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The Geometry of Mass Distributions

S. B. Davis *

Research Foundation of Southern California,
8861 Villa La Jolla Drive #13595, La Jolla, CA 92037.

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Abstract: Geometrical characteristics of mass distributions are defined and the relation with classical mechanics and general relativity is described. The classical stability of closed geodesic trajectories on surfaces of arbitrary genus is established. An iterative procedure for solving the N-body problem to a high degree of precision is introduced through a complexity minimization method.

Keywords: *center; geodesics; geometrical complexity; N-body problem.*

Mathematics Subject Classification (2010): 52C45, 58E10, 70S05.

1 Introduction

The equations of classical mechanics and general relativity describe the motion of particle in a geometry of three or four dimensions. The potential in general relativity is derived from the curvature of space-time which results from the energy-momentum tensor or mass distribution. The effect of the geometry of the mass distribution on the dynamics will be considered. A geometrical median will be given and verified for various curves and surfaces. It is proven in the two theorems of Section 2 that the geometrical median of a curve is located on the curve if it is a straight line in Euclidean space and a geodesic in curved space. These theorems remain valid for the barycentre which coincides with the center of mass of a uniform distribution. The role of the center of the mass distribution then will be described in classical mechanics and general relativity. It is known that mass distributions tend towards the center [8]. The local stability of geometrical configurations under the gravitational potential will follow for geodesics.

The stability of geodesics that can be identified with strings on a surface is considered. Given the tendency of uniform mass distributions towards the center of a geometrical configuration, it follows from the theorems of Section 2 that only closed geodesics will

* Corresponding author: mailto:sbdavis@resfndsca.org

be stable on a Riemann surface in a gravitational field. The geodesic flows on elliptic and hyperbolic surfaces are described. The effect of the flow on an elliptic surface is an infinitesimal displacement of the trajectory. On a hyperbolic surface, however, the transformation of the Jacobi field tends to introduce divergences. The geodesics are known to form a dense set on a hyperbolic Riemann surface [27] and the flow is metrically transitive [17]. The dynamics is invariant under quasi-isometries [12]. The stability of the geodesic flow under the action of the fundamental group has been proven for diffusion paths on a geometrially finite hyperbolic surface with finite [24] or infinite volume [10]. Gibbs measures for the dynamics of geodesic flow on negatively curved Riemannian manifolds have been developed to describe the equilibrium state in the presence of a potential [29]. It is related to the Patterson-Sullivan density at the boundary of Teichmüller space, which suffices for the Myrberg limit set with the full measure [33, 37], where singularities occur for an isometry in the interior.

Particles follow geodesics on the space-times shaped by energy-momentum tensors. A measure of geometrical complexity will be defined in Section 3 for spatial curves satisfying conditions of minimal complexity for geodesics and locally extremal values for curves of high symmetry in a neighbourhood in path space. A divergence is found to arise for the sets of points, with the same equivalence class of tangent vectors or covariant derivatives, having zero Lebesgue measure. The occurrence of these infinities is similar to that of the singularities in a theory of gravity or the elementary particles through point particles. A fundamental length scale may be introduced which would require, however, a theoretical basis. The sum representing this term in the intrinsic complexity is rendered finite through the removal of the singular term in a zeta function regularization method. Given this measure of the complexity, the geodesics paths of particles in curved spaces may be derived from an action principle with a Lagrange multiplier term.

The principle of complexity minimization in deterministic processes may be used to establish the time development of a configuration of masses. Its theoretical foundations are enunciated in the first law of classical mechanics and the geodesic free motion in general relativity. It is adapted in Section 3 to predict the dynamics of an N-body system of approximately equal masses, with an iterative procedure of replacing two masses by a single mass at the center of gravity. This subsequent motion can be placed in a general relativistic setting and the geodesics on the curved manifold representing the force fields would tend to reduce complexity of the system. The classical limit then would yield a configuration that also minimizes complexity.

2 The Geometrical Characteristic

The geometrical median of any continuous set S will be defined to be that point a which minimizes $\int_C r(s,a)ds$, where $r(s,a)$ is the distance from the point a to the point $s \in C$. For a discrete set of points, the sum $\sum_s r(s,a)ds$ is minimized [39] and a generalization to continuous sets has been given [13].

For a straight line of length L, $\int_0^L r(s)ds = 2\int_0^{\frac{L}{2}} rdr = 2\frac{\left(\frac{L}{2}\right)^2}{2} = \frac{L^2}{4}$ from the midpoint, while $\int_0^L r(s)ds = \int_0^L rdr = \frac{L^2}{2}$ from the endpoint. For the vertices of an equilateral triangle with sides of length L, the sum of the distances from the center is $3\frac{L}{\sqrt{3}} = \sqrt{3}L$, whereas the sum of the distances from any of the vertices equals $2L$.

From a point at a distance r_0 from the point of symmetry of a circle,

$$r(\theta) = \sqrt{R^2 + r_0^2 - 2r_0 R\, cos\ \theta}, \tag{1}$$

where θ is the angle subtended from the point of symmetry. When $r_0 = 0$, $\int r(\theta)ds = R^2 \int_{\theta=0}^{2\pi} d\theta = 2\pi R^2$. If $r_0 = R$,

$$\int r(\theta)ds = \sqrt{2}R^2 \int_0^{2\pi} (1 - cos\ \theta)^{\frac{1}{2}}\ d\theta = 8R^2. \tag{2}$$

The geometrical median therefore coincides with the conventional definition of the center for these sets of points.

