(t) + (p_2q_2z - c_2)E_2(t)\} + \tau_1\{rx\left(1 - \frac{x}{K}\right) - \tau_1 x \\ &+ \tau_2 y - \frac{axz}{a+x} - q_1E_1x\} + \lambda_2\{sy\left(1 - \frac{y}{t}\right) + \tau_1 x - \tau_2 y\} \\ &+ \lambda_3\{\frac{\beta\alpha xz}{a+x} - kz - q_2E_2z\}, \end{aligned} \tag{17}$$

where $\lambda_1(t)$, $\lambda_2(t)$, and $\lambda_3(t)$ denote the adjoint variables. Again, following Pontryagin's maximum principle, we set $\dot{\lambda_1} = -\frac{\partial H}{\partial x}$, $\dot{\lambda_2} = -\frac{\partial H}{\partial y}$, $\dot{\lambda_3} = -\frac{\partial H}{\partial z}$, with $\lambda_1(T) = \lambda_2(T) = \lambda_3(T) = 0$. Since the equation (17) is linear in E_1 and E_2 with the slope $\frac{\partial H}{\partial E_1} = e^{-\delta t}(p_1q_1x - c_1) - \lambda_1q_1x$ and $\frac{\partial H}{\partial E_2} = e^{-\delta t}(p_2q_2z - c_2) - \lambda_3q_2z$, we define the following to maximize H:

$$E_1^*(t) = \begin{cases} 0, e^{-\lambda t}(p_1q_1x - c_1) - \lambda_1q_1x < 0, \\ 1, e^{-\lambda t}(p_1q_1x - c_1) - \lambda_1q_1x \geq 0 \end{cases}$$

and

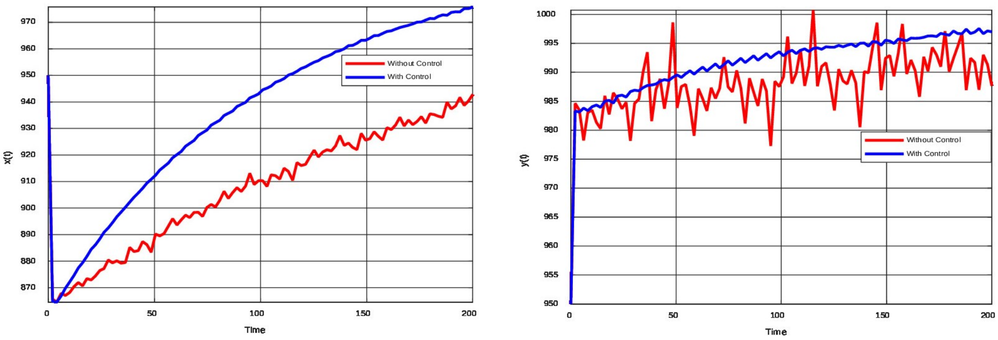
$$E_2^*(t) = \begin{cases} 0, e^{-\lambda t}(p_2q_2z - c_2) - \lambda_2q_2z < 0, \\ 1, e^{-\lambda t}(p_2q_2z - c_2) - \lambda_2q_2z \geq 0. \end{cases}$$

Because the Hamiltonian function H is linear in E_1 and E_2, the usual first order condition $\frac{dH}{dE_1} = \frac{dH}{dE_2} = 0$ is inapplicable in our search for $E_1^*(t)$ and $E_2^*(t)$, but here we define $E_1^*(t) = E_2^*(t) = 1$ when $\frac{dH}{dE_1} = \frac{dH}{dE_2} = 0$. The solution for the problem of finite horizon will be given using the forward-backward sweep numerical method to plot the optimal solution of $x^*(t)$, $y^*(t)$, $z^*(t)$, $E_1^*(t)$, and $E_2^*(t)$.

Example 5.2. Suppose that the hypothetical values of the paramaters of the model are given as $r = 1.8$, $a = 200$, $\tau_1 = 0.25$, $\tau_2 = 0.25$, $\beta = 0.15$, $K = 1000$, $\alpha = 0.5$, $s = 1.8$, $L = 1000$, $k = 0.01$, $q_1 = 0.01$, $q_2 = 0.01$ in appropriate units. Take $p_1 = 10$, $p_2 = 12$, $c_1 = 5$, $c_2 = 6$, $\delta = 0.005$, and $T = 200$. Set the initial and terminal conditions $x(0) = 950$, $y(0) = 950$, $z(0) = 600$, and $\lambda_1(T) = \lambda_2(T) = \lambda_3(T) = 0$. The curves of state, costate, and adjoint variables are plotted using a Matlab program as given in Figures 1–4.

