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Performance Comparison of Some Two-Dimensional Chaotic Maps for Global Optimization

A. Djaout [1], T. Hamaizia [1*] and F. Derouiche [2]

[1] *Department of Mathematics, Faculty of Exact Sciences, University of Constantine 1, Algeria.*
[2] *Department of Mathematics, Faculty of Exact Sciences and Science of Nature and Life, University of Oum El-Bouaghi, Algeria.*

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Abstract: This paper studied the performance of a new class of evolutionary algorithms called the chaos optimization algorithms (COA). It was originally proposed to solve nonlinear optimization problems with bounded variables by Caponetto et al. [1, 2]. Different chaotic mappings have been considered, combined with several working strategies. We propose four different 2-D chaotic maps in the optimization algorithm using a two-stage chaos optimization method and compare them. This study surveys and compares the chaotic optimization algorithms in the literature. Furthermore, a two-phase strategy is a technique commonly used in the COA to fine tune the solution and help escaping from local optimums. The performance study is conducted to understand their impact on the chaos optimization algorithm.

Keywords: *chaos; global optimization; chaotic map; chaos optimization algorithm.*

Mathematics Subject Classification (2010): 34D45, 70K55.

1 Introduction

The existence of chaotic systems is an accepted fact of science [3]. Chaos is a kind of characteristics of nonlinear systems and chaos theory studies the behavior of systems that follow deterministic laws but appear random and unpredictable. This theory brings many qualitative and quantitative tools, namely, ergodicity, entropy, expansivity, and sensitive dependence on initial conditions. Theory of chaos, since its evolution, has found application in various important areas such as engineering, medicine, biology, economy and many others. The application of the Chaotic Search strategy in engineering had its peak of popularity over the last few years [3–8]. This approach configured as an

* Corresponding author: mailto:el.tayyeb@umc.edu.dz

attractive option for global optimization. One of the modern optimization algorithms is the chaos-based optimization [9, 10].

The chaos optimization search as a novel method of global optimization has attracted much attention in nonlinear fields. The chaos optimization algorithm (COA) is an effective way to solve the optimization problem of a nonlinear multimodal function with boundary constraint. Due to the nonrepetition of chaos, it can carry out overall searches at higher speeds than stochastic searches, which depend on probabilities. The application of chaotic sequences instead of random sequences in the COA is a powerful strategy to improve the COA's performance in preventing premature convergence to local minima [11, 14].

In the present paper, a robust chaos optimization algorithm is applied to efficiently solve the problem of optimizing a nonlinear multimodal function. In most of the chaos optimization algorithms, chaos variables are generated by logistic mapping [15, 16], but the uneven distribution will weaken the ergodicity of chaos variables. To overcome this problem, we select 5 different two-dimensional maps and replace the chaos variable generator in one of the existing COAs [17–20] with them. The remainder of this paper is organized as follows. Section 2 is made for Chaotic maps. Then in Section 3, the chaos optimization algorithm is introduced, experiments and simulation results are shown in Section 4, and finally, the conclusion is presented in Section 5.

2 Two-Dimensional Maps

Non-linear systems with complex dynamics have lately been the subject of intense research and exploration, giving birth to *chaos theory.* Chaotic systems are deterministic systems that exhibit irregular behavior and a sensitive dependence on the initial conditions. Chaos theory studies the behavior of systems that follow deterministic laws but appear random and unpredictable, i.e., dynamical systems. Chaotic variables can go through all states in certain ranges according to their own regularity without repetition [3, 8].

A chaotic map is a map that exhibits some type of chaotic behavior. In this work, we applied five different chaotic maps that are common in the literature, namely, the Hénon map, Lozi map, Duffing map, Gingerbreadman map, and Zeraoulia map. The mathematical form of a chaotic two-dimensional map, which maps the unit square $I \times I$, where $I = [0, 1]$, onto itself in a one-to-one manner, is chosen.

Later on, we will use these maps in the chaotic searches.

2.1 The Hénon map

The Hénon map is a discrete-time dynamical system [21]. It is one of the most studied examples of dynamical systems that exhibit chaotic behaviour. The Hénon map takes a point (x_n, y_n) in the plane and maps it to a new point

$$\begin{cases} y_1(k) = 1 - a(y_1(k-1))^2 + by(k-1), \\ y(k) = y_1(k-1), \end{cases} \tag{1}$$

where k is the iteration number.