More generally, let $C(t) : [t_0, t_1] \to C[t_0, t_1]$ be a curve in Euclidean space. The distance from a point on the curve is

$$r(t) = \sqrt{(x(t) - x_c)^2 + (y - y_c)^2} \tag{3}$$

and the integral $\int_{t_0}^{t_1} r(t)dt$ is minimized when

$$\delta_{x_c} \int_{t_0}^{t_1} \sqrt{(x(t) - x_c)^2 + (y(t) - y_c)^2} dt = \delta_{y_c} \int_{t_0}^{t_1} \sqrt{(x(t) - x_c)^2 + (y(t) - y_c)^2} dt = 0. \tag{4}$$

Then

$$\int_{t_0}^{t_1} \frac{x(t) - x_c}{\sqrt{(x(t) - x_c)^2 + (y(t) - y_c)^2}}\ dt = 0, \tag{5}$$
$$\int_{t_0}^{t_1} \frac{y(t) - y_c}{\sqrt{(x(t) - x_c)^2 + (y(t) - y_c)^2}}\ dt = 0.$$

The condition for the center to be a point on the curve is that there exists $t' \in [t_0, t_1]$ such that

$$\int_{t_0}^{t_1} \frac{x(t) - x(t')}{\sqrt{(x(t) - x(t'))^2 + (y(t) - y(t'))^2}}\ dt = 0, \tag{6}$$
$$\int_{t_0}^{t_1} \frac{y(t) - y(t')}{\sqrt{(x(t) - x(t'))^2 + (y(t) - y(t'))^2}}\ dt = 0.$$

Theorem 2.1 *The only curves in two-dimensional Euclidean space with geometrical medians located on the curves are straight lines.*

Proof. The conditions (6) can be verified for a straight line $y = mx + b$. Let $x(t) = a_1 t + b_1$ and $y(t) = a_2 t + b_2$ such that $y(t) = \frac{a_2}{a_1}(a_1 t + b_1) + \left(b_2 - \frac{a_2}{a_1} b_1\right)$. Substituting these linear relations into (6) gives

$$\int_{t_0}^{t_1} \frac{a_1(t - t')}{\sqrt{(a_1^2(t - t')^2 + a_2^2(t - t')^2}}\ dt = \frac{a_1}{\sqrt{a_1^2 + a_2^2}} \int_{t_0}^{t_1} \frac{t - t'}{|t - t'|}\ dt \tag{7}$$
$$= \frac{a_1}{\sqrt{a_1^2 + a_2^2}} \int_{t_0}^{t_1} [\theta(t - t') - \theta(t' - t)]\ dt = 0, \tag{8}$$
$$\int_{t_0}^{t_1} \frac{a_2(t - t')}{\sqrt{(a_1^2(t - t')^2 + a_2^2(t - t')^2}}\ dt = \frac{a_2}{\sqrt{a_1^2 + a_2^2}} \int_{t_0}^{t_1} \frac{t - t'}{|t - t'|}\ dt$$
$$= \frac{a_2}{\sqrt{a_1^2 + a_2^2}} \int_{t_0}^{t_1} [\theta(t - t') - \theta(t' - t)]\ dt = 0,$$

which can be satisfied if

$$-(t' - t_0) + (t_1 - t') = 0, \quad t' = \frac{t_0 + t_1}{2}.$$

The use of nonlinear parameters for the straight line does not alter the result.

If $y = mx^\alpha + b$, where $\alpha \neq 1$, then the conditions on t' will not be satisfied by a single value of $t' \in [t_0, t_1]$. Therefore, the center would not occur on the curve for $\alpha \neq 1$. A similar conclusion is reached for a sum of terms with different exponents $\{\alpha_1, ..., \alpha_\ell\}$, where $\alpha_i \neq 1,\ i = 1, ..., \ell$.

The generalization of the definition of the geometric median of the curve would be the point in a manifold which minimizes the integral $\int_{t_0}^{t_1} \sqrt{g_{\mu\nu}(x(t) - x_0^\mu)(x(t) - x_0^\nu)}dt$.

Theorem 2.2 *The geometrical median is located on a curve in a manifold if and only if it is a geodesic.*

Proof. The geodesic extremizes the arc length of the curve $\int_{t_0}^{t_1} \sqrt{g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}}dt$ between two fixed points $x(t_0)$ and $x(t_1)$. Since $\frac{dx^\mu}{dt} = \lim_{t-t' \to 0} \frac{x^\mu(t) - x^\mu(t')}{t-t'}$ for any curve $x(t)$, the arc length equals

$$\lim_{t-t'' \to 0} \int_{t_0}^{t_1} \sqrt{\frac{g_{\mu\nu}(x^\mu(t) - x^\mu(t''))(x^\nu(t) - x^\nu(t''))}{(t-t')^2}}\, dt. \tag{9}$$

Since $\frac{dx^\mu(t)}{dt}$ is a continuous function, one of the two sets of inequalities

$$\begin{aligned} \frac{x^\mu(t) - x^\mu(t-\delta t)}{\delta t} &< \frac{dx^\mu(t)}{dt} < \frac{x^\mu(t+\delta t) - x^\mu(t)}{\delta t}, \\ \frac{x^\mu(t) - x^\mu(t-\delta t)}{\delta t} &> \frac{dx^\mu(t)}{dt} > \frac{x^\mu(t+\delta t) - x^\mu(t)}{\delta t}, \end{aligned} \tag{10}$$

is valid when the second derivative $\frac{d^2x^\mu(t)}{dt^2}$ does not vanish. It follows that, given a positive definite metric, either

$$\int_{t-\frac{\delta t}{2}}^{t} \left(g_{\mu\nu} \frac{(x^\mu(t) - x^\mu(t - \frac{\delta t}{2}))(x^\nu(t) - x^\nu(t - \frac{\delta t}{2}))}{(\frac{\delta t}{2})^2} \right)^{\frac{1}{2}} dt \tag{11}$$

$$< \lim_{t-t' \to 0} \int_{t-\frac{\delta t}{2}}^{t} \sqrt{\frac{g_{\mu\nu}(x^\mu(t) - x^\mu(t'))(x^\nu(t) - x^\nu(t'))}{(t-t')^2}} dt$$

$$\lim_{t-t' \to 0} \int_{t}^{t+\frac{\delta t}{2}} \sqrt{\frac{g_{\mu\nu}(x^\mu(t) - x^\mu(t'))(x^\nu(t) - x^\nu(t'))}{(t-t')^2}} dt$$

$$< \int_{t}^{t+\frac{\delta t}{2}} \left(g_{\mu\nu} \frac{(x^\mu(t + \frac{\delta t}{2}) - x^\mu(t))(x^\nu(t + \frac{\delta t}{2}) - x^\nu(t))}{(\frac{\delta t}{2})^2} \right)^{\frac{1}{2}} dt$$

or

$$\int_{t-\frac{\delta t}{2}}^{t} g_{\mu\nu} \frac{(x^\mu(t) - x^\mu(t-\frac{\delta t}{2}))(x^\nu(t) - x^\nu(t) - \frac{\delta t}{2}))}{(\frac{\delta t}{2})^2} dt \qquad (12)$$