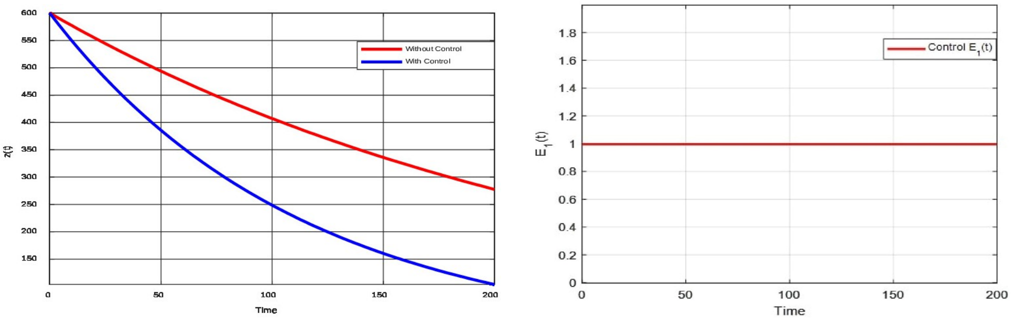
Figures 1(a), 1(b) and 2(a) show that when harvesting is not considered in the dynamical behavior of populations, the predator and prey will tend to the stable fixed point. From the previous analysis, we know that a certain condition is found, where the interior fixed point becomes globally and asymptotically stable. Harvesting efforts as control variables influence the dynamical behavior of the populations but the behavior is still similar to the behavior of the population model without harvesting. The dynamical behavior for preys with a control seems to increase with a little oscillation, while the dynamical behavior for predator remains decreasing smoothly.

Harvesting efforts as a control in the model make the predator population decline rapidly compared to the non-harvested one, but the predator population remains sustainable because when the population is very small, then the population will stop being harvested. The reduced predator population due to harvesting makes the effect of predation on the prey in the free area become ineffective. This gives an opportunity for the prey population to grow more rapidly. As a consequence, the prey population in the free and forbidden areas for harvesting grow faster than when there are no harvesting efforts



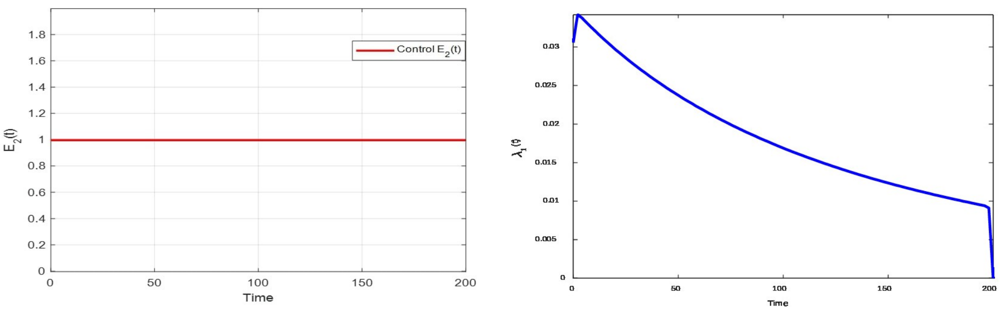
a b

Figure 1: a) plot curve of $x(t)$, b) plot curve of $y(t)$.



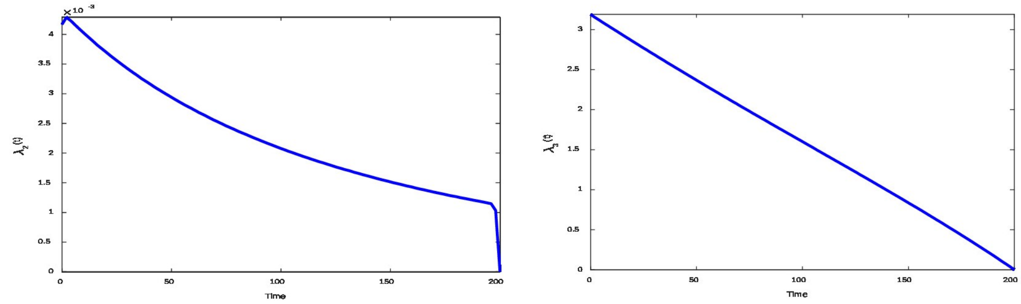
a b

Figure 2: a) plot curve of $z(t)$, b) plot curve of $E_1(t)$.



a b

Figure 3: a) plot curve of $E_2(t)$, b) plot curve of $\lambda_1(t)$.



a b

Figure 4: a) plot curve of $\lambda_2(t)$, b) plot curve of $\lambda_3(t)$.

in the model. In this example, the populations are harvested at the maximum level over the time interval $t \in [0, 200]$, see Figures 2(b) and 3(a). The optimal paths $x^*(t)$, $y^*(t)$, $z^*(t)$, $E_1^*(t)$, and $E_2^*(t)$ maximize the present value J for the problem of finite horizon.

6 Conclusion

The dynamical behavior of preys in the free and forbidden areas of harvesting and predator population with the Holling response function of type II has an interior fixed point when a specific condition is fulfilled. The interior fixed point both for the model with and without harvesting effort was analyzed and it was found that the interior fixed point is locally and globally asymptotically stable. The local stability of the interior fixed point was analyzed via the linearization approach and Routh-Hurwitz stability criteria. The Lyapunov function was constructed under a specific condition to guarantee the global stability of the interior fixed point in the first octant.