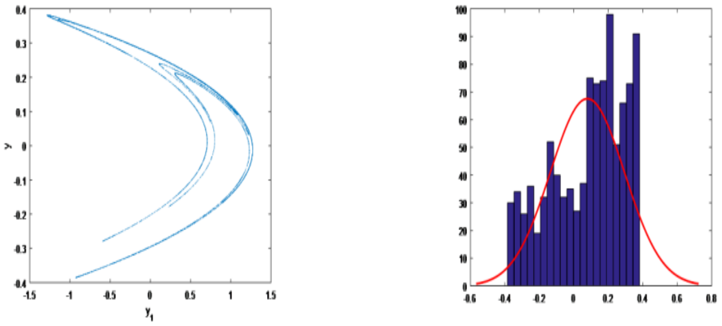

Figure 1: A chaotic Hénon attractor obtained for $a = 1.4$ and $b = 0.3$.

2.2 The Lozi map

Lozi map [22, 23] is a piecewise linear simplification of the Hénon map and it admits strange attractors. It is given by

$$\begin{cases} y_1(k) = 1 - a|(y_1(k-1))| + by(k-1), \\ y(k) = y_1(k-1). \end{cases} \tag{2}$$

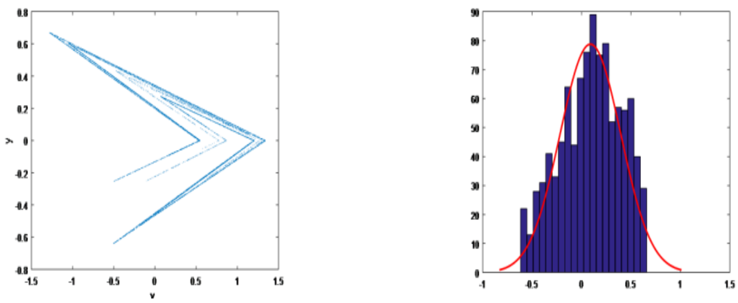

Figure 2: A chaotic Lozi attractor obtained for a = 1.7 and b = 0.5.

2.3 The Duffing map

The Duffing map (also called the 'Holmes map') [24] is a discrete-time dynamical system. It is an example of a dynamical system that exhibits chaotic behavior. The Duffing map takes a point (x_n, y_n) in the plane and maps it to a new point given by

$$\begin{cases} y_1(k) = y(k-1), \\ y(k) = -by_1(k-1) + y_1(k-1) - y(k-1)^3. \end{cases} \tag{3}$$

The map depends on two constants a and b. These are usually set to $a = 2.75$ and $b = 0.2$ to produce chaotic behaviour.

It is a discrete version of the Duffing equation.

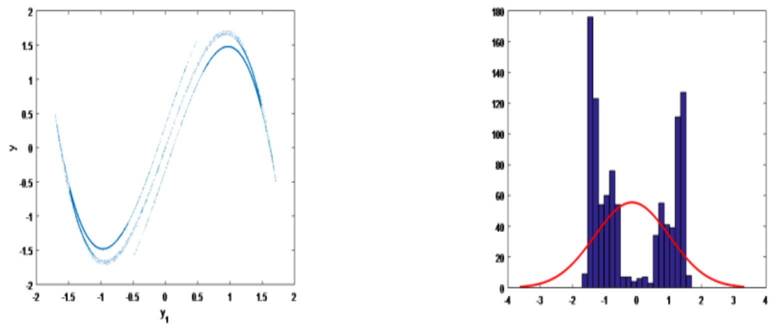

Figure 3: A chaotic Duffing attractor obtained for a = 2.75 and b = 0.2.

2.4 The Gingerbreadman map

In dynamical systems theory, the Gingerbreadman map [25] is a chaotic two-dimensional map. It is given by the piecewise linear transformation

$$\begin{cases} y_1(k) = 1 - a(y_1(k-1))^2 + by(k-1), \\ y(k) = y_1(k-1). \end{cases} \tag{4}$$

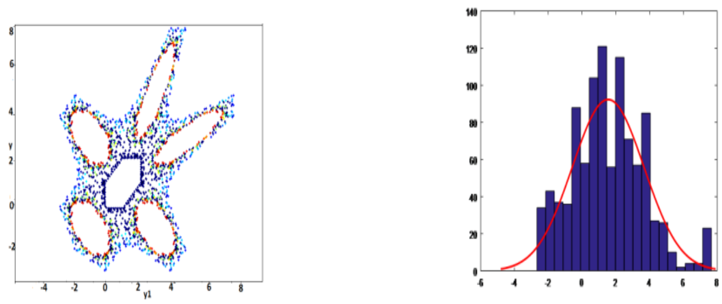

Figure 4: A chaotic Gingerbreadman attractor obtained for a = 1 and b = 1.