$$> \lim_{t-t'\to 0} \int_{t-\frac{\delta t}{2}}^{t} \frac{g_{\mu\nu}(x^\mu(t) - x^\mu(t'))(x^\nu(t) - x^\nu(t'))}{(t-t')^2} dt,$$

$$\lim_{t-t'\to 0} \int_{t}^{t+\frac{\delta t}{2}} \frac{g_{\mu\nu}(x^\mu(t) - x^\mu(t'))(x^\nu(t) - x^\nu(t'))}{(t-t')^2} dt$$

$$> \int_{t}^{t+\frac{\delta t}{2}} g_{\mu\nu} \frac{(x^\mu(t+\frac{\delta t}{2}) - x^\mu(t))(x^\nu(t+\frac{\delta t}{2}) - x^\nu(t))}{(\frac{\delta t}{2})^2} dt.$$

At the point $x^\mu(t)$, moving an infinitesimal distance δt in any other direction than the tangent vector to the geodesic will increase the integrals in the bounds (10) and (11). Therefore, by eliminating the fixed value of δt, the integral

$$\int_{t-\frac{\delta t}{2}}^{t+\frac{\delta t}{2}} (g_{\mu\nu}(x^\mu(t) - x'^\mu)(x^\nu(t) - x'^\nu))^{\frac{1}{2}} \qquad (13)$$

is minimized with respect to x'^μ, defined by a change of δt in the affine parameter along a curve which is derived by exponentiation of a vector field at the point $x^\mu(t)$, when this curve is the same geodesic $\{x^\mu(s)|t-\frac{\delta t}{2} < s < t+\frac{\delta t}{2}\}$ in the neighbourhood $N_{exp_{\frac{\delta t}{2}}(x^\mu(t))}$. By overlapping neighbourhoods $(t-\frac{\delta t}{2}, t+\frac{\delta t}{2})$ throughout the interval (t_0, t_1), it may be concluded that there exists a point x_0^μ on the path, equal to $x^\mu(t'')$, with t'' fixed, such that the integral

$$\int_{t_0}^{t_1} \sqrt{g_{\mu\nu}(x^\mu(t) - x_0^\mu)(x^\nu(t) - x_0^\nu)} dt \qquad (14)$$

achieves a minimal value.

Suppose that x_0^μ is not located on the geodesic $x(t)$ between $x(t_0)$ and $x(t_1)$. That would be equivalent to the existence of a path $\hat{x}(t)$ including x_0^μ which is not a geodesic between $x(t_0)$ and $x(t_1)$. By triangulation of the interior region between the two curves $x(t)$ and $\hat{x}(t)$, with $x_0^\mu = \hat{x}^\mu(t')$,

$$\int_{t_0}^{t_1} \sqrt{g_{\mu\nu}(\hat{x}^\mu(t) - \hat{x}(t'))(x^\nu(t) - \hat{x}^\nu(t'))}\, dt < \int_{t_0}^{t_1} \sqrt{g_{\mu\nu}(x^\mu(t) - x_0^\mu)(x^\nu(t) - x_0^\nu)}\, dt. \qquad (15)$$

The inequality

$$\int_{t_0}^{t_1} \sqrt{g_{\mu\nu}(\hat{x}^\mu(t) - \hat{x}^\mu(t'))(\hat{x}^\nu(t) - \hat{x}^\nu(t'))}\, dt \qquad (16)$$

$$> \int_{t_0}^{t_1} \sqrt{g_{\mu\nu}(x^\mu(t) - x^\mu(t'''))(x^\nu(t) - x^\nu(t'''))}\, dt$$

for a choice of t''' is valid by the integral form of the mean value theorem and the minimization of the integral by the geodesic. Then

$$\begin{aligned} &\int_{t_0}^{t_1} \sqrt{g_{\mu\nu}(x^\mu(t) - x_0^\mu)(x^\nu(t) - x_0^\nu)}\; dt \\ &\qquad > \int_{t_0}^{t_1} \sqrt{g_{\mu\nu}(x^\mu(t) - x^\mu(t'))(x^\nu(t) - x^\nu(t'))}\; dt \\ &\qquad > \int_{t_0}^{t_1} \sqrt{g_{\mu\nu}(x^\mu(t) - x^\mu(t''')(x^\mu(t) - x^\mu(t''')}\; dt, \end{aligned} \tag{17}$$

again, by triangulation. The inequality may be proven generally by overlapping neighbourhoods of the geodesic. It follows that $x(t''')$ is located on the geodesic, which, therefore, includes its center.

Paths which are not geodesics do not minimize the integral (14) for some x_0^μ on the curve, and therefore, by triangulation, there exists another curve through x_0^μ and the endpoints $x(t_0)$ and $x(t_1)$ which has a lesser integral. A slight perturbation of the second curve will produce a curve with nearly the same integral that does not include x_0^μ. Then x_0^μ will minimize the integral for a curve on which it is not located. Consequently, the only curves which include the geometrical medians are geodesics.

The centroid or center of mass has been defined for regions in Euclidean space and generalized to Riemannian manifolds [2,15,19]. The center of mass of an object occupying a volume in a Euclidean space has coordinates

$$x_{i,c.m} = \frac{\int \rho(x) x_i dV}{\int \rho(x) dV}. \tag{18}$$

The mass density $\rho(x)$ is constant for a uniform distribution and

$$x_{i,\ c.m} = \frac{\rho \int x_i dV}{\rho \int dV} = \frac{\int x_i dV}{\int dV}. \tag{19}$$

The barycenter minimizes the integral of the squared distance [20], [21] from a given point x_0 to the other points in the region

$$\int \sum_i (x_i - x_{i,0})^2 dV. \tag{20}$$

Extremizing this integral requires

$$\delta \int \sum_i (x_i - x_{i,0})^2 dV = 0. \tag{21}$$

Suppose $x_0 = x_{c.m.}$. Then

$$\begin{aligned} -\sum_i \int (x_i - x_{i,0}) \delta x_{i,0} dV &= -\sum_i \int x_i \delta x_{i,0} dV + \sum_i \int \frac{\int x_i dV'}{\int dV'} \delta x_{i,0} dV \\ &= -\sum_i \int x_i \delta x_{i,0} dV + \sum_i \int x_i \delta x_{i,0} dV \\ &= 0. \end{aligned}$$

Therefore, the center of mass of a uniform distribution coincides with the barycentre of the geometric configuration. It may be verified that the barycenter of a straight line $x(t) = a_1 t + b_1$ and $y(t) = a_2 t + b_2$ between two points $(x(t_0), y(t_0))$ and $(x(t_1), y(t_1))$ occurs at $\frac{t_0+t_1}{2}$. Similarly, the barycenter of a geodesic will be located on the geodesic since the integral $\int_{t_0}^{t_1} g_{\mu\nu}(x^\mu(t) - x_0^\mu)(x^\nu(t) - x_0^\nu)\, dt$ will be minimized when $x_0^\mu = x^\mu(t'')$, $t'' \in [t_0, t_1]$.