In the case of exploitation with the fixed efforts for the predator and the prey populations, there exists an interior fixed point. Under a specific condition, this fixed point becomes globally and asymptotically stable and also gives maximum profit, but the predator population is driven to extinction. By considering that there exists a minimum size of the predator population which is banned to be exploited, we found a pair value of the efforts and the suitable values of parameter to get a globally and asymptotically stable interior fixed point. The stable fixed point also maximizes the profit function for a certain span of time. Both predator and prey populations in the free and forbidden area of capture can be sustainable and also maximize the profit function forever even though the predator and the prey populations in the free area of capture are harvested with fixed efforts of harvesting.

For the problem of maximizing the present value of revenues, there exist extremal paths for harvesting efforts that maximize the present value of net revenues for finite and infinite horizon problems. The harvesting efforts as control variables via simulation show that the harvesting efforts can reduce the predator population and also, at the same time, can reduce the effect of predation on the prey population. The harvesting effect allows the preys to grow rapidly comparing to their growth without harvesting.

Acknowledgment

This research is supported by LPPM Hasanuddin University with the contract number: 915/UN4.22/PT.01.03/2021.

References

[1] L. J. Diblik and D. Khusainov. Stability of an Autonomous System with Quadratic Right-Hand Side in the Critical Case. *Nonlinear Dynamics and Systems Theory* **2** (2) (2002) 145–156.

[2] V.V. Vlasov. Asymptotic Behavior and Stability of the Solutions of Functional Differential Equations in Hilbert Space. *Nonlinear Dynamics and Systems Theory* **2** (2) (2002) 215–232.

[3] Firman, S. Toaha and M. Nur. Asymptotic Stability of Some Class of Affine Nonlinear Control Systems through Partial Feedback Linearization. *Nonlinear Dynamics and Systems Theory* **21** (3) (2021) 238–245.

[4] H. F. Hou, H.M. Jiang and X.Y. Meng. A Dynamic Model for Fishery Resource with Reserve Area and Taxation. *Journal of Applied Mathematics* **2012** (2012) 794719:1–794719:15.

[5] L. Yunfei, R. Yuang and Y. Pei. A Prey-Predator Model with Harvesting for Fishery Resource with Reserve Area. *Applied Mathematical Modelling* **37** (5) (2013) 3048–3062.

[6] M. N. Srinivas, M.A.S. Srinivas, K. Das and N.H. Gazi. Prey-Predator Fishery Model with Stage Structure in Two Patchy Marine Aquatic Environment. *Applied Mathematics* (2) (2011) 1405—1416.

[7] T. K. Kar. A Dynamic Reaction Model of a Prey-Predator System with Stage-Structure for Predator. *Modern Applied Science* **4** (5) (2010) 183—195.

[8] R. P. Gupta, M. Banerje and P. Chandra. Bifurcation Analysis and Control of Leslie–Gower Predator–Prey Model with Michaelis–Menten Type Prey-Harvesting. *Journal Differential Equations and Dynamical Systems* **22** (3) (2012) 339—366.

[9] Y. Qu and J. Wei. Bifurcation Analysis in a Predator-Prey System with Stage Stucture and Harvesting. *Journal of Franklin Institute* **347** (7) (2010) 1096—1113.

[10] K. Chakraborty, M. Chakraborty and T. K. Kar. Optimal Control of Harvest and Bifurcation of Prey-Predator Model with Stage Structure. *Applied Mathematics and Computation* **217** (2011) 8778—8792.

[11] K. Chakraborty, S. Jana and T. K. Kar. Global Dynamics and Bifurcation in a Stage Structured Prey-Predator Fishery Model with Harvesting. *Applied Mathematics and Computation* **218** (5) (2012) 9271—9290.

[12] G. M. V. Laksmi, M. Gunasekaran and S. Vijaya. Bifurcation Analysis of Predator Model with Harvested Predator. *International Journal of Engineering Research and Development* **10** (6), (2014) 42–51.

[13] S. Toaha, J. Kusuma, Khaeruddin and Mawardi. Stability Analysis and Optimal Harvesting Policy of Prey-Predator Model with Stage Structure for Predator. *Applied Mathematical Sciences* **8** (159) (2014) 7923–7934.

[14] L. E. Keshet. *Mathematical Models in Biology.* SIAM, Philadelphia, 2005.

[15] L. Chen and F. Chen. Global Stability of Leslie-Gower Predator-Prey Model with Feedback Controls. *Applied Mathematics Letter* **22** (9) (2009) 1330–1334.

[16] D. Grass, J. P. Caulkins, G. Feichtinger, G. Tragler and D.A. Behrens. *Optimal Control of Nonlinear Processes.* Springer-Verlag, Berlin Heidelberg, 2003.