2.5 The Zeraoulia map

In dynamical systems theory, the Zeraoulia map [26] is a chaotic two-dimensional map. It is given by the piecewise linear transformation

$$\begin{cases} y_1(k) = 1 - asin(y_1(k-1)) + y(k-1), \\ y(k) = by_1(k-1). \end{cases} \tag{5}$$

The choice of the term $sin(x)$ has an important role in that it makes the solutions bounded for the values of b such that $|b| \leq 1$, and all values of a, while they are unbounded for $|b| > 1$. The chosen parameter values are $a = 4$ and $b = 0.9$ as suggested in [26]. For these values the observed attractor shown in Figure 5.

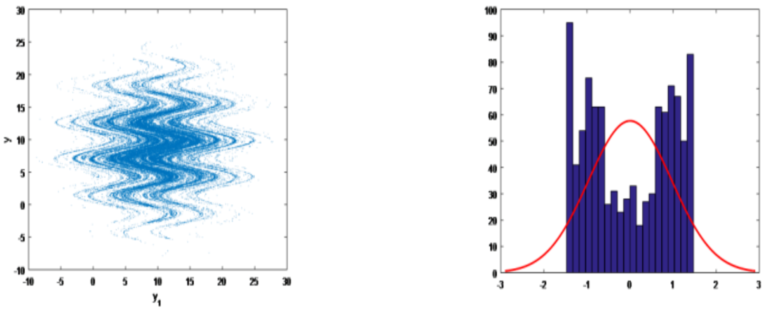

Figure 5: A chaotic Zraoulia attractor obtained for a = 4 and b = 0.9.

3 Chaos Optimization Search

The study of chaos has been rapidly developed and attracted a great attention due to a variety of applications in science and technology, e.g., chaos-based global optimization. The chaos optimization algorithm (COA) is one of the hot topics in recent years. The COA is an effective method to solve the optimization problem of a nonlinear multimodal function with boundary constraint. Many chaotic strategies in the COA generally include two major stages [17–19]: the global phase and the local phase. Firstly, during the global phase, chaotic points are drawn from the domain of searches $[L, U]$ according to a certain 2-D chaotic model. Then, the objective function is evaluated at these points and the point with the minimum objective function as the current optimum is chosen. Secondly, during the local phase, the current optimum is assumed to be close to the global optimum after certain iterations and it is viewed as the center with a little chaotic perturbation and the global optimum is obtained through the fine search.

Consider the following optimization problem on the minimum of functions. If the target function $f(x_i)$ is continuous and differentiable, the object problem to be optimized is find x_i to minimize $f(x_i); x_i \in [L_i, U_i]; i = 1, 2, ..., n$. The main procedures of this algorithm are shown as follows:

Input :

M_g : maximum number of iterations of the global search.

M_l : maximum number of iterations of the local search.
$M_l + M_g$: stopping criterion of the chaotic optimization method in iterations
λ : step size in the chaotic local search
Output :
X^* : best solution from the current run of the chaotic search.
f^* : best objective function (minimization problem).

- **Step 1 :** *Initialization* of the numbers M_g, M_l of steps of the chaotic search and initialization of the parameters λ and initial conditions. Set $k = 1, y(0), y_1(0).a = 1.4$ and $b = 0.3$ of the Henon map, $a = 1.7$ and $b = 0.5$ of the Lozi map, $a = 2.75$ and $b = 0.2$ of the Duffing map, $a = 4$ and $b = 0.9$ of the Zraoulia map. Set the initial best objective function $f^* = infini$.

- **Step 2 :** *algorithm of the chaotic global search:*
  Map the chaotic variables $z_i(k) = \frac{(x_i(k)-L_i)}{(U_i-L_i)}$ into the optimization variables $x_i(k)$ by the following equation in the chaotic map function:

  $$x_i(k) = L_i + (U_i - L_i)z_i(k),$$

  where $i = 1, 2, ..., n$.