The geometrical median may be compared with the barycenter for various compact sets including the circle [4, 40]. The variational conditions for this integral in Euclidean space are

$$\delta_{x_c} \int_{t_0}^{t_1} [(x(t) - x_c)^2 + (y(t) - y_c)^2] dt = 0, \qquad \delta_{y_c} \int_{t_0}^{t_1} [(x(t) - x_c)^2 + (y(t) - y_c)^2] dt = 0,$$

or

$$\int_{t_0}^{t_1} (x(t) - x_c) dt = 0, \qquad \int_{t_0}^{t_1} (y(t) - y_c) dt = 0.$$

These equations generally differ from Eq. (5). For the circle, with (x_c, y_c) located at the center and $\sqrt{(x(t) - x_c)^2 + (y(t) - y_c)^2}$ equal to a constant, the conditions are equivalent.

The tendency of uniform mass distributions towards the centers would cause the linear density of a geodesic on a Riemann surface to move towards a point on the curve. When it is closed, there are no distinguished points on the geodesic, which should be stable against variations satisfying classical equations. It may be identified, therefore, with a closed string state. However, a closed curve that is not a geodesic would have a center of mass located away from the path, and if it tends towards this point, the configuration will not be stable. It follows that there is an equivalence between closed string states and closed geodesics only.

The linear Poincare mapping of a closed geodesic translates the Jacobi field and its covariant derivative from one curve to another. The eigenvalue of this transformation has magnitude one when the geodesic is elliptic and stable and it is not equal to one if the geodesic is hyperelliptic and unstable [31]. This variation does not cause a geodesic to disintegrate. Instead, it is moved to a neighbourhood in the first class and diverges in the second category. The transformation only would represent a form of propagation of closed string states along the surface. Consequently, the geodesic flows differ at genus $g = 1$ and $g \geq 2$.

3 Complexity of a Curve and the Relation to the Center

Consider the Frenet frame of a curve spanned by the tangent, normal and binormal vectors $\vec{t}$, $\vec{n}$ and $\vec{b}$ and the resultant $\vec{v} = \vec{t} + \vec{n} + \vec{b}$. The integrals

$$\frac{-\int_\gamma ds \left[\frac{(\vec{v}\cdot\vec{t})^2}{|\vec{v}|^2} ln\ \frac{(\vec{v}\cdot\vec{t})^2}{|\vec{v}|^2} + \frac{(\vec{v}\cdot\vec{n})^2}{|\vec{v}|^2} ln\ \frac{(\vec{v}\cdot\vec{n})^2}{|\vec{v}|^2} + \frac{(\vec{v}\cdot\vec{b})^2}{|\vec{v}|^2} ln\ \frac{(\vec{v}\cdot\vec{n})^2}{|\vec{v}|^2} \right]}{\int ds} \tag{22}$$

and

$$\frac{1}{\int ds}\Bigg[-\int_\gamma ds\Bigg[\frac{(\nabla_{\vec{t}}\vec{t}\cdot\hat{t})^2}{|\nabla_{\vec{t}}\vec{t}|^2}ln\left(\frac{(\nabla_{\vec{t}}\vec{t}\cdot\hat{t})^2}{|\nabla_{\vec{t}}\vec{t}|^2}\right)+\frac{(\nabla_{\vec{t}}\vec{t}\cdot\hat{n})^2}{|\nabla_{\vec{t}}\vec{t}|^2}ln\left(\frac{(\nabla_{\vec{t}}\vec{t}\cdot\hat{n})^2}{|\nabla_{\vec{t}}\vec{t}|^2}\right) + \frac{(\nabla_{\vec{t}}\vec{t}\cdot\hat{b})^2}{|\nabla_{\vec{t}}\vec{t}|^2}ln\left(\frac{(\nabla_{\vec{t}}\vec{t}\cdot\hat{b})^2}{|\nabla_{\vec{t}}\vec{t}|^2}\right)\Bigg]\tag{23}$$

increase with the number of nonrepetitive windings of a spatial curve. When the curve γ is a geodesic, $\nabla_{\vec{t}}\vec{t}=0$ or it is proportional to $\vec{t}$ with a change of the affine parameter and $ln\frac{(\nabla_{\vec{t}}\vec{t}\cdot\hat{t})^2}{|\nabla_{\vec{t}}\vec{t}|^2}=0$, the integral vanishes.

The first integral for the circle is maximized amongst planar curves. Given the coordinates and the tangent vector

$$\begin{aligned}(x(t),y(t))&=(a\ cos\ t,\ a\ sin\ t),\\ \left(\frac{dx}{dt},\frac{dy}{dt}\right)&=(-a\ sin\ t,\ a\ cos\ t),\end{aligned}\tag{24}$$

the normal vector is $\vec{n}=(-a\ cos\ t,-a\ sin\ t)$. Then

$$\begin{aligned}\vec{v}&=(-a(sin\ t+cos\ t),\ a(cos\ t-sin\ t)),\\ |\vec{t}+\vec{n}|^2&=a^2((sin\ t+cos\ t)^2+(cos\ t-sin\ t)^2)=2a^2\end{aligned}\tag{25}$$

and

$$\begin{aligned}\vec{t}\cdot\vec{v}&=a^2,\\ \vec{n}\cdot\vec{v}&=a^2.\end{aligned}\tag{26}$$

It follows that

$$\frac{-\int_\gamma ds\left[\frac{(\vec{v}\cdot\vec{t})^2}{|\vec{v}|^2}ln\ \frac{(\vec{v}\cdot\vec{t})^2}{|\vec{v}|^2}+\frac{(\vec{v}\cdot\vec{n})^2}{|\vec{v}|^2}ln\ \frac{(\vec{v}\cdot\vec{n})^2}{|\vec{v}|^2}\right]}{\int ds}=\frac{2\pi a\left(-\frac{1}{2}ln\ \frac{1}{2}-\frac{1}{2}ln\ \frac{1}{2}\right)}{2\pi a}=ln\ 2.\tag{27}$$

If a series expansion $\sum_k\frac{1}{k!}I_k$ is considered, where

$$\begin{aligned}&I_k\\ &=\frac{-\int_C\left[\frac{\vec{t}^{(k)}\cdot\vec{v}^{(k)}}{|\vec{v}^{(k)}|^2}ln\left(\frac{\vec{t}^{(k)}\cdot\vec{v}^{(k)}}{|\vec{v}^{(k)}|^2}\right)+\frac{\vec{n}^{(k)}\cdot\vec{v}^{(k)}}{|\vec{v}^{(k)}|^2}ln\left(\frac{\vec{n}^{(k)}\cdot\vec{v}^{(k)}}{|\vec{v}^{(k)}|^2}\right)+\frac{\vec{b}^{(k)}\cdot\vec{v}^{(k)}}{|\vec{v}^{(k)}|^2}ln\left(\frac{\vec{b}^{(k)}\cdot\vec{v}^{(k)}}{|\vec{v}^{(k)}|^2}\right)\right]ds}{\int_C ds},\end{aligned}\tag{28}$$

it may be verified that, by the Frenet equations,

$$
\begin{aligned}
\vec{t}' &= \frac{d|\vec{t}|}{ds}\hat{t} + \kappa|\vec{t}|\hat{n}, \\
\vec{n}' &= -\kappa|\vec{n}|\hat{t} + \frac{d|\vec{n}|}{ds}\hat{n} + \tau|\vec{n}|\hat{b}, \\
\vec{b}' &= -\tau|\vec{b}|\hat{n} + \frac{d|\vec{b}|}{ds}\hat{b}, \\
\vec{v}' &= \left(\frac{d|\vec{t}|}{ds} - \kappa|\vec{n}|\right)\hat{t} + \left(\kappa|\vec{t}| + \frac{d|\vec{n}|}{ds} - \tau|\vec{b}|\right)\hat{n} + \left(\tau|\vec{n}| + \frac{d|\vec{b}|}{ds}\right)\hat{b},
\end{aligned}
\tag{29}
$$

and

$$
\begin{aligned}
I_1 = -\frac{1}{\int_C ds}\int_C ds \Bigg\{ & \frac{\left[\frac{d|\vec{t}|}{ds}\left(\frac{d|\vec{t}|}{ds} - \kappa|\vec{n}|\right) + \kappa|\vec{t}|\left(\kappa|\vec{t}| + \frac{d|\vec{n}|}{ds} - \tau|\vec{b}|\right)\right]}{\left[\left(\frac{d|\vec{t}|}{ds} - \kappa|\vec{n}|\right)^2 + \left(\kappa|\vec{t}| + \frac{d|\vec{n}|}{ds} - \tau|\vec{b}|\right)^2 + \left(\tau|\vec{n}| + \frac{d|\vec{b}|}{ds}\right)^2\right]} \\
& ln\left[\frac{\left[\frac{d|\vec{t}|}{ds}\left(\frac{d|\vec{t}|}{ds} - \kappa|\vec{n}|\right) + \kappa|\vec{t}|\left(\kappa|\vec{t}| + \frac{d|\vec{n}|}{ds} - \tau|\vec{b}|\right)\right]}{\left[\left(\frac{d|\vec{t}|}{ds} - \kappa|\vec{n}|\right)^2 + \left(\kappa|\vec{t}| + \frac{d|\vec{n}|}{ds} - \tau|\vec{b}|\right)^2 + \left(\tau|\vec{n}| + \frac{d|\vec{b}|}{ds}\right)^2\right]}\right] \\
& + \frac{\left[-\kappa|\vec{n}|\left(\frac{d|\vec{t}|}{ds} - \kappa|\vec{n}|\right) + \frac{d|\vec{n}|}{ds}\left(\kappa|\vec{t}| + \frac{d|\vec{n}|}{ds} - \tau|\vec{b}|\right) + \tau|\vec{n}|\left(\tau|\vec{n}| + \frac{d|\vec{b}|}{ds}\right)\right]}{\left[\left(\frac{d|\vec{t}|}{ds} - \kappa|\vec{n}|\right)^2 + \left(\kappa|\vec{t}| + \frac{d|\vec{n}|}{ds} - \tau|\vec{b}|\right)^2 + \left(\tau|\vec{n}| + \frac{d|\vec{b}|}{ds}\right)^2\right]} \\
& ln\left[\frac{\left[-\kappa|\vec{n}|\left(\frac{d|\vec{t}|}{ds} - \kappa|\vec{n}|\right) + \frac{d|\vec{n}|}{ds}\left(\kappa|\vec{t}| + \frac{d|\vec{n}|}{ds} - \tau|\vec{b}|\right) + \tau|\vec{n}|\left(\tau|\vec{n}| + \frac{d|\vec{b}|}{ds}\right)\right]}{\left[\left(\frac{d|\vec{t}|}{ds} - \kappa|\vec{n}|\right)^2 + \left(\kappa|\vec{t}| + \frac{d|\vec{n}|}{ds} - \tau|\vec{b}|\right)^2 + \left(\tau|\vec{n}| + \frac{d|\vec{b}|}{ds}\right)^2\right]}\right] \\
& + \frac{\left[-\tau|\vec{b}|\left(\kappa|\vec{t}| + \frac{d|\vec{n}|}{ds} - \tau|\vec{b}|\right) + \frac{d|\vec{b}|}{ds}\left(\tau|\vec{n}| + \frac{d|\vec{b}|}{ds}\right)\right]}{\left[\left(\frac{d|\vec{t}|}{ds} - \kappa|\vec{n}|\right)^2 + \left(\kappa|\vec{t}| + \frac{d|\vec{n}|}{ds} - \tau|\vec{b}|\right)^2 + \left(\tau|\vec{n}| + \frac{d|\vec{b}|}{ds}\right)^2\right]} \\
& ln\left[\frac{\left[-\tau|\vec{b}|\left(\kappa|\vec{t}| + \frac{d|\vec{n}|}{ds} - \tau|\vec{b}|\right) + \frac{d|\vec{b}|}{ds}\left(\tau|\vec{n}| + \frac{d|\vec{b}|}{ds}\right)\right]}{\left[\left(\frac{d|\vec{t}|}{ds} - \kappa + \frac{d|\vec{n}|}{ds} - \tau|\vec{b}|\right)^2 + \left(\tau|\vec{n}| + \frac{d|\vec{b}|}{ds}\right)^2\right]}\right]\Bigg\}.
\end{aligned}
\tag{30}
$$