  Equation $x_i(k) = L_i + (U_i - L_i)z_i(k)$ is suitable for most chaotic maps. It is determined by the range of the chaotic sequences generated by each chaotic map to select the equation. As the chaotic sequences generated by chaotic maps is the interval $(0, 1)$, equation $x_i(k) = L_i + (U_i - L_i)z_i(k)$ can map $(0, 1)$ into the interval (L, U) for optimization variables.

- **Step 3 :** compute the function value $f(x(k))$. If $f(x(k)) < f^*$, then $f^* = f(x(k))$ and the optimal solution $x^* = x(k)$.

- **Step 4 :** utilize a chaotic map function to generate next chaotic variables $z_i(k+1)$.

- **Step 5 :** $k = k + 1$. If $k \leq M_g$, turn to step 2, otherwise terminate the first stage search.

- **Step 6 :** *algorithm of the chaotic local search:*
  If $r < 0.5$, then (where r is a uniformly distributed random)
  Map the chaotic variables $z_i(k)$ into the optimization variables $x_i(k)$ by one of the following equations of the chaotic map function:

  $$x_i(k+1) = x_i^* + \lambda.z_i(k).|U_i - L_i^*|,$$

  $$x_i(k+1) = x_i^* - \lambda.z_i(k).|U_i - L_i^*|,$$

  where $i = 1, 2, \ldots, n$.

- **Step 7 :** compute the function value $f(x(k_1)), f(x(k_2))$. Take the minimum value of the two as $f(x(k))$. If $f(x(k_1)) < f(x(k_2))$, then $x(k) = x(k_1), f(x(k)) = f(x(k_1))$; otherwise $x(k) = x(k_2), f(x(k)) = f(x(k_2))$. Compare $f(x(k_1))$ with the optimal value, so far f^*. If $f(x(k)) < f^*$, then $f^* = f(x(k))$ and the optimal solution $x^* = x(k)$.

- **Step 8 :** utilize a chaotic map function to generate next chaotic variables $z_i(k+1)$.

- **Step 9 :** $k = k+1$. If $k \le M_g + M_l$, turn to step 6, otherwise terminate the second stage search.

4 Simulation Results

The proposed algorithm was tested on two benchmark functions, see Table 1, Figures 6, 7 . All the programs were run on a 2 GHz Pentium IV processor with 2 GB of random access memory in the MATLAB. The algorithm used for comparison is a two-stage chaotic optimization algorithm with five chaotic maps. The algorithm was executed with 50 runs; M_g=1000, M_l=400, and different values for the step size λ (such $\lambda = 0.01$, $\lambda = 0.001$ and $\lambda \in]0.001, 0.01[$). Tables 2,3,4 show the best solution, the mean of the solution and standard deviation. From Tables 2, 3, 4, all of the best solutions are exactly equal to the exact solution of the function 2. From Tables 3, 4, the Hénon map and Zeraoulia map have better solutions for $\lambda = 0.001$ and $\lambda \in]0.001, 0.01[$ of function 1 than other maps according to the best solution. The Hénon map, Lozi map and Gingerbreadman map have better solutions for $\lambda = 0.001$ and $\lambda \in]0.001, 0.01[$ of function 2 than other maps according to the best solution.

Function name	Expression	bounds	Opt	Modality
The Schaffer	$F_1(x_1, x_2) = -0.5\frac{(\sin\sqrt{(x_1^2+x_2^2)}^2)-0.5)}{(1+0.001(x_1^2+x_2^2)^2}$	$[-100, 100]$	-1	Multimodal
The Easom	$F_2(x_1, x_2) = \cos(x_1)\cos(x_2)\exp(-(x_1-\pi)^2-(x_2-\pi)^2)$	[-20,20]	-1	unimodal

Table 1: Properties of benchmark functions.

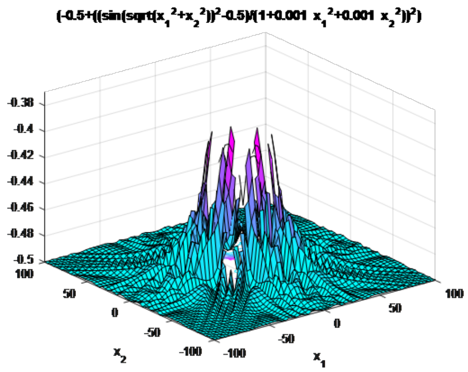

Figure 6: Plot of F_1.