For the circle,

$$
\begin{aligned}
I_1 & \\
&= -\frac{1}{\int ds}\int ds\left[\frac{\kappa^2|\vec{t}|^2}{\kappa^2|\vec{t}|^2 + \kappa^2|\vec{n}|^2} ln\frac{\kappa^2|\vec{t}|^2}{\kappa^2|\vec{t}|^2 + \kappa^2|\vec{n}|^2} + \frac{\kappa^2|\vec{n}|^2}{\kappa^2|\vec{t}|^2 + \kappa^2|\vec{n}|^2} ln\frac{\kappa^2|\vec{n}|^2}{\kappa^2|\vec{t}|^2 + \kappa^2|\vec{n}|^2}\right] \\
&= -\frac{1}{\int ds}\int ds\left[\frac{1}{2}ln\frac{1}{2} + \frac{1}{2}ln\frac{1}{2}\right] = ln\ 2,
\end{aligned}
\tag{31}
$$

since $|\vec{t}| = |\vec{n}| = a$, $\frac{d|\vec{t}|}{ds} = \frac{d|\vec{n}|}{ds} = 0$ and $|\vec{b}| = 0$. Given equal magnitudes of the integrals I_k, $k \geq 0$, the entire measure would be $\sum_{k=0}^{\infty} \frac{1}{k!} ln\ 2 = e\ ln\ 2$, which is the maximal bound for planar curves.

The second integral is significantly reduced because the projection of the covariant derivative of the tangent vector onto the vectors would be given in a polar diagram by

$$\begin{aligned} \hat{t} &= \frac{1}{r}\hat{\theta}, \qquad \hat{n} = \hat{r}, \\ \nabla_{\vec{t}}\vec{t} &= \Gamma^{\theta}{}_{\theta\theta}\hat{\theta} + \Gamma^{r}{}_{\theta\theta}\hat{r}, \\ \Gamma^{\theta}{}_{\theta\theta} &= 0, \qquad \Gamma^{r}{}_{\theta\theta} = \frac{1}{2}g^{rr}(g_{\theta\theta,r}) = r. \end{aligned} \tag{32}$$

Then

$$\mathcal{C}_{curve} = -\int ds \frac{(\nabla_{\vec{t}}\vec{t}\cdot\hat{n})^2}{|\nabla_{\vec{t}}\vec{t}|^2} ln\left(\frac{(\nabla_{\vec{t}}\vec{t}\cdot\hat{n})^2}{|\nabla_{\vec{t}}\vec{t}|^2}\right) = 0. \tag{33}$$

Even though the circle has maximal symmetry, it is not a geodesic in Euclidean space, and therefore, given the changing direction of the tangent vector, the introduction of a non-zero measure, less than that of neighbouring winding curves, may be considered.

The intrinsic complexity of a curve has been defined to be

$$\begin{aligned} &\sum_{k=1}^{\infty} \frac{1}{k!}\mathcal{C}_{int}^{(k)}, \\ &\mathcal{C}_{int}^{(k)} = -\int_{\mu=L_\delta} \frac{f_k(\ell)d\ell}{L} ln \frac{f_k(\ell)\ell_{i'\ min}^{(k)}}{L} - \sum_{i'} \frac{\ell_{i'}^{(k)}}{L} ln\ \frac{\ell_{i'}^{(k)}}{L}, \end{aligned} \tag{34}$$

where $f_k(\ell)$ equals the finite number of times that the $(k-1)^{th}$ covariant derivative of the tangent vector can be identified, the index i' labels arcs with identified $(k-1)^{th}$ derivatives and $\ell_{i'\ min}$ is the minimum length of these arcs of non-zero measure [9]. When the curve is a geodesic, $\nabla_{\vec{t}}\vec{t} = 0$, and the $(k-1)^{th}$ derivatives vanish for $k \geq 2$ and $\mathcal{C}_{int} = \mathcal{C}_{int}^{(1)} = 0$ because the tangent vectors may be identified through parallel transport. The angular component of this expression has been evaluated for a circle to be non-zero, while the radial component is found to vanish [9], representing a local minimum amongst neighbouring paths. If $\ell_{i'\ min} \neq 0$, it would be proportional to the arc length of the curve since a dilation of the curve increases $\ell_{i'\ min}$ and the length L by the same factor. When there are no points that can be identified, the second sum vanishes and $\ell_{i'\ min}$ would be set equal to $\overset{\circ}{\delta\ell}L$, where $\overset{\circ}{\delta\ell} = \frac{\delta\ell}{[\delta\ell]}$, which causes a divergence as $\delta\ell \to 0$. This infinity can be removed from the formula by equating $\ell_{i'\ min}$ to a constant for these curves, yielding a dependence on L that breaks dilatational invariance. Another possibility for $\ell_{i'\ min}$ would be $\lambda_\delta L$, where λ_δ is constant. Then, although dilatational invariance is preserved, the formula includes an arbitrary constant with no theoretical basis.

The measure $\delta\ell\ ln\ \overset{\circ}{\delta\ell}$, however, tends to zero, in this limit. Suppose that the variable η is defined by

$$\delta\eta = -\delta\ell\ ln\ \overset{\circ}{\delta\ell}. \tag{35}$$

The integral $\int_0^L d\ell = L$ may be regarded as the limit of a Riemann sum $\sum_{i=1}^{\{\frac{L}{\delta\ell}\}} 1\cdot\delta\ell =$

$\left\{\frac{L}{\delta\ell}\right\}\delta\ell$. By contrast, the sum of the infinitesimals $\frac{\delta\eta}{L}$ equals

$$-\sum_{i=1}^{\{\frac{L}{\delta\ell}\}} \frac{1}{L}\cdot \delta\ell\ ln\ \overset{\circ}{\delta\ell} = -\frac{1}{L}\left\{\frac{L}{\delta\ell}\right\}\delta\ell\ ln\ \overset{\circ}{\delta\ell} = -ln\ \overset{\circ}{\delta\ell}. \tag{36}$$

Given the approximation $\psi(z) \sim ln\ z$ for $z \gg 1$, this value may be replaced by

$$\sum_{k=1}^{\left\{\frac{\overset{\circ}{L}}{\overset{\circ}{\delta\ell}}\right\}} \frac{1}{k} - ln\ L \simeq \sum_{k=\{\overset{\circ}{L}\}}^{\{\frac{\overset{\circ}{L}}{\overset{\circ}{\delta\ell}}\}} \frac{1}{k}. \tag{37}$$

Independence with respect to L requires equality with $\sum_{k=1}^{\{\frac{1}{\overset{\circ}{\delta\ell}}\}}\frac{1}{k}$. Then, $\lim_{\overset{\circ}{\delta\ell}\to 0}\sum_{k=1}^{\{\frac{1}{\overset{\circ}{\delta\ell}}\}}\frac{1}{k} = \lim_{s\to 1}\zeta(s)$. Zeta function regularization would consist of removing the singular term in the expansion of the zeta function around $s = 1$, yielding $\lim_{s\to 1}\left[\zeta(s) - \frac{1}{s-1}\right] = \gamma$.

At a point (x, y) on the circle $x^2 + y^2 = r^2$, the polar coordinates are (r, θ), with r equal to a constant. The tangent vector has components $(-y, x)$ and

$$\begin{aligned}\vec{t} &= -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} = -y\left(\frac{x}{r}\frac{\partial}{\partial r} - \frac{y}{r^2}\frac{\partial}{\partial\theta}\right) + x\left(\frac{y}{r}\frac{\partial}{\partial r} + \frac{x}{r^2}\frac{\partial}{\partial\theta}\right) \\ &= \frac{xy - xy}{r}\frac{\partial}{\partial r} + \frac{x^2+y^2}{r^2}\frac{\partial}{\partial\theta} = \frac{\partial}{\partial\theta}.\end{aligned} \tag{38}$$

The r component of the gradient is $\frac{\partial}{\partial r}$, while the θ component is $\frac{1}{r}\frac{\partial}{\partial\theta}$. Therefore, the components of the tangent vector in this basis are $(0, r)$. There is no radial component of the tangent vector, the theta component is constant, and yet, the vector $\frac{\partial}{\partial\theta}$ keeps changing with θ_0 at the points $(1, \theta_0)$ since

$$\left.\frac{\partial}{\partial\theta}\right|_{\theta_0} = -rsin\ \theta_0\frac{\partial}{\partial x} + rcos\theta_0\frac{\partial}{\partial y} \tag{39}$$

in contrast with the fixed unit vectors $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$. The vanishing of the radial component of the tangent vector to the circle is sufficient to ensure a local minimum for $\mathcal{C}_r^{int}$ for this curve, while the angular component yields a non-zero value, without any further identification of the tangent vectors through Euclidean motions of the plane. The non-zero value is supported by the work that is required to move an object travelling at a constant velocity in a circular path, by contrast with a straight trajectory. It may be noted that this feature is evident also if the distances from the center constitute the sequence for the radial complexity. More generally, it would be necessary to evaluate the perpendicular component of the distance. Then, it would be equal to zero from the center to any other point on a straight line. Since the center is located on the geodesic in curved space by Theorem 2, it would follow that the perpendicular component of the distance to any point on this path and the radial complexity with respect to the center would vanish.

The reduction of the N-body problem to an $(N-1)$-body problem through the replacement of two masses by another centrally located mass introduces an approximation in the description of the motion [26]. The error will be increased by the method of induction culminating in a three-body problem, which has an analytic formulation and can be solved in the plane, where it is equivalent to a system of geodesic equations [34]. It may be reduced through a complexity minimization procedure [9]. The tendency of mass distributions to the center is a transition to a more symmetric and less complex distribution about this point. The introduction of this variational principle selects a classical configuration with each error range.

Theorem 3.1 *The final motions in an N-body problem may be reduced to a three-body problem for equal masses under the condition of the minimization of the complexity of the configuration.*

Proof. Lagrange multiplier terms may be added to give

$$L_N = \frac{1}{2}m\sum_i\sum_{k=1}^{N}\dot{x}_{i,k}^2 + Gm^2\sum_{k<\ell}^{N}\frac{1}{r_{k\ell}} + \sum_j \lambda_j \sum_{k=1}^{N}(x_{j,k}(t) - x_{c,N}(t)). \tag{40}$$

The minimization of complexity of the configuration of N masses is equivalent to the extremization of the sum of the distances to the center of mass for this system. When the masses of two bodies are replaced by the combined mass $m_{N-1,N}$, the geometrical center must be replaced by the center of mass $x_{c.m.,N-1}$ [18], and given a tendency towards this point, the Lagrangian may be formulated to be

$$\begin{aligned} L_{N-1} = {} & \frac{1}{2}m\sum_i\sum_{k=1}^{N-2}\dot{x}_{i,k}^2 + \frac{1}{2}m_{N-1,N}\sum_i \dot{x}_{i,(N-1,N)}^2 + Gm^2\sum_{k<\ell}^{N-2}\frac{1}{r_{k\ell}} \\ & + Gmm_{N-1,N}\sum_{k=1}^{N-2}\frac{1}{r_{k,(N-1,N)}} \\ & + \sum_j \lambda_j \left[\sum_{k=1}^{N-2}(x_{j,k}(t) - x_{c.m,N-1}(t)) + (x_{j,(N-1,N)}(t) - x_{c.m.,N-1})\right]. \end{aligned} \tag{41}$$

This averaging technique may be applied to the equations of motion derived from the Lagrangian, the nonlinear equations may be formulated with generalized derivatives which yield estimates of deviations from the exact configurations and ensure existence and convergence to the solution [25].