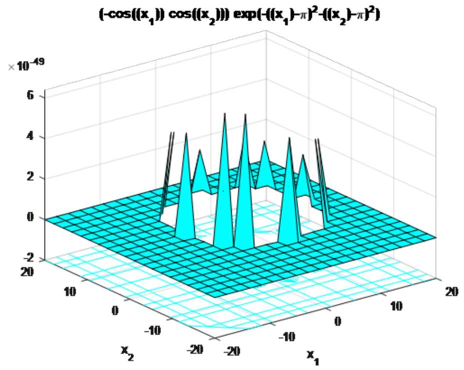

Figure 7: Plot of F_2.

$\lambda = 0.01$					
Fun-ction	Maps	Best fit	(Xbest,Ybest)	Mean fit	standard fit
F_1	Hénon	-0.9897	(3.1331,0.4282)	-0.9897	1.0e-15×0.1121
	Lozi	-0.8304	(5.7743,10.8775)	-0.8304	1.0e-14×0.1009
	Duffing	-0.9398	(0.4017,-6.1062)	-0.9398	1.0e-14×0.0224
	Gingerbreadman	-0.8217	(-11.7448,-11.7448)	-0.8217	0.0000
	Zraoulia	-0.9870	(3.1432,-0.5825)	-0.9870	1.0e-15×0.6729
F_2	Hénon	-0.9961	(3.1739 ,3.1812)	-09960	1.0e-03×0.0452
	Lozi	-0.9859	(3.0529,3.1054)	-0.9818	0.0021
	Duffing	-0.9963	(3.1819,3.1128)	-0.9961	0.0002
	Gingerbreadman	-0.9918	(3.0893,3.0893)	-0.9918	1.0e-15×0.4486
	zraoulia	-0.9961	(3.1006,3.1736)	-0.9960	1.0e-03×0.0718

Table 2: COA based five chaotic saerches so that M_g=1000, M_l=400, for 50 run.

$\lambda = 0.001$					
Function	Maps	Best fit	(Xbest,Ybest)	Mean fit	standard fit
F_1	Hénon	-0.9899	(3.1432,0.4698)	-0.9899	1.0e-14×0.0336
	Lozi	-0.8727	(5.4511 ,11.0601)	-08726	0.0001
	Duffing	-0.9628	(0.2366,-6.2706)	-0.9627	0.0001
	Gingerbreadman	-0.8218	(-11.4213,-11.4217)	-0.8214	0.0004
	Zeraoulia	-0.9903	(3.0870,-0.6231)	-0.9902	0.0001
F_2	Hénon	-1	(3.0706,3.0244)	-0.9983	0.0053
	Lozi	-0.9999	(3.1335,3.1428)	-0.9999	0.001
	Duffing	-1	(3.1393,3.1363)	-1	0.0000
	Gingerbreadman	-1	(3.0303,3.030)	-0.9971	0.0072
	Zeraoulia	-1	(3.1421,3.1388)	-1	1.0e-0.3×0.0020

Table 3: COA based five chaotic searches so that M_g=1000, M_l=400, for 50 run.

$\lambda \in]0.01, 0.001[$					
Function	Maps	Best fit	(Xbest,Ybest)	Mean fit	standard fit
F_1	Hénon	-0.9899	(3.0829,0.4698)	-0.9899	1.0e-14×0.0336
	Lozi	-0.8727	(5.4511 ,11.0601)	-0.8726	0.0001
	Duffing	-0.9628	(0.2366,-6.2706)	-0.9627	0.0001
	Gingerbreadman	-0.8218	(-11.4213,-11.4217)	-0.8214	0.0004
	Zeraoulia	-0.9903	(3.0870,-0.6231)	-0.9902	0.0001
F_2	Hénon	-1	(3.0706,3.0244)	-0.9984	0.0053
	Lozi	-0.9999	(3.1303,3.1385)	-0.9999	0.0001
	Duffing	-1	(3.1399,3.1363)	-1	0.0000
	Gingerbreadman	-1	(2.9741,2.9741)	-0.9931	0.0179
	Zeraoulia	-1	(3.1424,3.1388)	-1	1.0e-03×0.0020

Table 4: COA based five chaotic searches so that M_g=1000,M_l=400, for 50 run.

5 Conclution

In this paper, we have proposed some two-dimensional maps which can be used as search patterns in the chaos optimization algorithm. We use five chaotc map searches. Our main conclusion is made by comparing different search patterns based on the numerical simulation results. We exhibited the generated chaotic sequences and the obtained best chaotic sequences. Further, this algorithm is tested on a benchmark consisting of two known nonlinear objective functions.

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