Since there exists one mass in the new configuration with a different magnitude, the minimization of complexity would not coincide exactly with a tendency towards the center of mass. Nevertheless, the process can be continued over extended intervals progressing to a state of minimum complexity approximated by a tendency towards the location $x_{0,K}$ near the center of mass $x_{c.m.,K}$. Gradient transformation differential equation algorithms have been developed for the minimization of a scalar function that may be identified presently with the complexity [14].

Iteration of the process yields the Lagrangian

$$L_K = \frac{1}{2}m\sum_i\sum_{k=1}^{K-1}\dot{x}_{i,k}^2 + \frac{1}{2}m_{K,\dots,N}\sum_i \dot{x}_{i,(K,\dots,N)}^2 + Gm^2\sum_{k<\ell}^{K-1}\frac{1}{r_{k\ell}} \qquad (42)$$
$$+ Gmm_{K,\dots,N}\sum_{k=1}^{K-1}\frac{1}{r_{k,(K,\dots,N)}}$$
$$+\sum_j \lambda_j\left[\sum_{k=1}^{K-1}(x_{j,k}(t) - x_{0,K}(t)) + (x_{j,(K,\dots,N)} - x_{0,K}(t))\right],$$

where $m_{K,\dots,N}$ is the combined mass replacing the masses of $N-K+1$ bodies, $\{x_{i,(K,\dots,N)}\}$ is the location of center of mass for this system and $x_{c.m.,K}$ is the center of mass derived from the $m_{K,\dots,N}$ and the remaining $K-1$ masses.

When $K=3$,

$$L_3 = \frac{1}{2}m\sum_i\sum_{k=1}^{2}\dot{x}_{i,k}^2 + \frac{1}{2}m_{3.\dots,N}\sum_i \dot{x}_{i,(3,\dots,N)}^2 + Gm^2\sum_{k<\ell}^{2}\frac{1}{r_{k\ell}} \qquad (43)$$
$$+ Gmm_{3,\dots,N}\sum_{k=1}^{K-1}\frac{1}{r_{k,(3,\dots,N)}}$$
$$+\sum_j \lambda_j\left[\sum_{k=1}^{2}(x_{j,k}(t) - x_{0,3}(t)) + (x_{j,(3,\dots,N)} - x_{0,3}(t))\right]$$

with $m_{3,\dots,N}$ being the combined mass for $N-2$ bodies, $\{x_{i,(3,\dots,N)}\}$ is the center of mass for this system and $x_{0,3}$ is an attractor for the configuration of minimal complexity for the mass $m_{3.,\dots,N}$ and the two masses at x_1 and x_2, amongst those motions that are allowed by the equations of motion. The equations derived from this Lagrangian would be solvable.

Series solutions to the three-body problem [35] and the N-body problem [38] converge sufficiently slowly, and approximations are necessary over brief time intervals. The general instability of solutions in the nonhierarchical three-body problem, where there is a stratification of the masses and distances, requires statistical methods for a theoretical solution. The method derived from Theorem 3 would allow the errors to be reduced over longer time intervals, especially through stable repeating trajectories including the Lagrange-Euler family of solutions for three masses [11,23]. The replacement of two masses by another mass at the center of gravity resembles the description of the restricted three-body problem as a two-point boundary value problem [30]. The approximation introduced in this theorem would increase in precision given a longer time interval. It is necessary, therefore, to minimize the error for each replacement by the center of mass.

The complexity minimization principle provides a method for determining final states of classical systems. These motions in the three-body problem have been classified, including existence of five relative equilibria representing planar central configuration [5]. The classification may be extended to the N-body problem qualitatively [28], and the addition of a mass yields only finitely many relative equilibria [16], the finiteness of the number of equivalence classes of these critical points has not been determined generally for $N > 3$. By the above theorem, however, it would follow that this number is finite for equal masses since the calculation can be reduced to a three-body problem through an iteration of an algorithm consisting of the replacement of the location of two masses by the center of mass.

The simplification of calculations resulting from a minimization of complexity is similar to that of the virial theorem, equating the average total kinetic energy with half of the negative of the Newtonian potential energy [3]. The generalization of relative equilibrium motions to G-equivariant motions yields a classification of planar three-body motions by the symmetry groups. Given the action for a certain class of motions, its minimization for a certain subset of motions can be determined. It is found, for example, that the minimum of the action amongst motions with an isosceles symmetry of order 2 is achieved by the Lagrange configurations with a discrete invariance group of order 6 [36]. By contrast, the minimization of the action for choreographic motions is found to be given by relative equilibrium motion corresponding to a regular n-gon. The characterization of stable solutions to the N-body problem by symmetry has generated a classification of the equilibria [7]. The minimization of the gravitational action [6] may be supplemented by that of the complexity, which may be combined with integration techniques to give a description of the dynamics [1].

4 Conclusion

The complexity of a path in curved space would be minimized by geodesics. Prime geodesics are represented by closed curves on surfaces with handles. Consequently, it follows that the geodesic trajectories in two dimensions could represent the propagation of closed strings only on Riemann surfaces of arbitrary genus. The other curves would be unstable against classical perturbations given a tendency toward the center.

The consistency of the dynamics of closed string theory and gravitation therefore follows from the motion of free particles along geodesics on a metric which is a solution to the gravitational field equations. The string effective field equations that tend to the equations of general relativity coupled to matter in the classical limit represent conditions for the quantum conformal invariance. Given the propagation of the quantum string along the surface, the equilibrium configuration of the geodesic can be derived from the variation of an action that includes a Lagrange multiplier term for the minimization of complexity [9]. This auxiliary condition may be transferred from the worldsheet to geodesic motion in the embedding space.

The dynamics of mass distributions in classical mechanics and general relativity then can be described, given the condition of minimization of complexity. The initial motion of N masses in a gravitational field may be formulated in terms of an $(N-1)$-body problem after two masses are replaced by the combined system at the center of mass. The subsequent coordinates then can be computed by requiring the complexity of the configuration to be minimized, which would include the classical limit of geodesic trajectories in a manifold curved by a gravity. Iteration of this procedure eventually produces a Lagrangian for the solvable three-body problem. Then the motion of the N bodies is predicted by separating the combined masses and determining the time evolution of this and subsequent configurations along the geodesics on the curved manifold representing the gravitational field of the remaining masses.

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